PICTURE GROUPS AND MAXIMAL GREEN SEQUENCES

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Abstract. We show that picture groups are directly related to maximal green sequence. Namely, there is a bijection between maximal green sequences and positive expressions (words in the generators without inverses) for the coxeter element of the picture groups.

Contents

1. Introduction

We give the statements of the main theorems and an outline of the proofs. Although the results here are mainly in finite type, we have made the assumptions more general for applications to quivers of infinite type which is ongoing joint work with Thomas Brüstle and Steven Hermes. All quivers will be without oriented cycles.

1.1. **Basic definitions.** In order to state the theorems we need the basic definitions. We assume that Q is a valued quiver and that $\Phi_+(Q)$ is the set of (positive) real Schur roots of Q. These are the dimension vectors of the exceptional modules over any modulated quiver with underlying valued quiver Q. We need to order these roots in two different ways which we call "lateral" and "vertical" ordering. One example of "lateral" ordering is the left-to-right order

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of roots in the order they appear in the Auslander-Reiten quiver. One example of "vertical" ordering is by order of length.

Definition 1.1.1. Let $\mathcal{S} = (\beta_1, \cdots, \beta_m)$ be an ordered set of real Schur roots. We say that $\mathcal{S}$ is an *admissible* sequence of roots if it satisfies Conditions (1) and (2) below.

(1) If $\beta_m \in \mathcal{S}$ and $\beta' \subset \beta_m$ is any subroot of β_m then $\beta' = \beta_j$ for some $j \leq m$.
(2) If $\beta_m \in \mathcal{S}$ and $\beta'' \twoheadleftarrow \beta_m$ is any quotient root of β_m then $\beta'' = \beta_k$ for some $k \leq m$.

We say that $\mathcal{S}$ is *weakly admissible* if, instead of conditions (1), (2) we have the following weaker conditions.

(1') If $\beta_m \in \mathcal{S}$ and $\beta' \subset \beta_m$ then either $\beta' = \beta_j$ for some $j \leq m$ or all components of β_m/β' lie in $\mathcal{S}$ and come before β_m.
(2') If $\beta_m \in \mathcal{S}$ and $\beta'' \twoheadleftarrow \beta_m$ then either $\beta'' = \beta_k$ for some $k \leq m$ or all components of the kernel of $\beta_k \twoheadrightarrow \beta''$ lie in $\mathcal{S}$ and come before β_m.

We say that the roots are *vertically ordered/weakly vertically ordered* if the sets are admissible/weakly admissible and ordered as prescribed above.

For example, a finite set of real Schur roots closed under subroots and quotient roots is admissible if the roots are ordered by length. If we take any admissible set, remove one object and keep the same order on what is left, the result will be weakly admissible. If the last object β_m is removed, the resulting set will still be admissible. We will do induction on m to study maximal green sequences.

Definition 1.1.2. By a *lateral ordering* $\leq$ on a set of real Schur roots $\mathcal{S}$ we mean a total ordering on $\mathcal{S}$ satisfying the following two conditions.

(1) If $hom(\alpha, \beta) \neq 0$ then $\alpha \leq \beta$.
(2) If $ext(\alpha, \beta) \neq 0$ then $\alpha > \beta$.

In particular, if ω is a rightmost root in $\mathcal{S}$ then $ext(\beta, \omega) = 0$ for all $\beta \in \mathcal{S}$ and $hom(\omega, \beta') = 0$ for all $\beta' \neq \omega \in \mathcal{S}$. Note that ω, as well as the leftmost root α, will, in general, be in the middle of an admissible sequence of roots.

For example, the left-to-right order of preprojective roots as they occur in the Auslander-Reiten quiver is a lateral ordering. Regular roots might not admit a lateral ordering. Given any set of roots $\mathcal{S}$, a subset $\mathcal{R} \subseteq \mathcal{S}$ will be called *relatively closed under extension* if $\mathcal{R}$ contains all roots which are iterated extensions of roots in $\mathcal{R}$. Two examples are $\mathcal{S} \backslash \omega$ and $\mathcal{R}_-(\omega) := \{\beta \in \mathcal{S} : hom(\beta, \omega) = 0\}$ where ω is a rightmost root in $\mathcal{S}$. Induction using such subsets of $\mathcal{S}$ will be used to prove theorems about pictures and the picture group.

Definition 1.1.3. For admissible $\mathcal{S} = (\beta_1, \cdots, \beta_m)$, we define the *picture group* $G(\mathcal{S})$ as follows. There is one generator $x(\beta)$ for each $\beta \in \mathcal{S}$. There is the following relation for each pair β_i, β_j of hom-orthogonal roots with $ext(\beta_i, \beta_j) = 0$:

$$x(\beta_i)x(\beta_j) = \prod x(\gamma_k) \tag{1.1}$$

where γ_k runs over all roots in $\mathcal{S}$ which are linear combinations $\gamma_k = a_k\beta_i + b_k\beta_j$ in increasing order of the ratio a_k/b_k (going from $0/1$ where $\gamma_1 = \beta_j$ to $1/0$ where $\gamma_k = \beta_i$). For any $g \in G(\mathcal{S})$, we define a *positive expression* for g to be any word in the generators $x(\beta)$ (with no $x(\beta)^{-1}$ terms) whose product is g.

Remark 1.1.4. If β_i, β_j are hom-orthogonal and $\beta_i < \beta_j$ in lateral order then

$$x(\beta_i)x(\beta_j) = x(\beta_j)w$$

where w is a positive expression in letters γ where $\beta_i \leq \gamma < \beta_j$ in lateral order since $hom(\beta_i, \gamma) \neq 0$ and $hom(\gamma, \beta_j) \neq 0$ when $\gamma \neq \beta_i$.

An important case is when $j = m$. For β_i, β_m hom-orthogonal we get

$$x(\beta_i)x(\beta_m) = x(\beta_m)x(\beta_i).$$

since the other roots γ_k in the formula above come after β_m so do not lie in $\mathcal{S}$.

We recall that, for all roots β, there is a unique exceptional module M_β with dimension vector β. The subset $D(\beta) \subseteq \mathbb{R}^n$ is given by

$$D(\beta) = \{x \in \mathbb{R}^n : \langle x, \beta\rangle = 0 \text{ and } \langle x, \beta'\rangle \leq 0 \ \forall \beta' \subset \beta\}$$

where $\beta' \subset \beta$ means that M_β contains a submodule isomorphic to $M_{\beta'}$.

Definition 1.1.5. Given $\mathcal{S} = (\beta_1, \cdots, \beta_m)$ weakly admissible and $\epsilon = (\epsilon_1, \cdots, \epsilon_m) \in \{0, +, -\}^m$, let be the convex open set given by

$$\mathcal{U}_\epsilon = \{x \in \mathbb{R}^n : \langle x, \beta_i\rangle < 0 \text{ if } \epsilon_i = - \text{ and } \langle x, \beta_j\rangle > 0 \text{ if } \epsilon_j = +\}.$$

ϵ will be called *admissible* (with respect to $\mathcal{S}$) if for all $1 \leq k \leq m$ we have:

$$\epsilon_k = 0 \iff D(\beta_k) \cap \mathcal{U}_{\epsilon_1, \cdots, \epsilon_{k-1}} = \emptyset$$

When ϵ is admissible the open set $\mathcal{U}_\epsilon$ will be called an *$\mathcal{S}$-compartment*.

Definition 1.1.6. For any weakly admissible $\mathcal{S}$, we define a *maximal $\mathcal{S}$-green sequence* to be a maximal sequence of $\mathcal{S}$-compartments $\mathcal{U}_{\epsilon(0)}, \cdots, \mathcal{U}_{\epsilon(s)}$ so that every pair of consecutive compartments $\mathcal{U}_{\epsilon(i-1)}, \mathcal{U}_{\epsilon(i)}$ is separated by a wall $D(\beta_{k_i})$ so that $\epsilon(i-1)_{k_i} = -$ and $\epsilon(i)_{k_i} = +$. We define an *$\mathcal{S}$-green path* to be a continuous path, $\gamma : [0, s] \to \mathbb{R}^n$, so that $\gamma(t) \in \mathcal{U}_{\epsilon(i)}$ whenever $|t - i| < 1/2$ and $\gamma(t)$ goes from the negative side to the positive side of $D(\beta_{k_i})$ when t crosses the value $i - 1/2$. We say that the green path γ *represents* the green sequence $\mathcal{U}_{\epsilon(i)}$.

1.2. **Statement of the main results.** The main property of $\mathcal{U}_\epsilon$ is that it is convex and nonempty when $\mathcal{S}$ is weakly admissible. Furthermore, in the admissible case, the complements of the union of these regions forms a picture for $G(\mathcal{S})$. We use the notation $CL(\mathcal{S})$ for the union of all $D(\beta)$ where $\beta \in \mathcal{S}$ and $L(\mathcal{S})$ for the intersection of $CL(\mathcal{S})$ with the unit sphere S^{n-1} in $\mathbb{R}^n$.

Theorem 1.2.1. *When $\mathcal{S}$ is admissible, $L(\mathcal{S}) \subset S^{n-1}$ with labels $x(\beta)$ on $D(\beta) \cap S^{n-1}$ forms a picture for the picture group $G(\mathcal{S})$. The complement in $\mathbb{R}^n$ of $CL(\mathcal{S})$ is equal to the disjoint union of the $\mathcal{S}$-compartments $\mathcal{U}_\epsilon$.*

Proposition 1.2.2. *Every convex region $\mathcal{U}_\epsilon$ lies in a maximal $\mathcal{S}$-green sequence. Each maximal $\mathcal{S}$-green sequence begins with the region containing $-\underline{\dim}\,\Lambda$ and ends with the region containing $\underline{\dim}\,\Lambda$.*

Proof. Given any region $\mathcal{U}_\epsilon$, consider the straight line from a general point $v \in \mathcal{U}_\epsilon$ to the dimension vector $\underline{\dim}\,\Lambda$. Except in the case when $\underline{\dim}\,\Lambda \in \mathcal{U}_\epsilon$, this line will pass though some wall $D(\beta)$ of $\mathcal{U}_\epsilon$ and this wall is green since

$$\langle \underline{\dim}\,\Lambda, \beta\rangle > 0$$

for all positive roots β. Similarly, the straight line from v to $-\underline{\dim}\,\Lambda$ passes through a red wall of $\mathcal{U}_\epsilon$ except in the case when $-\underline{\dim}\,\Lambda \in \mathcal{U}_\epsilon$. $\square$

Lemma 1.2.3. *Take any maximal $\mathcal{S}$-green sequence and consider the sequence of walls $D(\beta_{k_1}), \cdots, D(\beta_{k_s})$ which are crossed by the sequence. Then the product of the corresponding generators $x(\beta_{k_i}) \in G(\mathcal{S})$ is equal to a fixed element of the picture group $G(\mathcal{S})$, namely:*

$$x(\beta_{k_1}) \cdots x(\beta_{k_s}) = \prod x(\alpha_i)$$

where $\alpha_i \in \mathcal{S}$ are the simple roots in Λ which lie in $\mathcal{S}$ in admissible order (i.e., $ext(\alpha_i, \alpha_j) = 0$ for $i < j$).

