HORIZONTAL AND VERTICAL MUTATION FANS

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Abstract. We introduce diagrams for m-cluster categories which we call "horizontal" and "vertical" mutation fans. These are analogous to the mutation fans (also known as "semi-invariant pictures" or "scattering diagrams") for the standard ($m = 1$) cluster case which are dual to the poset of finitely generated torsion classes. The purpose of these diagrams is to visualize mutations and analogues of maximal green sequences in the m-cluster category with special emphasis on the c-vectors (the "brick" labels).

1. Introduction

These are annotated notes from my lecture at Workshop on Cluster Algebras and Related Topics held at the Chern Institute of Mathematics July 10-13, 2017. Preliminaries discussing the standard "pictures" used to visualize maximal green sequences are added. Also, there are additional comments to address a question of Zhe Han right after my talk: Do the horizontal fans correspond to torsion classes? I said "yes", but I will answer this more completely in these notes using bounded t-structures and the "spots" notation of [24] as illustrated in the lecture of Osamu Iyama on torsion classes and support τ-tilting modules.

The lecture begins with a "preview" of the "horizontal" and "vertical" mutation fans. In the lecture I used A_2 as the preview. Here I use a rank 3 example: A_3 (see Figure 1). The horizontal fans are viewed as the "floors" of a building. In rank 3, each floor is subdivided into triangular "rooms" (in rank 2 each room has only two walls). Each wall of each room has a door to the next room and at most one set of stairs going either up or down. The vertical fans are sets of rooms connected only be stairs.

All figures for quivers with 3 vertices (Figures 1, 4, 8, 9, 10) are drawn in perspective. The three coordinate hyperplanes become great circles when intersected with the unit sphere $S^2 \subseteq \mathbb{R}^3$. The stereographic projection to $\mathbb{R}^2$ gives three overlapping circles. When the plane is drawn in perspective, all circles become ellipses. These figures should be interpreted as patterns on the ground viewed from the side.

2. Preliminaries

The goal of this project is to visualize "m-maximal green sequences". First, we will go over several definitions of a standard maximal green sequence and demonstrate some valuable insights derived from the "semi-invariant picture" which are more difficult to see in the Hasse diagram of the poset of torsion classes which contains the same information.

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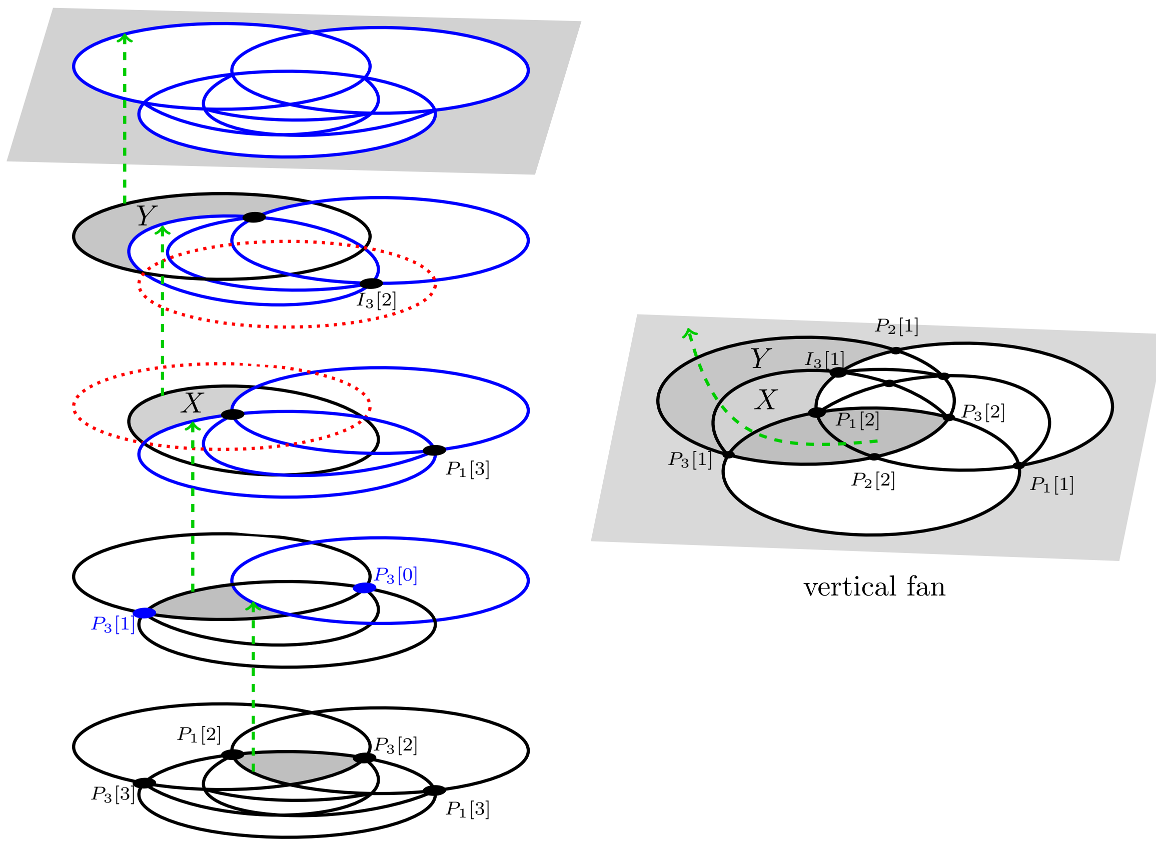


FIGURE 1. Horizontal fans are like floors of a building. Each floor is divided into triangular rooms (for $n = 3$). Each room is labeled with a bounded t-structure, the corners are labeled with components of the corresponding silting object, the walls are labelled with corresponding simple-minded components. Red dotted lines are absent walls with simple labels. Green dashed lines are "stairs" connecting shaded rooms which form 5 out of 14 rooms in the vertical fan shown on right. The wall at the top of each flight of stairs is blue.

2.1. **Basic definitions.** Let Λ be a finite dimensional hereditary algebra over a field K. We assume Λ is basic so that the dimension of the simple module S_i is equal to the dimension of the endomorphism ring of S_i and P_i, its projective cover. Call this f_i.

$$f_i := \dim_K S_i = \dim_K \mathrm{End}_\Lambda(P_i).$$

Let D be the $n \times n$ diagonal matrix with diagonal entries f_i. In the lecture, I assumed, for simplicity of notation, that K is algebraically closed and $D = I_n$, the identity matrix.

For any finitely generated right Λ-module M let $\underline{\dim}\, M \in \mathbb{N}^n$ be the *dimension vector* of M. The ith coordinate of $\underline{\dim}\, M$ is the number of times that S_i occurs in the composition series of M.

Suppose that M has a minimal projective presentation:

$$\coprod P_i^{b_i} \to \coprod P_i^{a_i} \to M$$

Then, the *g-vector* of M is the integer vector $g(M) \in \mathbb{Z}^n$ whose ith coordinate is $a_i - b_i$. For any two modules X, Y, the *Euler-Ringel pairing* is given by the dot product:

$$Dg(X) \cdot \underline{\dim}\, Y = g(X)^t D\, \underline{\dim}\, Y = \dim_K \mathrm{Hom}_\Lambda(X, Y) - \dim_K \mathrm{Ext}^1_\Lambda(X, Y)$$

One easy example is $g(P_i) = e_i$, the ith unit vector. The indecomposable objects in the bounded derived category of *mod*-Λ are $M[k]$ where M is an indecomposable module and $k \in \mathbb{Z}$. The g-vector of $M[k]$ is defined to be

$$g(M[k]) = (-1)^k g(M)$$

In particular, $g(P_i[1]) = -g(P_i) = -e_i$.