We denote this fixed element $c_{\mathcal{S}} = \prod x(\alpha_i)$.

Proof. This follows immediately from Theorem 1.2.1 and Proposition 1.2.2. It is a general property of pictures that there is a well defined group element $g(\mathcal{U}_\epsilon) \in G(\mathcal{S})$ associated to each region in such a way that the identity element of the group is assigned to a fixed region, in this case the region containing $-\underline{\dim}\,\Lambda$, and that when passing through a wall $D(\beta)$ the group label will change by multiplication by $x(\beta)$ or $x(\beta)^{-1}$ depending on which direction the wall is oriented. Therefore, the group element associated to the region containing $+\underline{\dim}\,\Lambda$ is equal to the product of the positive expression associated to any maximal $\mathcal{S}$-green sequence. Since one such sequence is the minimal such path which goes through the walls $D(\alpha_i)$ in increasing order of the simple roots α_i, the group element associated to the region containing $+\underline{\dim}\,\Lambda$ is the Coxeter element for the group $c_{\mathcal{S}} = x(\alpha_1) \cdots x(\alpha_n)$. □

This lemma can be rephrased as follows. Any maximal $\mathcal{S}$-green sequence gives a positive expression for $c_{\mathcal{S}}$ by reading the labels of the walls which are crossed by the sequence. The main theorem of this paper is the following theorem and its corollary.

Theorem 1.2.4. *Suppose that $\mathcal{S}$ is admissible and admits a lateral ordering. Then, the operation which assigns to each maximal $\mathcal{S}$-green sequence a positive expression for $c_{\mathcal{S}}$ is a bijection.*

It is clear that distinct maximal $\mathcal{S}$-green sequences give distinct positive expressions. Therefore, it suffices to show that every positive expression for $c_{\mathcal{S}}$ can be realized as a maximal $\mathcal{S}$-green sequence.

Corollary 1.2.5. *For any acyclic valued quiver Q of finite type there is a bijection between maximal green sequences and positive expressions for $c_Q = x(\alpha_1) \cdots x(\alpha_n)$.*

Proof. Let $\mathcal{S}$ be the set of positive roots of Q vertically arranged in order of length and laterally arranged according to the Auslander-Reiten quiver. Since this is admissible, the theorem applies. □

1.3. **Outline of proof of main theorem.** The proof is by induction on m, the size of the finite set $\mathcal{S}$. If $m = 1$, the root β_1 must be simple. So, the group $G(\mathcal{S})$ is infinite cyclic with generator $x(\beta_1)$ which is equal to $c_{\mathcal{S}}$. There are two compartments $\mathcal{U}_1, \mathcal{U}_{-1}$ separated by the single hyperplane $D(\beta_1) = H(\beta_1)$. And $\mathcal{U}_-, \mathcal{U}_+$ is the unique $\mathcal{S}$-green sequence. The associated positive expression is $x(\beta_1)$ which is the unique positive expression for $c_{\mathcal{S}}$. So, the result holds for $m = 1$.

A similar argument works when all the roots in $\mathcal{S}$ are simple. Then the hyperplanes $D(\beta_i) = H(\beta_i)$ divide $\mathbb{R}^n$ into 2^m compartments. The group $G(\mathcal{S})$ is free abelian of rank m. The Coxeter element $c_{\mathcal{S}}$ is the product of the generators. So, there are $m!$ positive expressions for $c_{\mathcal{S}}$. Each expression is realized by a maximal $\mathcal{S}$-green sequence. For example the sequence $x(\beta_1) \cdots x(\beta_m)$ is realized by the $\mathcal{S}$-green path $\gamma_t = (2t-1, 2t^2-1, \cdots, 2t^m-1)$ for $0 \le t \le 1$.

Now suppose that $m \geq 2$, the theorem holds for $m-1$ and β_m is not simple. Let $\mathcal{S}_0 = (\beta_1, \cdots, \beta_{m-1})$. Recall that there is an group epimorphism

$$\pi : G(\mathcal{S}) \twoheadrightarrow G(\mathcal{S}_0)$$

given by sending each $x(\beta_i) \in G(\mathcal{S})$ to the generator in $G(\mathcal{S}_0)$ with the same name when $i < m$ and sending $x(\beta_m)$ to 1. Then $\pi(c_\mathcal{S}) = c_{\mathcal{S}_0}$. Suppose that w is a positive expression for $c_\mathcal{S}$ in $G(\mathcal{S})$. Let $\pi(w) = w_0$ be the positive expression for $c_{\mathcal{S}_0}$ in $G(\mathcal{S}_0)$ given by deleting every instance of the generator $x(\beta_m)$ from w. By induction on m, there exists a unique maximal $\mathcal{S}_0$-green sequence $\mathcal{U}_{\epsilon(0)}, \cdots, \mathcal{U}_{\epsilon(s)}$ which realizes the positive expression w_0. These fall into two classes.

Class 1. Each region $\mathcal{U}_{\epsilon(i)}$ in the maximal $\mathcal{S}_0$-green sequence is disjoint from $D(\beta_m)$.

For maximal green sequences in this class, each $\mathcal{U}_{\epsilon(i)} = \mathcal{U}_{\epsilon'(i)}$ where $\epsilon'(i) = (\epsilon_1, \cdots, \epsilon_m)$ with $\epsilon_m = 0$ and $\epsilon(i) = (\epsilon_1, \cdots, \epsilon_{m-1})$. Therefore, the maximal $\mathcal{S}_0$-green sequence $\mathcal{U}_{\epsilon(i)}$ is also a maximal $\mathcal{S}$-green sequence and $w_0 = w$ by the following lemma proved in subsection 3.5. So, the positive expression w is realized by a maximal $\mathcal{S}$-green sequence.

Lemma 1.3.1. *Let w, w' be two positive expressions for the same element of the group $G(\mathcal{S})$. Suppose $\pi(w) = \pi(w')$, i.e., the two expressions are identical modulo the generator $x(\beta_m)$. Then $x(\beta_m)$ occurs the same number of times in w, w'. In particular, $x(\beta_m) \neq 1$ in $G(\mathcal{S})$.*

In the case at hand, $w' = w_0$ does not contain the letter $x(\beta_m)$. So, neither does w and we must have $w = w_0$ as claimed. So, the theorem hold when $w_0 = \pi(w)$ corresponds to a maximal $\mathcal{S}_0$-green sequence of Class 1.

Class 2. At least one region in the $\mathcal{S}_0$-green sequence meets $D(\beta_m)$.

For green sequences in this class, the regions which intersect $D(\beta_m)$ are consecutive:

Lemma 1.3.2. *Let $\mathcal{U}_{\epsilon(0)}, \cdots, \mathcal{U}_{\epsilon(s)}$ be a maximal $\mathcal{S}_0$-green sequence. Then*

1. *The $\mathcal{U}_{\epsilon(i)}$ which meet $D(\beta_m)$ are consecutive, say $\mathcal{U}_{\epsilon(p)}, \cdots, \mathcal{U}_{\epsilon(q)}$.*
2. *Let $D(\beta_{k_i})$ be the wall between $\mathcal{U}_{\epsilon(i-1)}$ and $\mathcal{U}_{\epsilon(i)}$ so that $w_0 = x(\beta_{k_1}) \cdots x(\beta_{k_s})$. Then β_{k_i} is hom-orthogonal to β_m if $p < i \leq q$ and $\beta_{k_p}, \beta_{k_{q+1}}$ are not hom-orthogonal to β_m.*
3. *For $p < i \leq q$ and $\delta \in \{+, -\}$, $D(\beta_{k_i})$ is also the wall separating $\mathcal{U}_{\epsilon(i-1),\delta}$ and $\mathcal{U}_{\epsilon(i),\delta}$.*

This lemma tells us: (1) The $\mathcal{S}_0$-compartments $\mathcal{U}_{\epsilon(r)}$ for $p \leq r \leq q$ are divided into two $\mathcal{S}$-compartments by the wall $D(\beta_m)$. (3) The same walls separating the $\mathcal{S}_0$-compartments $\mathcal{U}_{\epsilon(r)}$ also separate the sequence of $\mathcal{S}$-compartments $\mathcal{U}_{\epsilon(r),-}$ and the sequence of $\mathcal{S}$-compartments $\mathcal{U}_{\epsilon(r),+}$.

So, we can refine the maximal $\mathcal{S}_0$-green sequence to a maximal $\mathcal{S}$-green sequence, by staying on the negative side of $D(\beta_m)$ for until we reach the $\mathcal{S}$-compartment $\mathcal{U}_{\epsilon(r),-}$ for some $p \leq r \leq q$, then cross through $D(\beta_m)$ into $\mathcal{U}_{\epsilon(r),+}$ and continue in the given $\mathcal{S}_0$-compartments but on the positive side of $D(\beta_m)$. This gives the maximal $\mathcal{S}$-green sequence

$$\mathcal{U}_{\epsilon(0),0}, \cdots, \mathcal{U}_{\epsilon(p-1),0}, \mathcal{U}_{\epsilon(p),-}, \cdots, \mathcal{U}_{\epsilon(r),-}, \mathcal{U}_{\epsilon(r),+}, \cdots, \mathcal{U}_{\epsilon(q),+}, \mathcal{U}_{\epsilon(q+1),0}, \cdots, \mathcal{U}_{\epsilon(s),0}$$

of length $s+2$ giving the positive expression

$$w_r = x(\beta_{k_1}) \cdots x(\beta_{k_p}) \cdots x(\beta_{k_r}) x(\beta_m) x(\beta_{k_{r+1}}) \cdots x(\beta_{k_q}) \cdots x(\beta_{k_s}).$$

By the defining relations in the group $G(\mathcal{S})$, the generators $x(\beta)$ and $x(\beta_m)$ commute if β is hom-orthogonal to β_m. By (3) in the lemma this implies that w_r is a positive expression for $c_\mathcal{S}$ if $p \leq r \leq q$. We have just shown that each such w_r is realizable by a maximal $\mathcal{S}$-green sequence. So, it remains to show that the positive expression w that we started with is equal to one of these w_r.

By Lemma 1.3.1, $x(\beta_m)$ occurs exactly once in the expression w. We need to show that, if the generator $x(\beta_m)$ occurs in the "wrong place" then w is not a positive expression for $c_{\mathcal{S}}$, in other words, the product of the elements of w is not equal to $c_{\mathcal{S}}$. This follows from the following lemma proved in subsection 3.5.

Lemma 1.3.3. *Let*

$$\mathcal{R}(\beta_m) = \{\beta_i \in \mathcal{S}_0 : hom(\beta_i, \beta_m) = 0 = hom(\beta_m, \beta_i)\}.$$

Let $\beta_{j_1}, \cdots, \beta_{j_s}$ be elements of $\mathcal{S}_0$ which do not all lie in $\mathcal{R}(\beta_m)$. Then $x(\beta_m)$, $\prod x(\beta_{j_i})$ do not commute in the group $G(\mathcal{S})$.

By part (2) of the previous lemma, $\beta_{k_r} \in \mathcal{R}(\beta_m)$ if $p < r \le q$ and $\beta_{k_p}, \beta_{k_{q+1}} \notin \mathcal{R}(\beta_m)$. So, this lemma implies that w_r is a positive expression for $c_{\mathcal{S}}$ if and only if $p \le r \le q$. So, we must have $w = w_r$ for one such r and w is realizable. This concludes the outline of the proof of the main theorem. It remains only to prove the three lemmas invoked in the proof.

2. Properties of compartments $\mathcal{U}_\epsilon$

We derive the basic properties of the compartments $\mathcal{U}_\epsilon$ and prove Lemma 1.3.2. The basic property is the following.

Proposition 2.0.4. *For all weakly admissible $\mathcal{S}$ and all admissible ϵ the open region $\mathcal{U}_\epsilon$ is convex and nonempty. Moreover, when $\epsilon_m \neq 0$, or equivalently, when $D(\beta_m) \cap \mathcal{U}_{\epsilon_1,\cdots,\epsilon_{m-1}}$ is nonempty, we have the following.*

1. *$\langle x, \beta_i\rangle < 0$ for all subroots $\beta_i \subset \beta_m$ with $i < m$ and for all $x \in \mathcal{U}_{\epsilon_1,\cdots,\epsilon_i}$.*
2. *$\langle x, \beta_j\rangle > 0$ for all quotient roots $\beta_j \twoheadleftarrow \beta_m$ with $j < m$ and for all $x \in \mathcal{U}_{\epsilon_1,\cdots,\epsilon_j}$.*
3. $$D(\beta_m) \cap \mathcal{U}_{\epsilon_1,\cdots,\epsilon_k} = H(\beta_m) \cap \mathcal{U}_{\epsilon_1,\cdots,\epsilon_k}$$
   where k is the maximum of all i, j which appear in (1),(2) above. ($k = 0$ if there are no such i, j.)

Proof. When $m = 1$, β_1 is simple and $k = 0$ in (3). So, all statement are trivially satisfied. So, suppose $m \ge 2$ and all statements hold for $m - 1$.

To prove (1), suppose that $\beta_i \subsetneq \beta_m$ where $i < m$. By induction on i we may assume that (1) and (2) hold for all $i', j < i$. Suppose that $\langle y, \beta_i\rangle \ge 0$ for some $y \in \mathcal{U}_{\epsilon_1,\cdots,\epsilon_i}$. By assumption there also exists $v \in D(\beta_m) \cap \mathcal{U}_{\epsilon_1,\cdots,\epsilon_{m-1}}$. Then $\langle v, \beta_i\rangle \le 0$. So, there is a z on the straight line from v to y in the convex region $\mathcal{U}_{\epsilon_1,\cdots,\epsilon_i}$ so that $\langle z, \beta_i\rangle = 0$. This implies that $\epsilon_i = 0$.

<u>Claim</u> $z \in D(\beta_i)$. This implies that $\epsilon_i \neq 0$ which is a contradiction.

Proof of claim: Let $\beta' \subsetneq \beta_i$. Then either

(a) $\beta' = \beta_{i'}$ for some $i' < i$ or
(b) $\beta_i - \beta'$ is a sum of roots β_{j_s} where all $j_s < i$.

In case (a), $\langle z, \beta'\rangle < 0$ by (1) since $i' < i$. In case (b), $\langle z, \beta_{j_s}\rangle > 0$ for each s since $j_s < i$. So,

$$\langle z, \beta'\rangle = \langle z, \beta_i\rangle - \sum \langle z, \beta_{j_s}\rangle < 0$$

So, $z \in D(\beta_i)$.

This proves the claim. Therefore (1) holds. The proof of (2) is similar.

To prove (3), suppose that $w \in H(\beta_m) \cap \mathcal{U}_{\epsilon_1,\cdots,\epsilon_k}$. Then $\langle w, \beta_m\rangle = 0$ by definition of $H(\beta_m)$. For any proper subroot $\beta' \subsetneq \beta_m$ either

(a) $\beta' = \beta_i$ for some $i < m$ or
(b) $\beta_m - \beta'$ is a sum of roots β_{j_i} where all $j_i < m$.

In case (a), $\langle w, \beta' \rangle < 0$ by (1) since $w \in \mathcal{U}_{\epsilon_1,\cdots,\epsilon_k} \subseteq \mathcal{U}_{\epsilon_1,\cdots,\epsilon_i}$. In case (b), $\langle w, \beta_{j_i} \rangle > 0$ for all i by (2). So, $\langle w, \beta' \rangle = \langle w, \beta_m \rangle - \sum \langle w, \beta_{j_i} \rangle < 0$. So, $w \in D(\beta_m)$ by definition of $D(\beta_m)$.

Finally, we show that each $\mathcal{U}_\epsilon$ is convex and nonempty. If $\epsilon_m = 0$ then $\mathcal{U}_{\epsilon_1,\cdots,\epsilon_m} = \mathcal{U}_{\epsilon_1,\cdots,\epsilon_{m-1}}$ is convex and nonempty by induction on m. So, suppose $\epsilon_m \neq 0$. Then $D(\beta_m) \cap \mathcal{U}_{\epsilon_1,\cdots,\epsilon_{m-1}}$ is nonempty and equal to $H(\beta_m) \cap \mathcal{U}_{\epsilon_1,\cdots,\epsilon_{m-1}}$ by (3). By induction on m we know that $\mathcal{U}_{\epsilon_1,\cdots,\epsilon_{m-1}}$ is convex and nonempty. The hyperplane $H(\beta_m)$ divides this region into two nonempty convex subregions given by $\epsilon_m = 1, -1$. □

2.1. **Inescapable regions.** Suppose that $\mathcal{V}$ is the closure of the union of $\mathcal{S}_0$-compartments $\mathcal{U}_\epsilon$. Then $\mathcal{V}$ has internal and external walls. The *internal walls* of $\mathcal{V}$ are the ones between two of the compartments $\mathcal{U}_\epsilon$ in $\mathcal{V}$. $\mathcal{V}$ has points on both sides of the internal walls. The *external walls* of $\mathcal{V}$ are the ones which separate $\mathcal{V}$ from its complement. The region $\mathcal{V}$ will be called *inescapable* if it is on the positive side of all of its external walls. I.e., they are all red on the inside. Once an $\mathcal{S}_0$-green sequence enters such a region, it can never leave. Since $\mathcal{V}$ is closed, it contains all internal and external walls. We also consider open regions $\mathcal{W}$ which are inescapable regions minus their external walls. Then $\mathcal{W}$ is the complement of the closure of the union of all compartments not in $\mathcal{W}$.

Given an admissible sequence $\mathcal{S}$ with last object β_m which we assume to be nonsimple, let $\mathcal{S}_0 = (\beta_1, \cdots, \beta_{m-1})$. We will construct two inescapable regions $\mathcal{W}(\beta_m), \mathcal{V}(\beta_m)$ where the first is open and the second is closed. All maximal $\mathcal{S}_0$-green sequences start outside both regions, end inside both regions and fall into two classes: those that enter $\mathcal{W}(\beta_m)$ before they enter $\mathcal{V}(\beta_m)$ and those that enter $\mathcal{V}(\beta_m)$ before they enter $\mathcal{W}(\beta_m)$. And these coincide with the two classes of maximal $\mathcal{S}_0$-green sequences discussed in the outline of the main theorem (Corollary 2.2.1).

The first inescapable region is the open set

$$\mathcal{W}(\beta_m) := \{x \in \mathbb{R}^n : \langle x, \alpha \rangle > 0 \text{ for some } \alpha \subsetneq \beta_m\}.$$

Proposition 2.1.1. *The complement of $\mathcal{W}(\beta_m)$ in $\mathbb{R}^n$ is closed and convex. Furthermore:*

$$\mathcal{W}(\beta_m) \cap H(\beta_m) = H(\beta_m) - D(\beta_m). \tag{2.1}$$

Proof. The complement of $\mathcal{W}(\beta_m)$ is

$$\mathbb{R}^n - \mathcal{W}(\beta_m) = \{x \in \mathbb{R}^n : \langle x, \alpha \rangle \leq 0 \text{ for all } \alpha \subsetneq \beta_m\}$$

which is closed and convex since it is given by closed convex conditions $\langle x, \alpha \rangle \leq 0$.

For the second statement, suppose that $v \in H(\beta_m)$. Then $\langle v, \beta_m \rangle = 0$. By the stability conditions which we are using to define $D(\beta_m)$, $v \in D(\beta_m)$ if and only if $\langle v, \alpha \rangle \leq 0$ for all $\alpha \subset \beta_m$, in other words,

$$D(\beta_m) = H(\beta_m) \cap (\mathbb{R}^n - \mathcal{W}(\beta_m))$$

which is equivalent to (2.1). □

Proposition 2.1.2. *The region $\mathcal{W}(\beta_m)$ is inescapable. I.e., all external walls are red. Furthermore, each external walls of $\mathcal{W}(\beta_m)$ has the form $D(\alpha)$ for some $\alpha \subsetneq \beta_m$. Consequently, every $\mathcal{S}_0$-compartment is contained either in $\mathcal{W}(\beta_m)$ or in its complement.*

Proof. Take any external wall $D(\alpha)$ of $\mathcal{W}(\beta_m)$. Let v_t be a continuous path which goes through that wall from inside to outside. In other words, $v_t \in \mathcal{W}(\beta_m)$ for $t < 0$ and $v_t \notin \mathcal{W}(\beta_m)$ for

$t \geq 0$. By definition of $\mathcal{W}(\beta_m)$ this means that there is some $\alpha' \subsetneq \beta_m$ so that $\langle v_t, \beta\rangle$ changes sign from positive to nonpositive at t goes from negative to nonnegative.

By choosing v_t in general position, v_0 will not lie in $H(\alpha')$ for any $\alpha' \neq \alpha$. So, we must have $\alpha \subsetneq \beta_m$. And $\langle v_t, \alpha\rangle > 0$ for $t < 0$ and $\langle v_t, \alpha\rangle < 0$ for $t > 0$. Therefore, $\mathcal{W}(\beta_m)$ is on the positive (red) side of the external wall $D(\alpha)$. So, $\mathcal{W}(\beta_m)$ is inescapable.

Since each part of the boundary lies in $D(\alpha)$ for some $\alpha \in \mathcal{S}_0$, the boundary of $\mathcal{W}(\beta_m)$ is contained in the union of the boundaries of the $\mathcal{S}_0$-compartments. So, all such compartments are either entirely insider or entirely outside $\mathcal{W}(\beta_m)$. □

The second inescapable region is the closed set

$$\mathcal{V}(\beta_m) = \{y \in \mathbb{R}^n : \langle y, \gamma\rangle \geq 0 \text{ for all quotient roots } \gamma \text{ of } \beta_m\}$$

By arguments analogous to the ones above, we get the following.