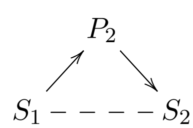
Definition 2.1.1. A module M is called *Schurian* (or a *brick*) if its endomorphism ring $\mathrm{End}_\Lambda(M)$ is a division algebra. Given such a module, the *stability set* $D_\Lambda(M)$ (also called *semi-invariant domain* [11]) is the subset of $\mathbb{R}^n$ given by

$$D_\Lambda(M) := \{x \in \mathbb{R}^n \mid x \cdot \underline{\dim}\, M = 0,\, x \cdot \underline{\dim}\, M' \le 0 \text{ for all } M' \subsetneq M\}$$

M is called *exceptional* if it is Schurian and rigid (*rigid* means without self-extensions). In the case when Λ is the path algebra $\Lambda = KQ$ of a Dynkin quiver Q, all indecomposable modules are exceptional.

These sets $D_\Lambda(M)$, sometimes called "walls", play a key role in visualizing cluster-tilting objects in the cluster category of Λ. These walls divide $\mathbb{R}^n$ into compartments corresponding to clusters and two adjacent compartments differ by a single mutation. I will explain this with an example.

Example 2.1.2. Let $\Lambda = KQ$ where Q is the quiver of type A_2: $1 \leftarrow 2$ and $f_1 = f_2 = 1$. There are three indecomposable modules arranged in the Auslander-Reiten quiver by:



The stability sets $D_\Lambda(S_1), D_\Lambda(S_1) \subset \mathbb{R}^2$ are the y and x-axes respectively:

$$D_\Lambda(S_1) = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \cdot \underline{\dim}\, S_1 = (x, y) \cdot (1, 0) = x = 0\} = (1, 0)^\perp$$

$$D_\Lambda(S_2) = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \cdot \underline{\dim}\, S_2 = y = 0\} = (0, 1)^\perp$$

However, $D_\Lambda(P_2)$ is only part of the hyperplane $(1, 1)^\perp$ since $S_1 \subset P_2$:

$$D_\Lambda(P_2) = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \cdot \underline{\dim}\, P_2 = x + y = 0,\ (x, y) \cdot \underline{\dim}\, S_1 = x \le 0\}$$

These walls are indicated in Figure 2. This figure also includes the position of the g-vectors of the five indecomposable objects $P_1 = S_1, P_2, S_2, P_1[1], P_2[1]$ of the cluster category of Λ.

2.2. **Cluster category.** We recall [2] that the *cluster category* $\mathcal{C}_\Lambda$ of Λ is the orbit category of the bounded derived category of *mod*-Λ by the functor $F = \tau^{-1}[1]$:

$$\mathcal{C}_\Lambda = \mathcal{D}^b(mod\text{-}\Lambda)/\tau^{-1}[1]$$

The indecomposable objects of $\mathcal{C}_\Lambda$ are represented by objects in $\mathcal{D}^b(mod\text{-}\Lambda)$ which are either indecomposable modules M or shifted indecomposable projective modules $P_i[1]$.

Definition 2.2.1. A *cluster-tilting object* in $\mathcal{C}_\Lambda$ is defined to be an object $T \in \mathcal{C}_\Lambda$ with n nonisomorphic components $T = \bigoplus T_i$ so that $\mathrm{Ext}^1(T,T) = 0$ in $\mathcal{C}_\Lambda$. This is equivalent to saying that T_i are either indecomposable modules, say $M_1, \cdots, M_m$, or shifted projective modules, say $P_{j_{m+1}}[1], \cdots, P_{j_n}[1]$, so that $\mathrm{Ext}^1_\Lambda(M_k, M_\ell)$ for all $k, l \le m$ and $\mathrm{Hom}_\Lambda(P_{j_i}, M_k) = 0$ for all $k \le m < i$. The module $M = M_1 \oplus \cdots \oplus M_m$ is called a *support tilting module.*

Cluster-tilting objects and stability sets are related by the following results [11].

Lemma 2.2.2. *For exceptional modules X, M, $Dg(X) \in D_\Lambda(M)$ if and only if*

$$\mathrm{Hom}_\Lambda(X, M) = 0 = \mathrm{Ext}^1_\Lambda(X, M)$$

Also, $-Dg(P_i) = Dg(P_i[1]) \in D_\Lambda(M)$ if and only if $\mathrm{Hom}_\Lambda(P_i, M) = 0$.

Theorem 2.2.3. *Let $T = T_1 \oplus \cdots \oplus T_n$ be a cluster-tilting object in $\mathcal{C}_\Lambda$. Then, there are unique exceptional modules $X_1, \cdots, X_n$ with the property that $Dg(T_i) \in D_\Lambda(X_j)$ if and only if $i \neq j$. Futhermore, any positive linear combination of the g-vectors $Dg(T_i)$ does not lie in any $D_\Lambda(M)$.*

Proof. The first statement is well-known [11]. The second statement is [15], Lemma A. □

Theorem 2.2.3 has the following interpretation. Given a cluster-tilting object $T = \bigoplus T_i$, the set of nonnegative linear combinations $\sum a_i Dg(T_i)$, $a_i \ge 0$ is bounded by n walls $D_\Lambda(X_i)$ and no walls meet the interior of this region. For example, in Figure 2, the five regions correspond to the 5 cluster-tilting objects (ordered counterclockwise from lower left):

$$P_1[1] \oplus P_2[1], \quad P_2[1] \oplus P_1, \quad P_1 \oplus P_2, \quad P_2 \oplus S_2, \quad S_2 \oplus P_1[1].$$

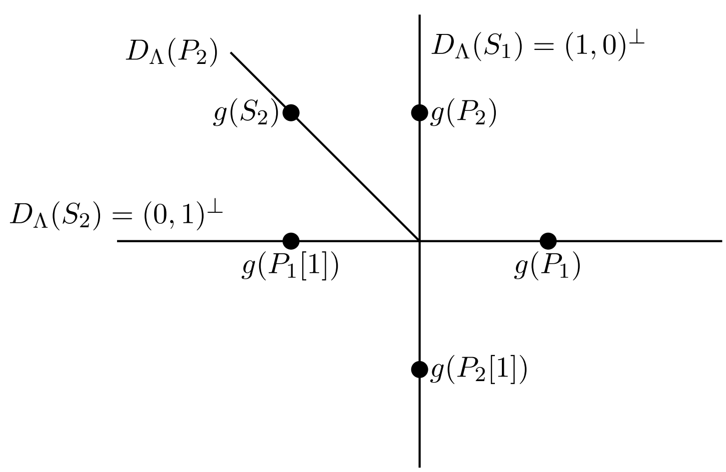


FIGURE 2. g-vectors of the components of each cluster-tilting object lie on the boundary of the corresponding region. For example, the upper right region corresponds to $P_1 \oplus P_2$ with $g(P_1)$ on one wall and $g(P_2)$ on the other. Here $D = I_2$. So, $g(X) = Dg(X)$.

To understand maximal green sequences (green dashed arrows in Figure 3), it helps to know that cluster-tilting objects for Λ are in bijection with certain torsion classes.

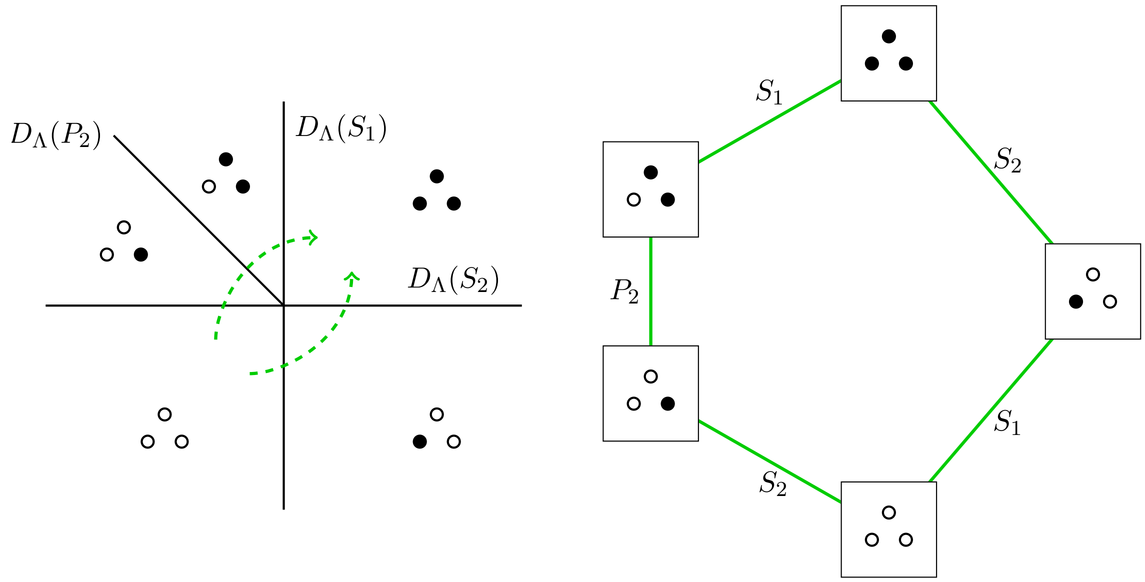


FIGURE 3. The poset of torsion classes on the right is dual to the semi-invariant picture on the left. The wall $D_\Lambda(M)$ separates two regions on the left iff the brick M labels the corresponding edge of the Hasse diagram. Maximal green sequences for Λ are maximal chains in the poset, equivalent to "green paths" from lower left to upper right in the picture. See [26].

Definition 2.2.4. A *torsion class* for Λ is a full subcategory $\mathcal{G}$ of *mod*-Λ with the property that $\mathcal{G}$ is closed under extension and quotient objects. We say that $\mathcal{G}$ is *finitely generated* if there exists a single module G in $\mathcal{G}$ so that $\mathcal{G}$ is the class of all quotients of direct sums of G. This conditions is often stated as "$\mathcal{G}$ is *functorially finite*" which means that the inclusion functor $\mathcal{G} \hookrightarrow$ *mod*-Λ has both a left and right adjoint.

The condition of being closed under extensions and quotients implies that the "trace" of $\mathcal{G}$ in any module M is also an object of $\mathcal{G}$. This is the sum of all images of all maps $X \to M$ where $X \in \mathcal{G}$. This gives a functorial short exact sequence:

$$0 \to A \to M \to B \to 0$$

where $A \in \mathcal{G}$ and $B \in \mathcal{F}$ where

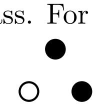
$$\mathcal{F} := \{B \in \textit{mod-}\Lambda \,|\, \mathrm{Hom}_\Lambda(X, B) = 0, \forall X \in \mathcal{G}\}.$$

This implies that $\mathcal{G} \hookrightarrow$ *mod*-Λ always has a right adjoint, namely, $M \mapsto A$.

For Λ of finite type, all torsion classes are clearly finitely generated. For Λ of infinite type, the full subcategory of preinjective Λ-modules is a torsion class which is not finitely generated.

Theorem 2.2.5. [24] *There is a 1-1 correspondence between (isomorphism classes of) cluster-tilting objects in $\mathcal{C}_\Lambda$ and functorially finite torsion classes for Λ.*

In Figure 3, the torsion classes corresponding to each of the five clusters is indicated by the spot diagram, also used by Iyama in his lecture, where the dark spots indicate the indecomposable objects which lie in the torsion class. For example,



Indicates the torsion class with indecomposable objects P_2, S_2 excluding $S_1 = P_1$.

2.3. **Maximal green sequences.** There are several equivalent definitions of a maximal green sequence. See [15].

Theorem 2.3.1. *Let Λ be a finite dimensional hereditary algebra over a field K of rank n. Let $\beta_1, \cdots, \beta_m \in \mathbb{N}^n$. Then the following are equivalent.*

(1) *$\beta_1, \cdots, \beta_m$ are the c-vectors of a maximal green sequence for Λ defined by a Fomin-Zelevinsky mutation sequence on positive c-vectors.*
(2) *β_i are (positive) real Schur roots and the unique modules M_i with $\underline{\dim}\, M_i = \beta_i$ have the following properties.*
   (a) $\mathrm{Hom}_\Lambda(M_i, M_j) = 0$ *for $i < j$.*
   (b) *(M_i) is maximal with property (a), i.e., for any other indecomposable module M, there exist $i < j$ so that* $\mathrm{Hom}(M_i, M) \neq 0$ *and* $\mathrm{Hom}(M, M_j) \neq 0$.
(3) *There is a generic green path in $\mathbb{R}^n$ which crosses the semi-invariant domains $D_\Lambda(M_i)$ for $i = 1, \cdots, m$ in increasing order of i.*
(4) *There is a finite maximal chain in the poset of functorially finite torsion classes on Λ labeled with bricks $M_1, \cdots, M_m$.*

By a *generic green path* we mean a C^1 path $\gamma : \mathbb{R} \to \mathbb{R}^n$ which starts in the negative octant (all coordinates of $\gamma(t)$ are negative for $t << 0$), ends in the positive octant and crosses only finitely many walls $D_\Lambda(M)$ at distinct times in the positive direction, i.e., so that $\frac{d\gamma}{dt}(t_0) \cdot \underline{\dim}\, M > 0$ when $\gamma(t_0) \in D_\Lambda(M)$. See [15].

Proof. The equivalence between (1),(2),(3) is shown in [15]. The equivalence of these with (4), leaving aside the brick labels for a moment, is well-known and due to Speyer and Thomas [24]. Figures similar to Figure 3 appear in [26]. The fact that the modules M_k with $\underline{\dim}\, M_k = \beta_k$ are the brick labels, call them B_k, of the corresponding edge in the Hasse diagram of functorially finite torsion classes follows from a lemma mentioned by Iyama in his lecture:

$$B_k = T_k / rad_{\mathrm{End}(T)} T_k.$$

Here $T = \bigoplus T_i$ is a support tilting module (same as support τ-tilting module in the hereditary case) corresponding to the torsion class generated by the extension closure of $\{M_1, \cdots, M_k\}$. Since T_k is an exceptional Λ-module, $rad_{\mathrm{End}(T)} T_k$ is equal to the trace in T_k of the other components T_i, $i \neq k$ of T. This formula implies that $\mathrm{Hom}_\Lambda(T_i, B_k) = 0$ for $i \neq k$. Also, $\mathrm{Ext}^1_\Lambda(T_i, B_k) = 0$ for $i \neq k$ by right exactness of Ext^1_Λ. Finally, the support of B_k is contained in the support of T_k. So, $\mathrm{Ext}^1(P_s[1], B_k) = 0$ for all s not in the support of T_k. This implies the stability wall $D_\Lambda(B_k)$ contains (D times) the $n-1$ g-vectors of $T_i, P_s[1]$. (See [11].). Thus $D_\Lambda(B_k) = D_\Lambda(M_k)$ making $B_k \in add\, M_k$. Since B_k is a brick, it must be equal M_k. □

There are two very useful consequences of the wall-crossing description of a maximal green sequence. One is an observation due to Greg Muller [21]:

Corollary 2.3.2. *Let $\Lambda = KQ$ be a path algebra and let Λ' be an algebra obtained from Λ by deleting arrows from KQ and adding relations. Then, any generic green path for Λ is also a generic green path for Λ'. In particular, when Λ' has no relations, the c-vectors of any maximal green sequence for Λ give the c-vectors of a maximal green sequence for Λ' when those vectors which are not real Schur roots of Λ' are deleted.*

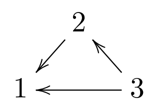
Proof. Since $Q' \subseteq Q$ be the quiver of Λ'. Then any Λ'-module M is also a KQ'-module which extends to a KQ-module by letting the other arrows act as 0. We have $\mathrm{End}_{\Lambda'}(M) = \mathrm{End}_\Lambda(M)$ and the Λ'-submodules $M' \subseteq M$ give Λ-submodules of M. Therefore:

$$D_{\Lambda'}(M) = D_\Lambda(M).$$

In other words, every wall for Λ' is a wall for Λ with the same dimension vector associated to it. The other walls of Λ simply disappear. So, a generic green path γ for Λ will be a (generally shorter) generic green path for Λ' passing through a subset of the original set of walls. $\square$

Another valuable insight is the visualization of maximal green sequences [5], [12], [21]. Figure 4 shows the union $\bigcup D_\Lambda(M)$ of all stability sets for $\Lambda = KQ$ of type $\tilde{A}_2$ (excluding an infinite number of stability sets accumulating onto the red line):

$Q:$



The idea behind this picture was generalized in the paper [5] to show that Figure 4 is typical for quivers of tame type and thus there are only finitely many MGSs.