Proposition 2.1.3. *$\mathcal{V}(\beta_m)$ is a closed convex inescapable region whose external walls all have the form $D(\gamma)$ where $\beta_m \twoheadrightarrow \gamma$. So, every $\mathcal{S}_0$-compartment is contained in $\mathcal{V}(\beta_m)$ or its complement. Furthermore,*

$$\mathcal{V}(\beta_m) \cap H(\beta_m) = D(\beta_m).$$

2.2. **Class 1 and Class 2 maximal $\mathcal{S}_0$-green sequences.** Recall that a maximal $\mathcal{S}_0$-green sequence with $\mathcal{S} = (\beta_1, \cdots, \beta_{m-1})$ is in:

(1) *Class 1* if each $\mathcal{S}_0$-compartment $\mathcal{U}_{\epsilon(i)}$ in the green sequence is disjoint from $D(\beta_m)$.
(2) *Class 2* if at least one $\mathcal{S}_0$-compartment, say $\mathcal{U}_{\epsilon(j)}$, in the $\mathcal{S}_0$-green sequence meets $D(\beta_m)$. So, $\mathcal{U}_{\epsilon(j)}$ is divided into two $\mathcal{S}$-compartments $\mathcal{U}_{\epsilon(j),-}$ and $\mathcal{U}_{\epsilon(j),+}$.

Corollary 2.2.1. *A maximal $\mathcal{S}_0$-green sequence is in Class 1 if and only if it passes through $\mathcal{W}(\beta_m)\backslash\mathcal{V}(\beta_m)$. It is in Class 2 if and only if it contains a compartment in*

$$\mathcal{V}(\beta_m)\backslash\mathcal{W}(\beta_m) = \{x \in \mathbb{R}^n : \langle x, \alpha\rangle \leq 0 \textit{ for all } \alpha \subsetneq \beta_m \textit{ and } \langle x, \gamma\rangle \geq 0 \textit{ for all } \beta_m \twoheadrightarrow \gamma\}.$$

Proof. Every maximal green sequence starts on the negative side of the hyperplane $H(\beta_m)$ and ends on its positive side. Therefore the maximal $\mathcal{S}_0$-green sequence must cross the hyperplane at some point. Since $\beta_m \notin \mathcal{S}_0$, none of the $\mathcal{S}_0$-compartments has $H(\beta_m)$ as a wall. So, there must be one compartment in the $\mathcal{S}_0$-green sequences which meets the hyperplane $H(\beta_m)$. Let $\mathcal{U}_\epsilon$ be the first such compartment. Then, either $\mathcal{U}_\epsilon \cap D(\beta_m)$ is empty or nonempty. In the first case, $\mathcal{U}_\epsilon$ is in $\mathcal{W}(\beta_m)$ and it is outside $\mathcal{V}(\beta_m)$. Since $\mathcal{W}(\beta_m)$ is inescapable and does not meet $D(\beta_m)$, the green sequence is in Class 1. In the second case, $\mathcal{U}_\epsilon$ is in $\mathcal{V}(\beta_m)$ and not in $\mathcal{W}(\beta_m)$ and the green sequence is in Class 2. So, these two cases correspond to Class 1 and Class 2 proving the corollary. □

Recall that $\mathcal{R}(\beta_m)$ is the set of all $\alpha \in \mathcal{S}_0$ which are hom-orthogonal to β_m.

Proposition 2.2.2. *If $\alpha \in \mathcal{S}_0$ and $D(\alpha)$ meets the interior of the closed region $\mathcal{V}(\beta_m)\backslash\mathcal{W}(\beta_m)$ then $\alpha \in \mathcal{R}(\beta_m)$. Conversely, if $\alpha \in \mathcal{R}(\beta_m)$ then $D(\alpha)$ is an internal wall in $\mathcal{V}(\beta_m)\backslash\mathcal{W}(\beta_m)$.*

Proof. Let $\mathcal{V}_0$ denote the interior of $\mathcal{V}(\beta_m)\backslash\mathcal{W}(\beta_m)$. Then

$$\mathcal{V}_0 = \{x \in \mathbb{R}^n : \langle x, \alpha\rangle < 0 \text{ for all } \alpha \subsetneq \beta_m \text{ and } \langle x, \gamma\rangle > 0 \text{ for all } \beta_m \twoheadrightarrow \gamma, \gamma \neq \beta_m\}.$$

Suppose that $x \in D(\alpha) \cap \mathcal{V}_0$ and $hom(\beta_m, \alpha) \neq 0$. Then there is a subroot α' of α which is also a quotient root of β_m: $\beta_m \twoheadrightarrow \alpha' \subset \alpha$. Since $\alpha \in \mathcal{S}_0$ we cannot have $\beta_m \subset \alpha$. Therefore α' is a proper quotient of β_m. Then $\langle x, \alpha'\rangle > 0$ since $x \in \mathcal{V}_0$ and $\langle x, \alpha'\rangle \leq 0$ since $x \in D(\alpha)$

and $\alpha' \subset \alpha$. This is a contradiction. So, $hom(\beta_m, \alpha) = 0$. A similar argument shows that $hom(\alpha, \beta_m) = 0$. So, $\alpha \in \mathcal{R}(\beta_m)$.

Conversely, if $\alpha \in \mathcal{R}(\beta_m)$ then α, β_m span a rank 2 wide subcategory $\mathcal{A}(\alpha, \beta_m)$ of mod-Λ. Choose any tilting object T in the left perpendicular category ${}^{\perp}\mathcal{A}(\alpha, \beta_m)$ (for example the sum of the projective objects). Then $\underline{\dim}\, T$ lies in the interior of both $D(\alpha)$ and $D(\beta_m)$ and therefore also lies in $\mathcal{V}_0$. So, $D(\alpha)$ meets $\mathcal{V}_0$. ($\underline{\dim}\, T$ lies in the interior of $D(\alpha)$ since T can be completed to a cluster tilting object of ${}^{\perp}\alpha$ in two different ways making $\underline{\dim}\, T$ the center point of the common face of two full dimensional simplices in $D(\alpha)$.) □

Corollary 2.2.3. *The open region $\mathcal{V}_0$ contains no vertices of the picture $L(\mathcal{S}_0)$.*

Proof. Suppose that $v \in \mathcal{V}_0$ is a vertex of $L(\mathcal{S}_0)$. Then v is a positive scalar multiple of a real Schur root β (not necessarily an element of $\mathcal{S}$). Let $\mathcal{A} = \beta^{\perp}$. This is rank $n-1$ wide subcategory of mod-Λ. Let $\alpha_1, \cdots, \alpha_{n-1}$ be the dimension vectors of the simple objects of $\mathcal{A}$. Since v is a vertex of the picture $L(\mathcal{S}_0)$, these roots α_i must be elements of $\mathcal{S}_0$ and $v = \beta$ is the unique point in the intersection of the sets $D(\alpha_i)$.

By the proposition each α_i is hom-orthogonal to β_m. This implies that $\alpha_1, \cdots, \alpha_{n-1}$ together with β_m form the simple roots of a wide subcategory of rank n. But this must be all of mod-Λ. So, β_m must be a simple root contrary to our initial assumption. Therefore $\mathcal{V}_0$ contains no vertices of $L(\mathcal{S}_0)$. □

Corollary 2.2.4. *Let $\alpha_1, \cdots, \alpha_k$ be pairwise hom-orthogonal elements of $\mathcal{R}(\beta_m)$ then the intersection $D(\alpha_1) \cap \cdots \cap D(\alpha_k) \cap D(\beta_m) \cap \mathcal{V}_0$ is nonempty.*

Proof. More precisely, let $\mathcal{A}(\alpha_1, \cdots, \alpha_k, \beta_m)$ be the rank $k+1$ wide subcategory of mod-Λ with simple objects $M_{\alpha_i}, M_{\beta_m}$. Let $T = T_1 \oplus \cdots \oplus T_{n-k-1}$ be any cluster tilting object of the cluster category of ${}^{\perp}\mathcal{A}(\alpha_1, \cdots, \alpha_k, \beta_m)$. Then we claim that for $r_i > 0$, the vector $\sum r_i \underline{\dim}\, T_i$ is a point in $D(\alpha_1) \cap \cdots \cap D(\alpha_k) \cap D(\beta_m)$ which lies in the interior of $D(\beta_m)$. This can be proved by induction on k using the argument in the proof of Proposition 2.2.2. □

2.3. **Proof of Lemma 1.3.2.** We will show that maximal $\mathcal{S}_0$-green sequences satisfy the three properties listed in Lemma 1.3.2.

Proposition 2.3.1. *An $\mathcal{S}_0$-compartment $\mathcal{U}_\epsilon$ meets $D(\beta_m)$ if and only if $\mathcal{U}_\epsilon \subseteq \mathcal{V}_0$.*

Before proving this we show that this implies the first property in Lemma 1.3.2. Recall that this states:

(1) *In every maximal $\mathcal{S}_0$-green sequence in Class 2, the compartments which meet $D(\beta_m)$ are consecutive.*

Proof. Let $\mathcal{U}_{\epsilon(i)}$ be a maximal $\mathcal{S}_0$-green sequence. Let p, q be minimal so that $\mathcal{U}_{\epsilon(p)} \subseteq \mathcal{V}(\beta_m)$ and $\mathcal{U}_{\epsilon(q)} \subseteq \mathcal{W}(\beta_m)$. When the green sequence is in Class 2, $p < q$. Since $\mathcal{V}(\beta_m)$ is inescapable, $\mathcal{U}_{\epsilon(i)} \subseteq \mathcal{V}(\beta_m)$ iff $p \leq i$. Since $\mathcal{W}(\beta_m)$ is inescapable, $\mathcal{U}_{\epsilon(i)} \subseteq \mathcal{V}_0$ iff $p \leq i < q$. So, the compartments of the green sequence which lie in $\mathcal{V}_0$ are consecutive. By the proposition these are the compartments which meet $D(\beta_m)$. □

Proof of Proposition 2.3.1. Let $\mathcal{U}_\epsilon$ be an $\mathcal{S}_0$-compartment in $\mathcal{V}_0$. Let $x \in \mathcal{U}_\epsilon$. If $\langle x, \beta_m \rangle = 0$ then $x \in H(\beta_m) \cap \mathcal{V}_0 \subset D(\beta_m)$ and we are done. So, suppose $\langle x, \beta_m \rangle \neq 0$. Pick a point $y \in D(\beta_m) \cap \mathcal{V}_0$ and take the straight line from x to y. Since $\mathcal{V}_0$ is convex, this line is entirely contained in $\mathcal{V}_0$. If the line is not in $\mathcal{U}_\epsilon$ then it must meet an internal wall $D(\alpha)$ on the boundary of $\mathcal{U}_\epsilon$. By Proposition 2.2.2, $\alpha \in \mathcal{R}(\beta_m)$.

Let k be maximal so that the closure of $\mathcal{U}_\epsilon$ contains a point $z \in D(\alpha_*) = D(\alpha_1) \cap \cdots \cap D(\alpha_k)$ where $\alpha_1, \cdots, \alpha_k \in \mathcal{R}(\beta_m)$ are pairwise hom-orthogonal. Then, by Corollary 2.2.4, $D(\alpha_*) \cap D(\beta_m) \cap \mathcal{V}_0$ is nonempty. Let w be an element. Since $D(\alpha_*)$ and $\mathcal{V}_0$ are both convex, $D(\alpha_*) \cap \mathcal{V}_0$ contains the straight line $\gamma(t) = (1-t)z + tw$, $0 \le t \le 1$.