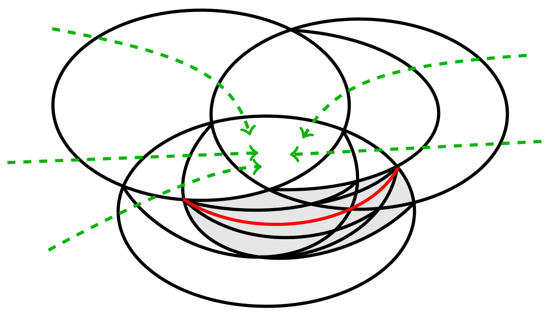


FIGURE 4. This is a perspective view of the stereographic projection of the intersection with S^2 of the "picture" $\bigcup D_\Lambda(M) \subset \mathbb{R}^3$ for $\Lambda = KQ$ of type $\tilde{A}_2$. The red line is the domain of the null root. This is the accumulation set of an infinite family of stability sets making the shaded region impassible. The curvature of the lines shows the positive direction in which green paths must be pointed. So, we see that there are only 5 maximal green sequences as indicated. This number of MGSs is given in this case in [17], for general $\tilde{A}_n$ in [1].

Definition 2.3.3. The *semi-invariant picture* (also called "cluster fan" or "scattering diagram") for Λ is defined to be the subset

$$L(\Lambda) := \bigcup D_\Lambda(M) \subset \mathbb{R}^n$$

where the union is over all Shurian Λ-modules M.

3. m-MAXIMAL GREEN SEQUENCES

We continue to let Λ denote a basic finite dimensional hereditary algebra over a field K. Let $m \geq 1$. There are several equivalent definitions of an m-maximal green sequence. In my lecture I used the technical definition that it is a maximal sequence of "backward mutations" in the derived category of *mod*-Λ starting with $\Lambda[m]$ and ending with $\Lambda[0]$. After subsequent lectures by Iyama, Demonet, Buan and discussion with Zhu Han it became clear that the definition in terms of t-structures might be more appropriate. I will give both of these here.

3.1. **m-clusters.** We review the definitions of an m-cluster and two other types of objects which are in bijection with m-clusters. The aim is to explain the definition of an m-maximal green sequence and the reason that each term in such a sequence has three types of labels. A popular reference is [20], but we take the point of view found in [3] and [19]. We use the base case $m = 0$ as a very easy example of each.

The original definition of the *m-cluster category* of Λ [25] is as an orbit category of the bounded derived category of *mod*-Λ:

$$\mathcal{C}_m(\Lambda) := \mathcal{D}^b(mod\text{-}\Lambda)/\tau^{-1}[m]$$

where τ is the Auslander-Reiten translation. An *m-cluster-tilting object* is an object T of this category with n nonisomorphic components so that

$$\mathrm{Ext}_{\mathcal{D}^b}(T, T[\ell]) = 0$$

for $0 < \ell \leq m$. The key point is that, if one component T_k of T is removed, there are, up to isomorphism, exactly m ways to replace it with something different, i.e., there are m mutations of T in the k-direction. For $m = 1$, this describes cluster-tilting objects in the cluster category $\mathcal{C}_1(\Lambda) = \mathcal{C}(\Lambda)$ of Λ [2].

One convenient way to look at m-cluster-tilting objects is to represent them by objects of the fundamental domain $\mathcal{F}D_m \subset \mathcal{D}^b(mod\text{-}\Lambda)$ of the functor $\tau^{-1}[m]$ given by

$$\mathcal{F}D_m := mod\text{-}\Lambda \coprod mod\text{-}\Lambda[1] \coprod \cdots \coprod mod\text{-}\Lambda[m-1] \coprod \Lambda[m]$$

This is the additive full subcategory of $\mathcal{D}^b(mod\text{-}\Lambda)$ whose indecomposable objects are $M[\ell]$ where M is a Λ-module and $0 \leq \ell \leq m$ with the additional restriction that M is projective when $\ell = m$. We call ℓ the *level* of $M[\ell]$. Recall that $M[\ell]$ is *exceptional* if M is an exceptional Λ-module.

Definition 3.1.1. An *m-cluster* (also called a *silting object* in $\mathcal{F}D_m$) is an object $T \in \mathcal{F}D_m$ having n nonisomorphic exceptional summands T_k so that $\mathrm{Hom}_{\mathcal{D}^b}(T, T[\ell]) = 0$ for $\ell > 0$.

In the base case $m = 0$, $\mathcal{F}D_0$ consists only of projective modules in level 0 and $T = \Lambda$ is the only 0-cluster.

In [3] it is shown that (isomorphism classes of) m-clusters are in 1-1 correspondence with "m-$\mathrm{Hom}_{\leq 0}$-configurations" defined as follows.

Definition 3.1.2. An m-$\mathrm{Hom}_{\leq 0}$*-configuration*, or simply *m-configuration*, (also called a *simple minded collection*) is a set of n exceptional objects $X_1[\ell_1], \cdots, X_n[\ell_n]$ in $\bigcup_{\ell=0}^m mod\text{-}\Lambda[\ell]$ so that, for all $i \neq j$ and all $k \leq 0$ we have

$$\mathrm{Hom}_{\mathcal{D}^b}(X_i[\ell_i], X_j[\ell_j + k]) = 0$$

and so that the exceptional Λ-modules $X_1, \cdots, X_n$ form an exceptional sequence in some order.

In the base case $m = 0$, $X_k \in mod\text{-}\Lambda$ must be the simple Λ-modules.

Finally, we recall the definition of a bounded t-structure.

Definition 3.1.3. A *t-structure* in a triangulated category $\mathcal{D}$ is a pair of additive full subcategory $(\mathcal{P}, \mathcal{Q})$ so that

(1) $\mathcal{P}[1] \subset \mathcal{P}$ and $\mathcal{Q} \subset \mathcal{Q}[1]$
(2) $\mathrm{Hom}_{\mathcal{D}}(P, Q) = 0$ for all $P \in \mathcal{P}, Q \in \mathcal{Q}$ and

(3) For every $X \in \mathcal{D}$ there is a distinguished triangle

$$P \to X \to Q \to P[1]$$

where $P \in \mathcal{P}, Q \in \mathcal{Q}$.

$\mathcal{P}$ is called an *aisle* [18]. It determines the t-structure since $\mathcal{Q}$ is the full subcategory of all $X \in \mathcal{D}$ so that $\mathrm{Hom}_{\mathcal{D}}(P, X) = 0$ for all $P \in \mathcal{P}$. We use the aisle to give the partial ordering on t-structures. Thus $(\mathcal{P}, \mathcal{Q}) \leq (\mathcal{P}', \mathcal{Q}')$ iff $\mathcal{P} \subseteq \mathcal{P}'$.

The *heart* of a t-structure $(\mathcal{P}, \mathcal{Q})$ is $\mathcal{P} \cap \mathcal{Q}[1]$. This is always an abelian category [4]. A t-structure is call *bounded* if every object of $\mathcal{D}$ is in $\mathcal{P}[-k] \cap \mathcal{Q}[k]$ for some integer k.

We are interested in the t-structures whose hearts $\mathcal{H}$ are *length categories*, i.e., so that the Jordan-Holder Theorem holds in $\mathcal{H}$. In $\mathcal{D}^b(mod\text{-}\Lambda)$, there is a 1-1 correspondence between bounded t-structures with length hearts and simple-minded collections. The correspondence sends a t-structure to the collection of simple objects in its heart. See [20].

3.2. **Graded tropical duality.** We need graded g-vectors and "slope vectors" to formulate the graded version of the tropical duality formula of [22].

Definition 3.2.1. The *graded g-vector* of $M[k]$ is defined to be the vector $\tilde{g}(M[k]) \in \mathbb{Z}[t]^n$ whose ith coordinate is $t^k(n_i - m_i)$ where

$$\coprod P_i^{m_i} \to \coprod P_i^{n_i} \to M \to 0$$

is the minimal projective presentation of M. Thus

$$D\tilde{g}(M[k]) = t^k f_i(n_i - m_i)_i \in \mathbb{Z}[t]^n$$

where D is the diagonal matrix with diagonal entries $f_i = \dim_K \mathrm{End}_\Lambda(P_i)$.

Definition 3.2.2. The *graded c-vector* of $N[k]$ is

$$\tilde{c}(N[k]) = t^{m-k}\, \underline{\dim}\, N \in \mathbb{Z}[t]^n$$

We define the *slope* of $N[k]$ to be $m - k$ and k is its *level* (or *degree*) if $N \in mod\text{-}\Lambda$. (The name comes from the fact that $m - k_i$ is the slope of the ith edge in the m-noncrossing tree corresponding to an m-cluster [13].)

Theorem 3.2.3 (graded tropical duality)**.** *Let $T = \coprod T_i$ be an m-cluster tilting object ("silting object") and let $X = \coprod X_j$ be the corresponding m-$\mathrm{Hom}_{\leq 0}$-configuration ("simple minded collection"). Then, we have:*

$$\tilde{g}(T_i)^t D\tilde{c}(X_j) = t^m f_i \delta_{ij} \text{ or } -t^{m-1} f_i \delta_{ij} \tag{3.1}$$

Furthermore, T and X uniquely determine each other by this relation.

Remark 3.2.4. In my lecture, I assumed, for simplicity of notation, that $K = \overline{K}$ and $D = I_n$, the identity matrix. Then this formula is a dot product:

$$\tilde{g}(T_i) \cdot \tilde{c}(X_j) = t^m \delta_{ij} \text{ or } -t^{m-1} \delta_{ij} \tag{3.2}$$

which implies that the *g-vectors* (the values of $\tilde{g}(T_i)$ at $t = -1$) lie in the hyperplane perpendicular to $c(X_j)$ for $i \neq j$. In general, it is $Dg(T_i)$ which lies in this hyperplane.

Proof. As stated, this theorem follows from the proof in [3]. In the Garside braid move which transforms T_i into X_i, the degree of X_i changes at most once and when it does it increases by one. (When X_i changes level, it becomes relatively projective and stays that way. When T_i is projective, the degree cannot shift by this process.) So, the slope decreases by zero or one which gives the two cases. (See [14] for more details.)

There is a very nice explanation of this in [19] in the simply laced case. King and Qiu show in that case that T_i are the projective objects of the heart $\mathcal{H}$. Since X_i are the simple objects of $\mathcal{H}$ we get:

$$\mathrm{Hom}_{\mathcal{D}^b}(T_i, X_j) = \delta_{ij} K \tag{3.3}$$

in the algebraically closed case. Since $T_i = M_i[\ell_i]$ is a projective complex in degrees ℓ_i and $\ell_i + 1$ (making $\tilde{g}(T_i)$ a vector in $t^{\ell_i}\mathbb{Z}^n$), the simple-minded component $X_i = N_i[k_i]$ must be in degree $k_i = \ell_i$ or $\ell_i + 1$ (putting it at level $m - \ell_i$ or $m - \ell_i - 1$). So, the formula (3.3) is equivalent to (3.2).

To see that T, X determine each other, we first set $t = -1$. Then the equation becomes

$$g(T)^t Dc(X) = (-1)^m D$$

which gives, e.g., $g(T) = (-1)^m D(c(X)^t)^{-1} D^{-1}$. Since exceptional modules are detemined by their dimension vectors, this gives each T_i up to its degree. The equation (3.1) gives only two possibilities for the level of T_i but only one of them has the correct sign. So, each T_i is uniquely determined by X. Similarly, the X_j are uniquely determined by T. □

3.3. **m-maximal green sequences.** In parallel with the discussion of standard maximal green sequences as maximal chains in the poset of functorially finite torsion classes in *mod*-Λ, an m-MGS can be described as a maximal chain in the poset of bounded t-structures with length heart starting with (aisle equal to)

$$\mathcal{D}^b_m(\Lambda) = \bigcup_{k \geq m} mod\text{-}\Lambda[k]$$

which is the smallest aisle containing $\Lambda[m]$ and ending with $\mathcal{D}^b_0(\Lambda)$.

In my lecture, I defined an m-maximal green sequence to be a sequence of "negative mutations" of m-clusters starting with $\Lambda[m]$ and ending with $\Lambda = \Lambda[0]$. Then I reinterpreted this as a sequence of "positive mutations" of m-configurations starting with $X = \bigoplus S_i[m]$ and ending with $\bigoplus S_i$. This is "positive" when using the slope $m - k$ instead of the level k of $X[k]$. These three notions of an m-maximal green sequence are equivalent since the bijection between bounded t-structures with length heart, silting objects and simple-minded objects respects the partial ordering (see [20]).

Various formulas for these mutations are known ([3], [16], [20]). I use the following numerical formula equivalent to the one in [19]. This is a modified version of the Fomin-Zelevinsky mutation formula for extended exchange matrices [10] using the tropical duality formula $g(T)^t Dc(X) = D$ of [22].

An *exchange matrix* is defined to be an $n \times n$ skew symmetrizable integer matrix B, i.e., DB is skew-symmetric for some positive diagonal matrix D. The matrix D is fixed, but B is mutable. The starting value of B will be denoted B_0 and called the *initial exchange matrix*. An *extended exchange matrix* will be a $(2n+1) \times n$ integer matrix $\widetilde{B}$ with initial value $\widetilde{B}_0$:

$$\widetilde{B} = \begin{bmatrix} B \\ \hline |C| \\ \hline s \end{bmatrix}, \qquad \widetilde{B}_0 = \begin{bmatrix} B_0 \\ \hline I_n \\ \hline 0 \end{bmatrix}$$

where B is an exchange matrix, $|C|$, is an $n \times n$ matrix with nonnegative integer entries and the last row $s = (s_1, \cdots, s_n)$ is an integer vector with entries $0 \leq s_i \leq m$. The $\tilde{c}_j$*-vector* is defined to be

$$\tilde{c}_j = t^{s_j} |c_j| \in \mathbb{Z}[t]^n$$

where $|c_j|$ is the jth column of $|C|$. We use the notation $c_j = (-1)^{s_j}|c_j|$ and we let C be the matrix with jth column c_j. The initial extended exchange matrix $\widetilde{B}_0$ uses the initial exchange matrix B_0, $C = I_n$ the identity matrix and 0 indicates the $1 \times n$ null matrix.

Definition 3.3.1. Let $\widetilde{B}$ be as above so that $B = D^{-1}C^tB_0C$. When $s_k < m$, the *positive mutation* $\widetilde{B}' = \mu_k^+\widetilde{B}$ of $\widetilde{B}$ in the *k-th direction* is defined as follows.