Let δ be a very small vector so that $\langle \delta, \beta_m \rangle = 0$ and $z + \delta \in \mathcal{U}_\epsilon$. Consider the line $\gamma(t) + \delta$. This is in $\mathcal{U}_\epsilon$ for $t = 0$ and lies in $D(\beta_m)$ when $t = 1$. This proves the proposition if $\gamma(t)+\delta \in \mathcal{U}_\epsilon$ for all $0 \le t \le 1$. So, suppose not. Let t_0 be minimal so that this open condition fails. Then the line $\gamma(t)$ meets another wall at $t = t_0$ and $\gamma(t_0)$ will be a point in the closure of $\mathcal{U}_\epsilon$ which meets a codimension $k+1$ set $D(\alpha_0) \cap D(\alpha_1) \cap \cdots \cap D(\alpha_k)$ where $\alpha_0 \in \mathcal{S}_0$ is hom-orthogonal to the other roots α_i. (Take α_0 of minimal length among the new roots so that $\gamma(t_0) \in D(\alpha_0)$.) This contradicts the maximality of k. So, there is no point t_0 and $\gamma(1) + \delta \in \mathcal{U}_\epsilon \cap D(\beta_m)$ as claimed. $\square$

We have already shown property (2) in Lemma 1.3.2: Any maximal $\mathcal{S}_0$-green sequence of Class 2 crosses a wall $D(\gamma)$ at some point to enter region $\mathcal{V}_0$, passes through several internal walls of $\mathcal{V}_0$, then exists $\mathcal{V}_0$ by a wall $D(\alpha)$ of $\mathcal{W}(\beta_m)$. By Propositions 2.1.3, 2.1.2 γ is a quotient root of β_m and α is a subroot of β_m, both not hom-orthogonal to β_m. By Proposition 2.2.2 and the internal walls of $\mathcal{V}_0$ are $D(\beta)$ where $\beta \in \mathcal{R}(\beta_m)$. So, property (2) in Lemma 1.3.2 holds.

The last property in Lemma 1.3.2 is the following.

(3) *Suppose that the two $\mathcal{S}_0$-compartments $\mathcal{U}_{\epsilon(1)}$ and $\mathcal{U}_{\epsilon(2)}$ meet along a common internal wall $D(\beta_j)$. Then the $\mathcal{S}$-compartments $\mathcal{U}_{\epsilon(1),+}, \mathcal{U}_{\epsilon(2),+}$ meet along the common internal wall $D(\beta_j)$ and the $\mathcal{S}$-compartments $\mathcal{U}_{\epsilon(1),-}, \mathcal{U}_{\epsilon(2),-}$ also meet along $D(\beta_j)$.*

Proof. Let $\mathcal{S}'$, $\mathcal{S}'_0$ be $\mathcal{S}, \mathcal{S}_0$ with β_j deleted. Then $\mathcal{S}', \mathcal{S}'_0$ are weakly admissible. Since $\beta_j \notin \mathcal{S}'_0$, the two $\mathcal{S}_0$-compartments $\mathcal{U}_{\epsilon(1)}$ and $\mathcal{U}_{\epsilon(2)}$ merge to form one $\mathcal{S}'_0$-compartment $\mathcal{U}_\epsilon$. This compartment meets $D(\beta_m)$ so it breaks up into two $\mathcal{S}'$-compartments $\mathcal{U}_{\epsilon,+}$ and $\mathcal{U}_{\epsilon,-}$. We know that $D(\beta_j)$ must divide these two $\mathcal{S}'$-compartments into four $\mathcal{S}$-compartments. Since $\mathcal{S}'$-compartments are convex by Proposition 2.0.4, this can happen only if $D(\beta_j)$ meets both $\mathcal{S}'$-compartments and forms the common wall separating the two halves of each. $\square$

3. Planar pictures and group theory

In this section we will use 2-dimensional pictures to prove the two properties of the group $G(\mathcal{S})$ that we are using: Lemmas 1.3.1 and 1.3.3. The key tool will be the "sliding lemma" which comes from the first author's PhD thesis [3]. Unless otherwise stated, all pictures in this section will be 2-dimensional. We begin with a review of the topological definition of a (planar) picture with special language coming from the fact that all relations in our group $G(\mathcal{S})$ are commutator relations. Since this section uses only planar diagrams, we feel that theorems can be proven using diagrams and topological arguments. Algebraic versions of the arguments using HNN extensions, geometric realizations of categories and cubical CAT(0) categories can be found in other papers which prove similar results for pictures of arbitrary dimension ([8], [5], [6]).

3.1. Planar pictures.

Suppose that the group G has a presentation $G = \langle \mathcal{X} \mid \mathcal{Y} \rangle$. This means there is an exact sequence

$$R_\mathcal{Y} \hookrightarrow F_\mathcal{X} \twoheadrightarrow G$$

where $F_\mathcal{X}$ is the free group generated by the set $\mathcal{X}$ and $R_\mathcal{Y} \subseteq F_\mathcal{X}$ is the normal subgroup generated by the subset $\mathcal{Y} \subseteq F_\mathcal{X}$. Then G is the fundamental group of a 2-dimensional CW-complex X^2 given as follows. Let X^1 denote the 1-dimensional CW-complex having a single 0-cell e^0, one 1-cell $e^1(x)$ for every generator $x \in \mathcal{X}$ attached on e^0. Then $\pi_1 X^1 = F_\mathcal{X}$ and any $f \in F_\mathcal{X}$ gives a continuous mapping $\eta_f : S^1 \to X^1$ given by composing the loops corresponding to each letter in the unique reduced expression for f. Here $S^1 = \{z \in \mathbb{C} : ||z|| = 1\}$, $1 \in S^1$ is the basepoint and S^1 is oriented counterclockwise.

Let X^2 denote the 2-dimensional CW-complex given by attaching one 2-cell $e^2(r)$ for every relations $r \in \mathcal{Y}$ using an attaching map

$$\eta_r : S^1 \to X^1$$

homotopic to the one described above. We choose each mapping η_r so that it is transverse to the centers of the 1-cells of X^1. So, the inverse images of these center points are fixed finite subsets of S^1. The relation r is given by the union of these finite sets, call it E_r, together with a mapping $\lambda : E_r \to \mathcal{X} \cup \mathcal{X}^{-1}$ indicating which 1-cell the point goes to and in which direction the image of η_r traverses that 1-cell. Then we have:

$$r = \prod_{x \in E_r} \lambda(x) \in F_\mathcal{X}.$$

The circle S^1 is the boundary of the unit disk $D^2 = \{x \in \mathbb{C} : ||x|| \leq 1\}$. Let

$$CE_r := \bigcup_{x \in E_r} \{ax \in D^2 : 0 \leq a \leq 1\}.$$

Example:
$r = xyx^{-1}z^{-1}$

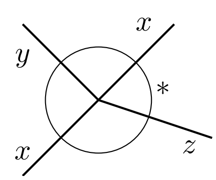


FIGURE 1. The cone of E_r in D^2 is the part inside the circle S^1. The asterisks $*$ indicates the position of the basepoint $1 \in S^1$ in the event the figure is rotated. The labels are drawn on the positive side of each edge.

A picture is a geometric representation of a continuous pointed mapping $\theta : S^2 \to X^2$ where pointed means preserving the base point. A (pointed) deformation of a picture represents a homotopy of such a mapping. Deformation classes of pictures form a module over the group ring $\mathbb{Z}G$.

Definition 3.1.1. Given a group G with presentation $G = \langle \mathcal{X} \,|\, \mathcal{Y} \rangle$ and fixed choices of $E_r \subset S^1$, $\lambda : E^4 \to \mathcal{X} \cup \mathcal{X}^{-1}$, a *picture* for G is defined to be a graph L embedded in the plane $\mathbb{R}^2$ with circular edges allowed, together with:

1. a label $x \in \mathcal{X}$ for every edge in L,
2. a normal orientation for each edge in L,
3. a label $r \in \mathcal{Y} \cup \mathcal{Y}^{-1}$ for each vertex in L,
4. for each vertex v, a smooth (C^∞) embedding $\theta_v : D^2 \to \mathbb{R}^2$ sending 0 to v

satisfying the following where $E(x)$ denotes the union of edges labeled x.

(a) Each $E(x)$ is a smoothly embedded 1-manifold in $\mathbb{R}^2$ except possibly at the vertices.
(b) For each vertex $v \in L$, $\theta_v^{-1}(E(x))$ is equal to the cone of $E_r^{-1}(\{x, x^{-1}\})$.

The image of $1 \in S^1$ under $\theta_v : D^2 \to \mathbb{R}^2$ will be called the *basepoint direction* of v and will be indicated with $*$ when necessary.

The embedding θ_v has positive, negative orientation when $r \in \mathcal{Y}$, $r \in \mathcal{Y}^{-1}$, respectively.

One easy consequence of this definition is the following.

Proposition 3.1.2. *Given a picture L for G, there is a unique label $g(U) \in G$ for each component U of the complement of L in $\mathbb{R}^2$ having the following properties.*

(1) $g(U_\infty) = 1$ *for the unique unbounded component* U_∞.
(2) $g(V) = g(U)x$ *if the regions U, V are separated by an edge labeled x and oriented towards V.*

Proof. For any region U, choose a smooth path from ∞ to any point in U. Make the path transverse to all edge sets. Then let $g(U) = x_1^{\epsilon_1} \cdots x_m^{\epsilon_m}$ if the path crosses m edges labeled $x_1, \cdots, x_m$ with orientations given by ϵ_i. This is well defined since any deformation of the path with push it through a vertex and the paths on either side of the vertex have edge labels giving a relation in the group and therefore give the same product of labels. □

Remark 3.1.3. Any particular smooth path γ from U_∞ to U gives a lifting $f_\gamma(U)$ of $g(U)$ to the free group $F_\mathcal{X}$.

The main theorem about general pictures is that the set of deformation classes of pictures for any group G is a $\mathbb{Z}G$-module. The action of the group G is very easy to describe. Given any picture L and any generator $x \in \mathcal{X}$, the pictures $xL, x^{-1}L$ are given by enclosing the set L with a large circle, labeling the circle with x and orienting it inward or outward. Addition of pictures is given by disjoint union of translates of the pictures.