(1) μ_k^+ increases the slope s_k of $\tilde{c}_k$ by 1: $\tilde{c}_k' = t\tilde{c}_k$.
(2) If $s_j \neq s_k, s_k+1$ or if $b_{kj} \le 0$ then $\tilde{c}_j$ does not change: $\tilde{c}_j' = \tilde{c}_j$.
(3) If $b_{kj} > 0$ and $s_j \in \{s_k, s_k+1\}$ we convert slopes to signs using the rule:

slope s_j		$c_j^+ = (-1)^{s_j - s_k}\|c_j\|$
s_k	$\longrightarrow$	$+$
s_k+1	$\longrightarrow$	$-$

do the following mutation, and convert signs back to slopes. The mutation is

$$c_j^{+\prime} = c_j^+ + |c_k|b_{kj}.$$

Thus, $s_j' = s_j - 1$ only when $s_j = s_k + 1$ and $c_j^{+\prime}$ is negative.

(4) The new value of B is $B' = D^{-1}(C')^tDB_0C'$ where C' is the matrix with jth column c_j', the value of $\tilde{c}_j'$ at $t = -1$.

Given a hereditary algebra Λ, its *Euler matrix* E is defined to have entries

$$E_{ij} = \dim_K \mathrm{Hom}_\Lambda(S_i, S_j) - \dim_K \mathrm{Ext}^1_\Lambda(S_i, S_j)$$

The diagonal matrix D has entries $f_i = \dim_K \mathrm{End}_\Lambda(S_i)$. The exchange matrix of Λ is

$$B_\Lambda = D^{-1}(E^t - E).$$

This is a skew-symmetrizable integer matrix. When restricted to $m = 1$, Definition 3.3.1 gives the formula for "green" mutations of the clusters in the cluster category of *mod*-Λ. (See [11].) We are claiming that Definition 3.3.1 is the extension of this formula to m-clusters. For the proof of this claim, see [14].

Example 3.3.2. Let Q be the quiver $1 \leftarrow 2$. Then $D = I_2$ and $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. So, $B_0 = E^t - E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Let $m = 2$. Then we can perform μ_1^+ twice:

$$\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right] \xrightarrow{\mu_1^+} \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \\ \hline 1 & 0 \end{array}\right] \xrightarrow{\mu_1^+} \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \\ \hline 2 & 0 \end{array}\right]$$

In the first mutation we note that the sign of $|c_1|$ does not change. Instead we increase its slope by 1. When we mutate the second time, $s_1 = 1$. So, the c-vector with slope 0 (not equal to s_1 or $s_1 + 1$) is unchanged. When $s_1 = 2$, the mutation μ_1^+ is no longer allowed.

4. Horizontal and vertical algebras

Horizontal and vertical mutation fans are semi-invariant pictures for horizontal and vertical algebras which are associated to each m-configuration. We give definitions and examples.

4.1. **Definitions.** Let $X = \bigoplus X_i$ be a fixed m-configuration. We use round brackets to indicate slope. Thus

$$X_i = E_i(s_i) = E_i[m - s_i].$$

Lemma 4.1.1. [3] *The components $X_i = E_i(s_i)$ can be numbered so that*

1. *$(E_1, \cdots, E_n)$ is a complete exceptional sequence, i.e., $\mathrm{Hom}_\Lambda(E_j, E_i) = 0 = \mathrm{Ext}^1_\Lambda(E_j, E_i)$ for $i < j$.*
2. *$s_1 \le \cdots \le s_n$.*

We define the *span* of an exceptional sequence $(M_1, \cdots, M_m)$ in *mod*-Λ to be

$$span(M_1, \cdots, M_m) = \left({}^\perp M\right)^\perp$$

where $M = \bigoplus M_i$ and ${}^\perp M$ denotes the *left hom-ext-perpendicular* subcategory of *mod*-Λ of all X so that $\mathrm{Hom}_\Lambda(X, M) = 0 = \mathrm{Ext}^1_\Lambda(X, M)$ and similarly for $M^\perp$. It is a well-known property of exceptional sequences ([7], [23]) that the span of an exceptional sequence of length m is equivalent to *mod-H* for some hereditary algebra H of rank m.

Definition 4.1.2. For $E_i(s_i)$ as above and $0 \le i < j \le n$, let Λ_{ij} be an algebra so that

$$mod\text{-}\Lambda_{ij} \cong \mathcal{A}_{ij} := span(E_{i+1}, \cdots, E_j).$$

For each s let $H_s(X)$ and $V_s(X)$ be the algebras $H_s(X) = \Lambda_{2s,2s+1}$ and $V_s(X) = \Lambda_{2s-1,2s}$. Let $H = \prod H_s$ and $V = \prod V_s$. These are the *horizontal* and *vertical algebras* of X.

We have the well-known properties $\mathcal{A}_{jn}^\perp = \mathcal{A}_{0j}$ and $\mathcal{A}_{jn} = {}^\perp\mathcal{A}_{0j}$. This implies

$$\mathcal{A}_{ij} = \mathcal{A}_{jn}^\perp \cap {}^\perp\mathcal{A}_{0i}.$$

Theorem 4.1.3. *Let $X_k = E_k(s_k)$ and let $X' = \mu_k^+(X)$. Then*

1. *$H(X') = H(X)$ if s_k is even.*
2. *$S(X') = S(X)$ if s_k is odd.*

Proof. This follows immediately from the formula for the mutation μ_k^+ given in Definition 3.3.1. Suppose s_k is even. All objects X_j with slope other than s_k and $s_k + 1$ are unchanged and all components with slope s_k or s_{k+1} are replaced with other components with slope s_k or s_{k+1}. So, $\mathcal{A}_{0,s_k-1}$ and $\mathcal{A}_{s_k+2,n}$ are unchanged. So, $H_s(X') = H_s(X)$ for $s \ne s_k/2$. And $H_{s_k/2}(X)$ is also unchanged since $\mathcal{A}_{s_k,s_k+1} = {}^\perp\mathcal{A}_{0,s_k} \cap \mathcal{A}_{s_k+2,n}^\perp$ is unchanged. The proof for s_k odd is similar. □

We call the mutation $X' = \mu_k^+(X)$ (and $X = \mu_k^-(X')$) a *horizontal mutation* if s_k even, so that $H(X) = H(X')$. When $V(X) = V(X')$ it is a *vertical mutation.*

Definition 4.1.4. The *horizontal mutation fan* of X is defined to be the semi-invariant picture $L(H(X))$ for the algebra $H(X)$. The *vertical mutation fan* of X is defined to be $-L(V(X))$, i.e., the set of all $x \in \mathbb{R}^n$ so that $-x \in L(V(X))$.

Assume that m is odd. (In the examples, $m = 3$ and $m = 1$ in the classical case.) The claim is that the compartment in $L(H(X))$ corresponding to X is equal as a subset of $\mathbb{R}^n$ to the compartment in $-L(V(X))$ corresponding to X. This is because the vertices (corners) of that compartment are the g-vectors of the m-cluster $T = \bigoplus T_i$. The sign reversal for $L(V(X))$ is due to the g-vectors having the "wrong" sign. This comes from the implicit sign convention: objects in *mod*-$V(X)$ are put in degree 0 when they are actually in odd degrees by definitions. When m is even, all the signs should be reversed. So, the claim still holds.

We will do two examples with $m = 3$. In this case, $H(X) = H_0(X) \times H_1(X)$, where we color the second factor blue, and $V(X) = V_0(X) \times V_1(X) \times V_2(X)$ for any X.

4.2. **Example:** A_2**.** Let Λ be the algebra of type A_2 from Example 2.1.2. Figure 2 and the left part of Figure 3 show the semi-invariant picture for Λ. There are five horizontal algebras:

(1) $A_2 \times 0$ our notation for $\Lambda \times 0$.
(2) $0 \times A_2$
(3) $P_2 \times S_2$ which is shorthand for H so that *mod*-H = add $P_2 \times add\, S_2$
(4) $S_1 \times P_2$
(5) $S_2 \times S_1$

Figure 5 shows the semi-invariant pictures for these five algebras. The first two are of type A_2 so have 5 clusters each. The last three are semi-simple of type $A_1 \times A_1$ so have 4 clusters. The total is $5 \times 2 + 4 \times 3 = 22$ m-clusters for $m = 3$. The formula for the number of m-clusters of type A_n is the "Fuss-Catalan number" [9], [14]:

$$\prod_{e=1}^{n} \frac{m(n+1)+e+1}{e+1} = \frac{1}{m(n+1)+1}\binom{(m+1)(n+1)}{n+1}$$

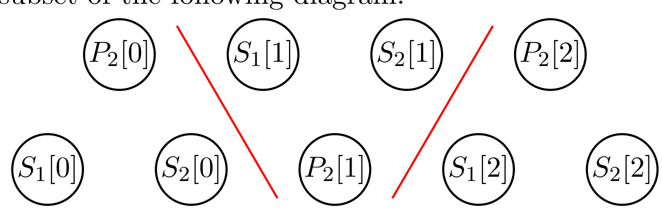
equal to 22 for $(n, m) = (2, 3)$ and 140 for $(n, m) = (3, 3)$.

For each horizontal fan in Figure 5 the walls are $D_H(X)$ where

$$X \in mod\text{-}(H_0 \times H_1) = mod\text{-}H_0 \coprod mod\text{-}H_1$$

The wall $D_H(X)$ is colored blue if $X \in mod\text{-}H_1$.

Each compartment in each horizontal fan has two labels. The letters: $a, b, c, a^+, b^+, c^+, 1, 2, 3, 4, 5, 6$ label the 12 vertical fans. For example, the shaded regions form vertical fan (2). The spot diagram indicates a subset of the following diagram:



Filled spots indicate that the object is in the aisle corresponding to the m-cluster of the compartment. Also all objects in *mod*-$\Lambda[k]$ for $k \geq 3$ lie in all aisles. When the spot diagram has fewer than 9 spots, the ones on the left are missing (and not filled in). The green lines in Figure 5 indicate the partial ordering of the shaded regions which are the compartments in the vertical fan $0 \times A_2 \times 0$ assembled in Figure 6.

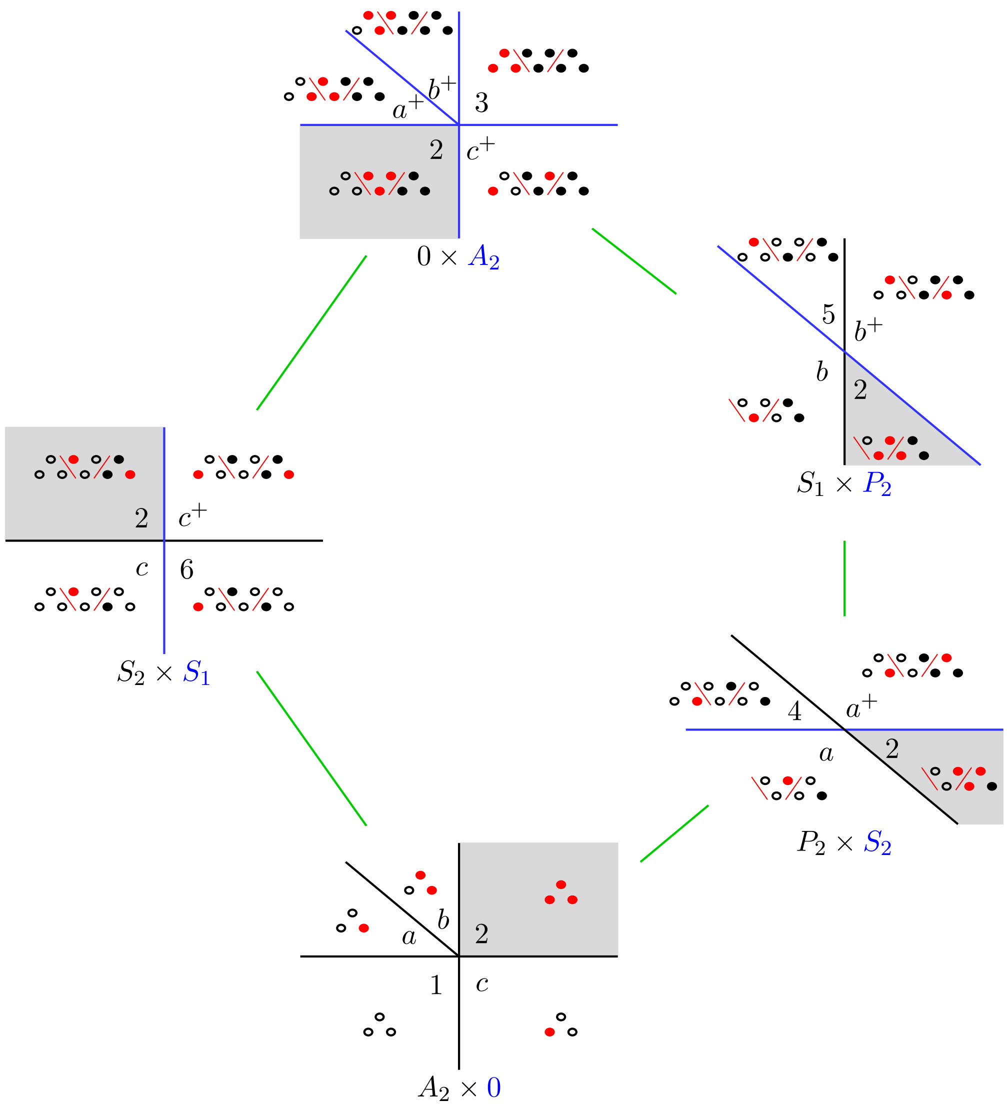


FIGURE 5. These are 5 horizontal fans for $A_2 : 1 \leftarrow 2$ which contain the 22 m-clusters for $m = 3$. The figure shows the corresponding bounded t-structures (with heart in red) which increase as we go northeast in each horizontal fan.

$A_2 : 1 \leftarrow 2$ has the following 12 vertical fans whose pieces are labeled in Figure 5. For example, vertical fan (a) has two m-clusters which appear in horizontal fans $A_2 \times 0$ and $P_2 \times S_2$. But only one of the vertical fans is complete, namely $0 \times A_2 \times 0$, shown in Figure 6.

(1) $A_2 \times 0 \times 0$ with 1 m-cluster
(2) $0 \times A_2 \times 0$ with 5 m-clusters
(3) $0 \times 0 \times A_2$ with 1 m-cluster
(a) $P_2 \times S_2 \times 0$ with 2 m-clusters.
(b) $S_1 \times P_2 \times 0$ with 2 m-clusters
(c) $S_2 \times S_1 \times 0$ with 2 m-clusters
(a^+) $0 \times P_2 \times S_2$ with 2 m-clusters

(b^+) $0 \times S_1 \times P_2$ with 2 m-clusters
(c^+) $0 \times S_2 \times S_1$ with 2 m-clusters
(4) $P_2 \times 0 \times S_2$ with 1 m-cluster
(5) $S_1 \times 0 \times P_2$ with 1 m-cluster
(6) $S_2 \times 0 \times S_1$ with 1 m-cluster

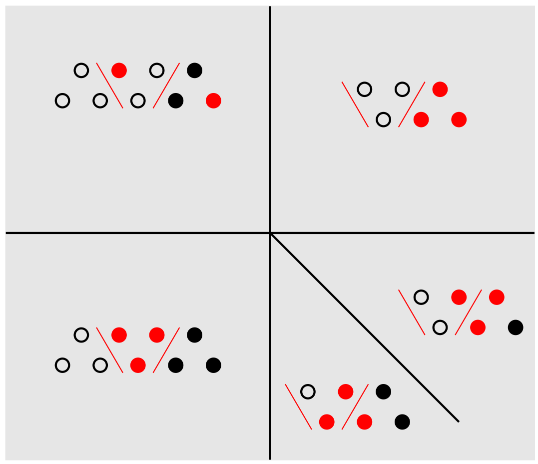


FIGURE 6. Vertical fan for $0 \times A_2 \times 0$. Indicated aisles are the union of torsion classes in mod-$\Lambda[1]$ with mod-$\Lambda[k]$ for all $k \geq 2$. Objects in the heart are red.