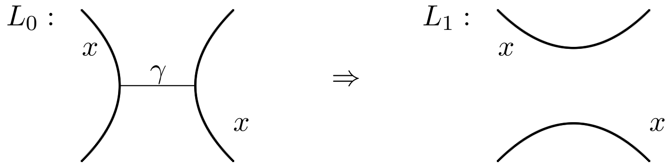
To define the equivalence relation which we call "deformation equivalence" of pictures, it is helpful to associate to each picture L an element $\psi(L) \in \mathbb{Z}G\langle\mathcal{Y}\rangle$ where $\mathbb{Z}G\langle\mathcal{Y}\rangle$ is the free $\mathbb{Z}G$ module generated by the set of relations $\mathcal{Y}$. This is given by

$$\psi(L) = \sum_{v_i} g(v_i) \langle r_i \rangle$$

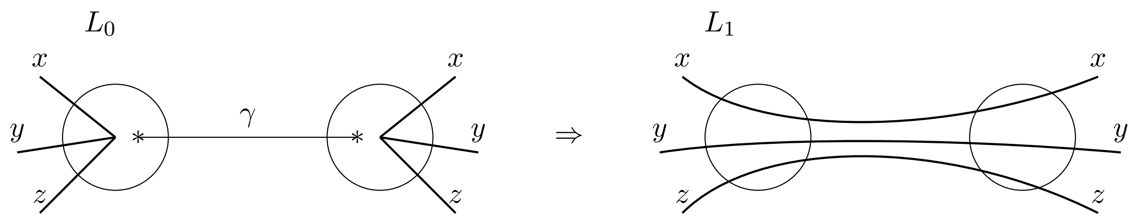
where the sum is over all vertices v_i of L, $r_i \in \mathcal{Y} \cup \mathcal{Y}^{-1}$ is the relation at v_i, $g(v_i) \in G$ is the group label at the basepoint direction of v_i and $\langle r^{-1} \rangle = -\langle r \rangle$ by definition.

Definition 3.1.4. A *deformation* $L_0 \simeq L_1$ of pictures for G is defined to be a sequence of allowable moves given as follows.

(1) Isotopy. $L_0 \simeq L_1$ if there is an orientation preserving diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ so that $L_1 = \varphi(L_0)$ with corresponding labels. By isotopy we can make the images of the embeddings $\theta_v : D^2 \to \mathbb{R}^2$ disjoint and arbitrarily small.
(2) Smooth concordance of edge sets. There are two concordance moves:
    (a) If L_0 contains a circular edge E with no vertices and L_0 does not have any point in the region enclosed by E then $L_0 \simeq L_1$ where L_1 is obtained from L_0 by deleting E.
    (b) If U is a connected component of $\mathbb{R}^2 - L_0$ and two of the walls of U have the same label x oriented in the same way (inward towards U or outward) then, choose a path γ in U connecting points on these two edges then perform the following modification of L_0 in a neighborhood of γ to obtain $L_1 \simeq L_0$.



(3) Cancellation of vertices. Suppose that two vertices v_0, v_1 of L_0 have inverse labels r, r^{-1}. Suppose that there is a path γ disjoint from L_0 connecting the basepoint directions of v_0, v_1. Let V be the union of the $\theta_{v_0}(D^2), \theta_{v_1}(D^2)$ and a small neighborhood of the path γ. We can choose V to be diffeomorphic to D^2. Then $L_0 \simeq L_1$ if L_0, L_1 are identical outside of the region V and L_1 has no vertices in V. (The two vertices in $V \cap L_0$ cancel.)



Concordance means L_0, L_1 have the same vertex sets and are equal in a neighborhood of each vertex and that $f_{\gamma_i} \in F_{\mathcal{X}}$ are equal for L_0, L_1 for some (and thus every) choice of paths γ_i disjoint from vertices from ∞ to the basepoint direction of each vertex of L_0. The same paths work for L_1 since L_0, L_1 have the same vertex set.

Theorem 3.1.5. [4] *L_0, L_1 are deformation equivalent if and only if $\psi(L_0) = \psi(L_1)$. Furthermore, the set of possible values of $\psi(L)$ for all pictures L is equal to the kernel of the mapping*

$$\mathbb{Z}G\langle\mathcal{Y}\rangle \xrightarrow{d_2} \mathbb{Z}G\langle\mathcal{X}\rangle$$

where $d_2\langle r\rangle = \sum \partial_x r\langle x\rangle$, where ∂_x is the Fox derivative of r with respect to x.

The *Fox derivative* of $w \in F_{\mathcal{X}}$ is given recursively on the reduced length of w by

(1) $\partial_x(x) = 1$, $\partial_x(x^{-1}) = -x^{-1}$.
(2) $\partial_x(y) = 0$ if $y \in \mathcal{X} \cup \mathcal{X}^{-1}$ is not equal to x, x^{-1}.
(3) $\partial_x(ab) = \partial_x a + a\partial_x b$ for any $a, b \in F_{\mathcal{X}}$.

The *picture group* $P(G)$ is defined to be the group of deformation classes of pictures for G.

Corollary 3.1.6. *There is an exact sequence of $\mathbb{Z}G$-modules*

$$0 \to P(G) \to \mathbb{Z}G\langle\mathcal{Y}\rangle \xrightarrow{d_2} \mathbb{Z}G\langle\mathcal{X}\rangle \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $d_1 \sum a_i\langle x_i\rangle = \sum a_i(x_i - 1)$ and $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ is the augmentation map.

Remark 3.1.7. The chain complex $\mathbb{Z}G\langle\mathcal{Y}\rangle \xrightarrow{d_2} \mathbb{Z}G\langle\mathcal{X}\rangle \xrightarrow{d_1} \mathbb{Z}G$ is the cellular chain complex of the universal covering $\widetilde{X^2}$ of the 2-dimensional CW complex X^2 constructed above. Since $\widetilde{X^2}$ is simply connected, we have

$$P(G) = H_2(\widetilde{X^2}) = \pi_2(\widetilde{X^2}) = \pi_2(X^2).$$

Therefore, $P(G) = \pi_2(X^2)$ as claimed at the beginning of this subsection.

We also “partial pictures”. These are given by putting a picture in half using a straight line transverse to the picture.

Definition 3.1.8. Let w be a word in $\mathcal{X} \cup \mathcal{X}^{-1}$ given by a finite subset W of the x-axis together with a mapping $W \to \mathcal{X} \cup \mathcal{X}^{-1}$. A *partial picture* with boundary w is defined to be a closed subset L of the upper half plane so that the intersection of L with the x-axis is equal to W together with labels on L so that the union of L and its mirror image L_- in the lower half plane is a picture for G and so that the labels on the edges which cross the x-axis agree with the given mapping $W \to \mathcal{X} \cup \mathcal{X}^{-1}$. We call $L \cup L_-$ the *double* of L.

A *deformation* of a partial picture L is defined to be any deformation of its double which is, at all times, transverse to the x-axis. It is clear that deformation of partial pictures preserves its boundary $\partial L = w$ and that w lies in the relation group $R_\mathcal{Y} \subseteq F_\mathcal{X}$. The main theorem about partial pictures is the following.

Theorem 3.1.9. [4] *The set of deformation classes of partial pictures forms a (noncommutative) group $Q(G)$ given by generators and relations as follows.*

1. *The generators of $Q(G)$ are pairs (f, r) where $f \in F_\mathcal{X}$ and $r \in \mathcal{Y}$.*
2. *The relations in $Q(G)$ are given by*

$$(f, r)(f', r')(f, r)^{-1} = (frf^{-1}f', r')$$

Note that there is a well defined group homomorphism

$$\varphi : Q(G) \to F_\mathcal{X}$$

given by $\varphi(f, r) = frf^{-1}$. The image is $R_\mathcal{Y}$, the normal subgroup generated by all $r \in \mathcal{Y}$.

3.2. **Pictures with good commutator relations.** If the same letter, say x, occurs more than twice in a relation r, then, at the vertex v, the edge set $E(x)$ cannot be a manifold. (For example, if $G = \langle x \,|\, x^3 \rangle$ then $E(x)$ will not be a manifold.) However, this does not happen in our case because our relations are “good”.

We define a *good commutator relation* to be a relation of the form

$$r(a, b) := ab(bc_1, \cdots, c_k a)^{-1}$$

where $a, b, c_1, \cdots, c_k$ are distinct elements of $\mathcal{X}$ and $k \geq 0$. The letters a, b will be called *X-letters* and the letters c_j will be called *Y-letters* in the relation. In the picture, the two X-letters in any commutator relation form the shape of the letter “X” since the lines labeled with these letters go all the way through the vertex. Call theses *X-edges* at the vertex. The edges labeled with the Y-letters go only half way and stop at the vertex. Call these *Y-edges* at the vertex. (See Figure 2.) In the definition of a picture we can choose the sets $E_r \subset S^1$ so that the points labeled a, a^{-1} (and b, b^{-1}) are negatives of each other. Then the edge sets $E(a), E(b)$ will be manifolds. (Since a, b, c_j are all distinct there are no other coincidences of labels at the vertices.)

We have the following trivial observation.

Proposition 3.2.1. *Suppose that $G = \langle \mathcal{X} \,|\, \mathcal{Y} \rangle$ is a group having only good commutator relations. Then, given any label x, the edge set $E(x)$ in L is a disjoint union of smooth simple closed curves and smooth paths. At both endpoints of each path, x occurs as a Y-letter. It occurs as x at one end and x^{-1} at the other.*

Example:
$r = r(a, b) = ab(bc_1c_2c_3a)^{-1}$

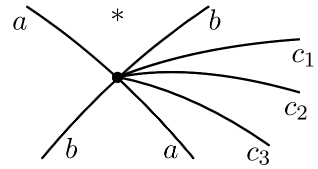


FIGURE 2. The X-letters a, b have edge sets which are smooth at the vertex. The basepoint direction is on the negative side of both X-edges $E(a), E(b)$.

Corollary 3.2.2. *Suppose that G has only commutator relations. Then, for any picture L for G and any label x, the number of vertices of L having x as Y-letter is equal to the number of vertices of L having x^{-1} as Y-letter.*

3.3. **Atoms.** Let $\mathcal{S} = (\beta_1, \cdots, \beta_m)$ be an admissible sequence of real Schur roots for a hereditary algebra Λ. Then $G(\mathcal{S})$ has only good commutator relations. We need the Atomic Deformation Theorem which says that every picture in $G(\mathcal{S})$ is a linear combination of "atoms". In other words atoms generate $P(G)$. The definition comes from [6] and [8] but is based on [7] where similar generators of $P(G)$ are constructed for a torsion-free nilpotent group G.

Suppose that $\mathcal{S}$ is admissible and $\alpha_* = (\alpha_1, \alpha_2, \alpha_3)$ is a sequence of three hom-orthogonal roots in $\mathcal{S}$ ordered in such a way that $ext(\alpha_i, \alpha_j) = 0$ for $i < j$. Let $\mathcal{A}(\alpha_*)$ be the rank 3 wide subcategory of *mod*-Λ with simple objects α_*. One easy way to describe this category is

$$\mathcal{A}(\alpha_*) = ({}^{\perp}M_{\alpha_*})^{\perp}$$

where $M_{\alpha_*} = M_{\alpha_1} \oplus M_{\alpha_2} \oplus M_{\alpha_3}$. In other words, $\mathcal{A}(\alpha_*)$ is the full subcategory of *mod*-Λ of all modules X having the property that $\mathrm{Hom}(X, Y) = 0 = \mathrm{Ext}(X, Y)$ for all Y having the property that $\mathrm{Hom}(M_{\alpha_*}, Y) = 0 = \mathrm{Ext}(M_{\alpha_*}, Y)$. The objects of $\mathcal{A}(\alpha_*)$ are modules M having filtrations where the subquotients are M_{α_i}. Since $ext(\alpha_i, \alpha_j) = 0$ for $i < j$, the modules M_{α_1} occur at the bottom of the filtration and M_{α_3} occurs at the top of the filtration. Let $ab(\alpha_*)$ denote the set of all dimension vectors of the objects of $\mathcal{A}(\alpha_*)$. The elements of $ab(\alpha_*)$ are all nonnegative integer linear combinations of the roots α_i. These are elements of the 3-dimensional vector space $\mathbb{R}\alpha_*$ spanned by the roots α_*.

Let $L(\alpha_*) \subseteq S^2$ be the semi-invariant picture for the category $\mathcal{A}(\alpha_*)$. We recall ([6], [8], [5]) that $L(\alpha_*)$ is the intersection with the unit sphere $S^2 \subseteq \mathbb{R}\alpha_* \cong \mathbb{R}^3$ with the union of the 2-dimensional subset $D(\beta)$ of $\mathbb{R}\alpha_*$ where $\beta \in ab(\alpha_*)$ given by the semi-invariants conditions:

$$D(\beta) := \left\{x \in \mathbb{R}\alpha_* : \langle x, \beta\rangle = 0, \left\langle x, \beta'\right\rangle \geq 0 \text{ for all } \beta' \subset \beta,\ \beta' \in ab(\alpha_*)\right\}$$

When we stereographically project $L(\alpha_*) \subset S^2$ into the plane $\mathbb{R}^2$ we get a picture for the group $G(ab(\alpha_*))$ according to the definitions in this paper.

Definition 3.3.1. Let $\mathcal{S}, \alpha_*$ be as above. Then the *atom* $A_{\mathcal{S}}(\alpha_*) \subset \mathbb{R}^2$ is defined to be the picture for $G(\mathcal{S})$ given by taking the semi-invariant picture $L(\alpha_*) \subset S^2$, stereographically projecting it away from the point $-\sum \underline{\dim}\, P_i \in \mathbb{R}\alpha_*$ where P_i are the projective objects of $\mathcal{A}(\alpha_*)$ and deleting all edges having labels $x(\gamma)$ where $\gamma \notin \mathcal{S}$.

Figure 3 gives an example. We need to prove that the certain aspects of the shape are universal.

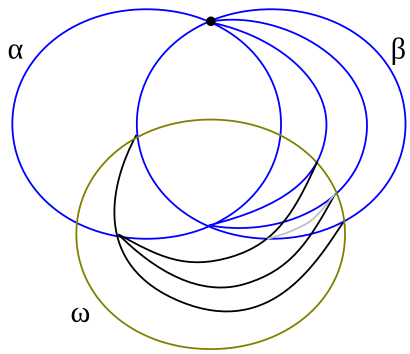


FIGURE 3. The atom $A_{\mathcal{A}}(\alpha, \beta, \omega)$. There are three circles labeled α, β, ω. There is only one vertex (black dot) outside the brown circle labeled ω. There is only one vertex inside the α circle. The faint gray line is deleted since, in this example, its label is not in the set $\mathcal{S}$.

Proposition 3.3.2. *Any atom $A_{\mathcal{S}}(\alpha_1, \alpha_2, \alpha_3)$ has three circles $E(\alpha_i) = D(\alpha_i)$ with labels $x(\alpha_i) \in G$ and all other edge sets have two endpoints. There is exactly one vertex v outside the α_3 circle. This vertex has the relation $r(\alpha_1, \alpha_2)$. Dually, there is exactly one vertex inside the α_1 circle with relation $r(\alpha_2, \alpha_3)^{-1}$.*

We use the notation $r(\alpha, \beta)$ for $r(x(\alpha), x(\beta))$ For example, the blue lines in Figure 2 meet at two vertices giving the relations

$$r(\alpha, \beta) = x(\alpha)x(\beta)\left(x(\beta)x(\gamma_1)x(\gamma_2)x(\alpha)\right)^{-1}$$

at the top and $r(\alpha, \beta)^{-1}$ in the middle of the brown $x(\omega)$ circle.

Proof. The only objects of $\mathcal{A}(\alpha_*)$ which do not map only M_{α_3} are the objects of $\mathcal{A}(\alpha_1, \alpha_2)$ which are the objects $M_{\alpha_1}, M_{\alpha_2}$ and extensions M_{γ_j}. These give the terms in the commutator relation $r(\alpha_1, \alpha_2)$ and these lines meet at only two vertices in the atom. All other edges of the atom have at least one abutting edge with a label γ where $\gamma \twoheadrightarrow \alpha_3$. By the stability condition defining $D(\gamma)$, these points must be inside or on the α_3 circle as claimed. $\square$

3.4. **Sliding Lemma and Atomic Deformation Theorem.** We will prove the Sliding Lemma and derive some consequences such as the Atomic Deformation Theorem which says that every picture for $G(\mathcal{S})$ is a linear combination of atoms. First, some terminology. We say that L' is an *atomic deformation* of L if L' is a deformation of L plus a linear combination of atoms. Thus the Atomic Deformation Theorem states that every picture has an atomic deformation to the empty picture.

Suppose that $\mathcal{S}$ is an admissible set of roots with a fixed lateral ordering and let $\omega \in \mathcal{S}$. Then let $\mathcal{R}_-(\omega)$ be set of all $\beta \in \mathcal{S}$ with $\beta < \omega$ in the lateral ordering so that β, ω are hom-orthogonal. Since $\beta < \omega$ implies $hom(\omega, \beta) = 0 = ext(\beta, \omega)$ the condition that β, ω are hom-perpendicular is equivalent to the linear condition:

$$\langle \beta, \omega \rangle = 0.$$

Lemma 3.4.1. *The homomorphism $G(\mathcal{R}_-(\omega)) \to G(\mathcal{S})$ induced by inclusion has a retraction ρ given on generators by*

$$\rho(x(\beta)) = \begin{cases} x(\beta) & \text{if } \beta \in \mathcal{R}_-(\omega) \\ 1 & \text{otherwise} \end{cases}$$

Furthermore, ρ takes pictures and partial pictures L for $G(\mathcal{S})$ and gives a picture or partial picture $\rho(L)$ for $G(\mathcal{R}_-(\omega))$ by simply deleting all edges with labels $x(\beta)$ where $\beta \notin \mathcal{R}_-(\omega)$.

Proof. Since $\mathcal{R}_-(\omega)$ is given by a linear condition and any two letters in any relation are linearly independent, if two letters in any relation in $G(\mathcal{S})$ lie in $\mathcal{R}_-(\omega)$ then all the letters lie in $\mathcal{R}_-(\omega)$. Thus, if only part of the relation survives under the retraction it must be a single letter. This letter cannot be a Y-letter: If neither X letter in a relation lies in $\mathcal{R}_-(\omega)$ then the condition $hom(\beta, \omega) \neq 0$ implies $hom(\gamma, \omega) = 0$ for all extensions γ of α by β. So, none of the letters in such a relation will lie in $\mathcal{R}_-(\omega)$. Therefore, the retraction $\mathcal{R} \to \mathcal{R}_-(\omega)$ sends relations to relations and induces a retraction of groups $G(\mathcal{R}) \to G(\mathcal{R}_-(\omega))$.

Given any picture or partial picture L for $G(\mathcal{S})$, each vertex has a relation r which has the property that either $\rho(r) = r$ or $\rho(r)$ is an unreduced relation of the form xx^{-1} or $\rho(r)$ is empty. In the second case we consider the vertex as part of the smooth curve $E(x)$. Removal of all edges with labels not in $\mathcal{R}_-(\omega)$ therefore keeps L looking locally like a picture for $\mathcal{R}_-(\omega)$. But pictures and partial pictures are defined by local conditions. □

We can now state and prove the key lemma about pictures for $G(\mathcal{S})$. Recall that $E(\omega)$ is the union of the set of edges with label $x(\omega)$ and that, for any root $\beta \in \mathcal{R}_-(\omega)$, any vertex with relation $r(\beta, \omega)$ or $r(\beta, \omega)^{-1}$ has Y-edges on the positive side of the X-line $E(\omega)$. (For example, in Figure 3, α, β and all letters γ_i in $r(\alpha, \beta)$ lie in $\mathcal{R}_-(\omega)$. So the edges corresponding to the commutator relations $r(\gamma, \omega)$ for all letters γ in $r(\alpha, \beta)$ lie in the interior of the brown circle $E(\omega) = D(\omega)$. Since Figure 3 is an atom, are lines are curved in the positive direction.) We also note that the base point direction is on the negative side of both X-lines.

Lemma 3.4.2 (Sliding Lemma)**.** *Suppose that L is a picture for $G(\mathcal{S})$ and $\Sigma \subseteq E(\omega)$ is a simple closed curve. Let U be the region in $\mathbb{R}^2$ enclosed by Σ. Suppose that U is on the negative side of Σ and that all edges in $L \cap U$ have labels $x(\beta)$ for $\beta \in \mathcal{R}_-(\omega)$. Then there is an atomic deformation $L \sim L'$ which alters L only in an arbitrarily small neighborhood V of $U \cup \Sigma$ so that $L' \cap V$ contains no edges with labels $\geq \omega$ in lateral order.*

Proof. Every edge which crosses Σ has a label $x(\beta)$ where $\beta \in \mathcal{R}_-(\omega)$. This implies that all Y-edges at all vertices on Σ lie outside the region U. So, at each vertex of Σ, only one edge $E(\beta)$ goes into the region U. Also, all basepoint directions of all vertices on Σ lie inside U.

The proof of the lemma is by induction on the number of vertices in the open region V. Suppose first that this number is zero. Then Σ has no vertices and $L \cap U$ is a union of disjoint simple closed curves which can be eliminated by concordance one at a times starting with the innermost scc. This includes Σ. The result has no edges with labels $\geq \omega$.

Suppose next that L has vertices on the set Σ but no vertices in the region U enclosed by Σ. Then every edge of L in U is an arc connecting two vertices on Σ and the negative side of each arc has a path connecting the two basepoint directions at these two vertices. So, we can cancel all pair of vertices and we will be left with no vertices in V. As before, we can then eliminate all closed curves in V including Σ which has now become a union of simple closed curves.

Finally, suppose that U contains a vectex v having relation $r(\alpha, \beta)^{\pm}$. So, v contributes $\pm g \langle r(\alpha, \beta) \rangle$ to the algebraic expression for L. Then $\alpha, \beta \in \mathcal{R}_-(\omega)$ by assumption. Now add the atom $\mp A(\alpha,\beta,\omega)$ (which resembles Figure 3) in the region containing the basepoint direction of v. This adds $\mp gA(\alpha,\beta,\omega)$ to the algebraic expression for L. The atom has a circle labeled $x(\omega)$ oriented inward with one vertex outside this circle with relation $r(\alpha,\beta)^{\mp}$ (the mirror image of the relation at v) by Proposition 3.3.2. The new vertex cancels the vertex v. Repeating this process eliminates all vertices in the region U. However, it changes U into a shape which is not contractible. So, we need to revisit the argument in the previous paragraph and realize that it works when Σ is a union of simple closed curves so that one connected region U bounded by Σ is on the negative side of Σ at all points. Then the arcs of L in U can be used to cancel all vertices in Σ thereby eliminating all vertices inside the contractible region V.

Since the entire process was a sequence of picture deformations and addition of $\mathbb{Z}G(\mathcal{S})$ multiples of atoms, it is an atomic deformation. □

Theorem 3.4.3 (Atomic Deformation Theorem)**.** *Suppose that $\mathcal{S}$ is an admissible set of real Schur roots which admits a lateral ordering. Then any picture for $G(\mathcal{S})$ has a null atomic deformation. I.e., it is deformation equivalent to a $\mathbb{Z}G$ linear combination of atoms.*

This theorem follows from the Sliding Lemma and it implies Lemma 1.3.1.

Proof. Let $\mathcal{S}=(\beta_1,\cdots,\beta_m)$ be an admissible set of roots. Let $\beta^1,\cdots,\beta^m$ be the same set rearranged in lateral order. Let $\mathcal{R}^k$ be the set of all elements of $\mathcal{S}$ which are $\leq \beta^k$ in lateral order. Take k minimal so that the labels which occurs in L all lie in $\mathcal{R}^k$. If $k=1$ then L has no vertices and is a disjoint union of simple closed curve (scc) which is null homotopic. By induction, it suffices to eliminate $\omega=\beta^k$ as a label from the picture L by picture deformations and addition of atoms without introducing labels β^j for $j>k$.

Since ω is a rightmost element in the set $\mathcal{R}^k$, $x(\omega)$ does not occur as a Y-letter at any vertex of L. Therefore the edge set $E(\omega)$ is a disjoint union of simple closed curves. Let Σ be one such scc. The curves that cross Σ are labeled $x(\beta)$ where

$$\beta\in\mathcal{R}^k(\omega):=\{\beta\in\mathcal{R}^k : hom(\beta,\omega)=0\}$$

Since $ext(\beta,\omega)=0$ for all $\beta\in\mathcal{R}^{k-1}$, the condition $hom(\beta,\omega)=0$ defining $\mathcal{R}^k(\omega)$ is equivalent to the linear condition

$$\langle\beta,\omega\rangle=0.$$

Also, the Y-letters at each vertex on Σ give edges which all lie on the positive side of Σ and on the negative side of each $E(\beta)$ which crosses Σ.

Let Σ' be a scc which is parallel to Σ and lies slightly to the negative side of Σ. Then the labels on the edges which cross Σ' all lie in $\mathcal{R}^k(\omega)$ giving a word w_0 in the letters $x(\beta)$ and their inverses. The scc Σ' cuts the picture L into two partial pictures $L=L_0\cup L_1$ where L_0,L_1 have boundary equal to this word w_0.

We now claim that L is deformation equivalent to a sum of two pictures $L=L'+L''$ given as follows. $L'=\rho(L_0)\cup L_1$ and $L''=L_0\cup(-\rho(L_0))$ where $-\rho(L_0)=\rho(-L_0)$ is the mirror image of $\rho(L_0)$ using Σ' as mirror and ρ is the retraction given in Lemma 3.4.1. (To do this, take a new point at infinite for $S^2=\mathbb{R}^2\cup\infty$ that lies on the curve Σ', then deform Σ' into a straight line. Now take the usual mirror image of $\rho(L_0)$ along the straight line Σ'. Then deform Σ' back to its original shape to get $\rho(L_0)^{-1}$.) Since $\rho(w_0)=w_0$, L',L'' are pictures for $G(\mathcal{R}^k)$. The reason that $L=L'+L''$ is because, algebraically, we have:

$$L=L_0+L_1=\rho(L_0)+L_1+L_0-\rho(L_0)=L''+L'.$$

This is a calculation in the $\mathbb{Z}G(\mathcal{R}^k)$-module $Q(\mathcal{R}^k)$.

The scc Σ lies either in L' or L''. In either case, the Sliding Lemma implies that Σ can be "slid off" of the picture by an atomic deformation. The result will have one fewer component of $E(\omega)$. Eventually, this set becomes empty. The new equivalent picture has labels in $\mathcal{R}^{k-1}$. So, be induction, we are done. The entire picture can be deformed into nothing by atomic deformation. □

3.5. **Proofs of the lemmas.** The proofs of the Lemmas 1.3.1 and 1.3.3 very similar.

Proof of Lemma 1.3.1. Suppose that w, w' are expressions for the same element of $G(\mathcal{S})$ and $\pi(w)$, $\pi(w')$ are equal as words in the generators of $G(\mathcal{S}_0)$. This means that $\pi(w^{-1}x')$ reduces to the trivial (empty) word in $G(\mathcal{S}_0)$.

Let L be a partial picture giving the proof that $w^{-1}w'$ is trivial in $G(\mathcal{S})$. Then $\pi(L)$ can be completed to a true picture L_0 for the group $G(\mathcal{S}_0)$ by joining together cancelling letters in $\pi(w^{-1}x')$. By the Atomic Deformation Theorem 3.4.3, L_0 is equivalent to a sum of atoms. However, each atom A for $G(\mathcal{S}_0)$ can be lifted to an atom $\tilde{A}$ for $G(\mathcal{S})$ by definition of the atoms. Therefore, up to deformation equivalence, L can be lifted to a picture $\tilde{L}$ for $G(\mathcal{S})$. By Corollary 3.2.2, the number of vertices of $\tilde{L}$ having $x(\beta_m)$ as Y-letter is equal to the number of vertices having $x(\beta_m)^{-1}$ as Y-letter. This implies that the number of vertices in L_0 lifting to ones in $\tilde{L}$ having $x(\beta_m)^{-1}$ as Y-letter is equal to the number of vertices in L_0 lifting to ones in $\tilde{L}$ having $x(\beta_m)^{-1}$ as Y-letter are equal. So, the number of times $x(\beta_m), x(\beta_m)^{-1}$ occur as Y-letters in L are equal. So, the number of times that $x(\beta_m), x(\beta_m)^{-1}$ occur in the word $w^{-1}w'$ are equal. So, $x(\beta_m)$ occurs the same number of times in the words w, w' as claimed. □

Proof of Lemma 1.3.3. Let w be a positive expression in $G(\mathcal{S})$ so that at least one letter of w does not lie in $\mathcal{R}(\beta_m) \cup \{\beta_m\}$. Suppose that $x(\beta_m)w = wx(\beta_m)$ in $G(\mathcal{S})$. In lateral order, $\beta_m = \beta^k$ is in the middle and $i \leq k \leq j$ where β^i, β^j are the leftmost and rightmost labels which occur in w. The proof will be by induction on $j - i$, the statement being vacuously true in the impossible case $j - i = 0$.

Given $j - i > 0$ minimal and fixed, let w be of minimal length. Then, neither the first nor last letter of w will lie in the set $\mathcal{R}(\beta_m) \cup \{\beta_m\}$. Also, $w_0 = x(\beta_m)^{-1}w^{-1}x(\beta_m)w$ is an expression for the identity in $G(\mathcal{S})$. Thus we have a partial picture L in $G(\mathcal{S})$ having w_0 as boundary. Since $\pi(w_0) = \pi(w^{-1}w)$ reduces to the empty word, $\pi(L) = L_0$ is a picture for $G(\mathcal{S}_0)$.

Let β^p, β^q be the leftmost and rightmost letters that occur in the picture L_0. If $q > j$ then we can remove the set $E(\beta^q)$ using the same argument as in the proof of Lemma 1.3.1. This is a deformation of the partial picture L which does not alter the boundary of L. Similarly, if $p < i$ then we can remore the set $E(\beta^p)$. Therefore, we may assume that $p = i$ and $q = j$.

Since $i < j$ either $i < k$ or $k < j$. By symmetry assume that $k < j$ and let $\omega = \beta^j$ be the rightmost letter which occurs in w. There are two cases. Either $\omega \in \mathcal{R}(\beta_m)$ or not.

Case 1: $\omega \notin \mathcal{R}(\beta_m)$. Then choose the scc $\Sigma \subset E(\omega)$ so that it encloses the free endpoint $x(\beta_m)$ in the partial picture L. Since β_m does not lie in $\mathcal{R}_-(\omega)$ in Case 1, the retraction ρ eliminating all edges inside Σ' parallel to Σ which do not lie in $\mathcal{R}_-(\omega)$ will eliminate the edge set $E(\beta_m)$. Then the new L will have only one loose endpoint labeled $x(\beta_m)^{-1}$. But this implies $x(\beta_m) = 1$ in $G(\mathcal{S})$ which contradicts Lemma 1.3.1 proved above. Therefore, Case 1 is not possible.

Case 2: $\omega = \beta^j \in \mathcal{R}(\beta_m)$. Then $w = w_1x(\omega)w_2$ where the last letter of w_2 does not lie in $\mathcal{R}(\beta_m)$. As in Case 1 we choose a scc $\Sigma \subseteq E(\omega)$ so that it encloses the free endpoint $E(\beta_m)$ and the arcs labeled with the letters of the word w_1. Apply the retraction ρ on the region enclosed by Σ. This eliminates all letters in w_1 which do not commute with $x(\omega)$. The new partial picture gives the proof that

$$x(\beta_m)\rho(w_1)x(\omega)w_2 = \rho(w_1)x(\omega)w_2x(\beta_m)$$

in $G(\mathcal{S})$. By Remark 1.1.4, $\rho(w_1)x(\omega) = x(\omega)w_3$ where w_3 is a positive expression in letters $x(\beta^s)$ where $i \le s < j$. This gives the equation

$$x(\beta_m)x(\omega)w_3w_2 = x(\omega)w_3w_2x(\beta_m)$$

in $G(\mathcal{S})$. Since $x(\omega)$ commutes with $x(\beta_m)$, we can cancel $x(\omega)$ from both sides of the equation. Repeating the process if $x(\omega)$ is a letter in w_2 we will obtain the relation

$$x(\beta_m)w_3w_2 = w_3w_2x(\beta_m)$$

where w_3w_2 is a positive expression in letters $x(\beta^s)$ where $i \le s < j$ with last letter not in the set $\mathcal{R}(\beta_m) \cup \{\beta_m\}$. This contradicts the minimality of $j - i$ and proves the lemma. $\square$

References

[1] Aslak Bakke Buan, Robert J. Marsh, Idun Reiten, and Gordana Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), no. 2, 572–618.

[2] Vlastimil Dlab and Claus Michael Ringel, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc. **6** (1976), no. 173, v+57.

[3] Kiyoshi Igusa, *The $Wh_3(\pi)$ obstruction for pseudoisotopy*, PhD thesis, 1979.

[4] Kiyoshi Igusa, *The generalized Grassmann invariant*, unpublished preprint, 1978.

[5] Kiyoshi Igusa, *The category of noncrossing partitions*, unpublished preprint, 2014.

[6] Kiyoshi Igusa, Kent Orr, Gordana Todorov, and Jerzy Weyman, *Picture groups of finite type*, unpublished preprint, 2014.

[7] Kiyoshi Igusa and Kent E. Orr, *Links, pictures and the homology of nilpotent groups*, Topology **40** (2001), no. 6, 1125–1166.

[8] Kiyoshi Igusa and Gordana Todorov, *Signed exceptional sequences and the cluster morphism category*, unpublished preprint, 2014.

[9] T. Brüstle, G. Dupont, M. Pérotin, *On maximal green sequences*, Preprint arXiv: 1205.2050v1.

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