The m-maximal green sequence of maximal length is easy to visualize in Figures 5 and 6: Start at the lower left in horizontal fan $A_2 \times 0$, go clockwise to (2) at the upper right. This is in the vertical fan $0 \times A_2 \times 0$ in Figure 6. Go clockwise around that fan to the lower left. This is in horizontal fan $0 \times A_2$. In that fan, go clockwise to the maximum chamber in the upper right. This has 9 steps which add the 9 objects in the Auslander-Reiten quiver from right to left. The following example of an m-MGS was explained in the lecture by pointing to relevant parts of Figure 5.

$$\underset{(1)}{\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right]} \xrightarrow[H]{\mu_2^+} \underset{(a)}{\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \\ \hline 1 & 0 \\ 1 & 1 \\ \hline 0 & 1 \end{array}\right]} \xrightarrow[V]{\mu_2^+} \underset{(a)}{\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \\ \hline 1 & 0 \\ 1 & 1 \\ \hline 0 & 2 \end{array}\right]} \xrightarrow[H]{\mu_2^+} \underset{(4)}{\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \\ \hline 1 & 0 \\ 1 & 1 \\ \hline 0 & 3 \end{array}\right]} \xrightarrow[H]{\mu_1^+}$$

$$\underset{(a^+)}{\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \\ \hline 1 & 0 \\ 1 & 1 \\ \hline 1 & 3 \end{array}\right]} \xrightarrow[V]{\mu_1^+} \underset{(a^+)}{\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \\ \hline 1 & 0 \\ 1 & 1 \\ \hline 2 & 3 \end{array}\right]} \xrightarrow[H]{\mu_1^+} \underset{(b^+)}{\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \\ \hline 1 & 1 \\ 1 & 0 \\ \hline 3 & 2 \end{array}\right]} \xrightarrow[H]{\mu_1^+} \underset{(3)}{\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \\ \hline 0 & 1 \\ 1 & 0 \\ \hline 3 & 3 \end{array}\right]}$$

This starts at the bottom left compartment of the $A_2 \times 0$ horizontal fan. The first mutation is horizontal and goes to (a) in the same horizontal fan. Then, we move southwest to compartment (a) in $P_2 \times S_2$. Then NE in that horizontal fan to (a^+). Moving SW we get to (a^+) in $0 \times A_2$. Then we move NE in that horizontal fan to the maximal t-structure at (3).

4.3. **Example:** A_3**.** Let $\Lambda = KQ$ where Q is the quiver $1 \leftarrow 2 \rightarrow 3$. The poset of torsion classes is indicated in Figure 7. Each torsion class corresponds to a cluster for Λ. The compartments of the vertical fan $0 \times A_3 \times 0$ which is shown in Figure 8 correspond to these torsion classes. But the g-vectors have the opposite sign because of the shift in degree.

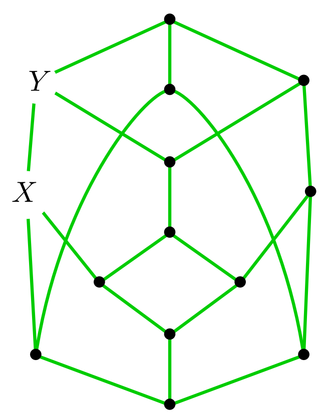


FIGURE 7. Poset of 14 torsion classes for $A_3 : 1 \leftarrow 2 \rightarrow 3$.

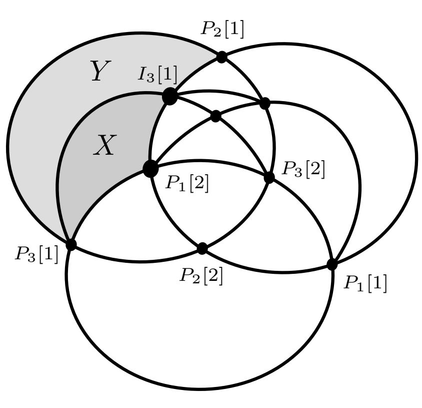
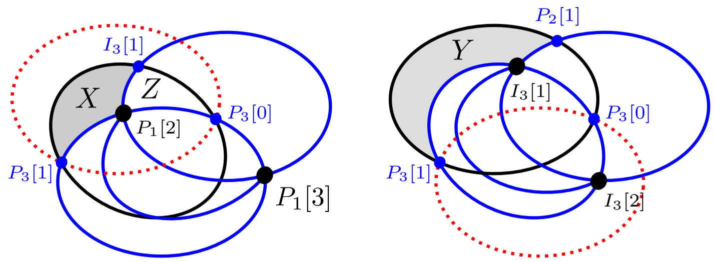


FIGURE 8. The vertical fan $0 \times A_3 \times 0$ (the stereographic projection to $\mathbb{R}^2$ of $-L(V(X)) \cap S^2$). Each region corresponds to a cluster in the cluster category, e.g., X corresponds to $I_3 \oplus P_1[1] \oplus P_3$. But the figure shows g-vectors for the corresponding m-clusters which have the opposite sign, e.g, $g(I_3[1]) = -g(I_3)$.

There are 55 vertical mutation fans. Only the one for $0 \times A_3 \times 0$ is shown.

Figure 10 shows all 14 horizontal fans for A_3. The regions corresponding to the m-cluster in the vertical fan $0 \times A_3 \times 0$ from Figure 8 are shaded and the green lines indicate the ordering



$I_1 \times {}^\perp I_1$ $\qquad$ $P_1 \times {}^\perp P_1$

$$\left[\begin{array}{ccc} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ \hline 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 2 & 1 & 3 \end{array}\right] \xleftarrow{\mu_3^+} \begin{array}{r} B: \\ \\ C: \\ \text{slope} \end{array} \left[\begin{array}{ccc} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ \hline 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 2 & 1 & 2 \end{array}\right] \xrightarrow{\mu_2^+} \left[\begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 2 & 2 \end{array}\right]$$

$Z : S_2(2) \oplus I_1(1) \oplus S_3(3)$ $\qquad$ $X : S_2(2) \oplus I_1(1) \oplus S_3(2)$ $\qquad$ $Y : S_1(1) \oplus I_1(2) \oplus S_3(2)$

$T_Z : I_3[1] \oplus P_1[2] \oplus P_3[0]$ $\qquad$ $T_X : I_3[1] \oplus P_1[2] \oplus P_3[1]$ $\qquad$ $T_Y : I_3[1] \oplus P_2[1] \oplus P_3[1]$

FIGURE 9. At top: the horizontal fans $H(X) = H(Z) : I_1 \times {}^\perp I_1$ and $H(Y) : P_1 \times {}^\perp P_1$. $Y = \mu_2^+(X)$ is a vertical mutation. $Z = \mu_3(X)$ is a horizontal mutation. Matrix B mutates in an unexpected way since the only change in c-vectors under μ_3^+ is changing the sign of c_3.

of these shaded m-clusters. Thus, the horizontal fans are placed at the nodes of the Hasse diagram in Figure 7.

Figure 1 in the introduction shows the five horizontal fans on the left side of Figure 10: $A_3 \times 0$, $P_3^\perp \times P_3$, $I_1 \times {}^\perp I_1$, $P_1 \times {}^\perp P_1$ and $0 \times A_3$. One can visualize several m-maximal green sequences in the figure as follows. On the bottom floor $A_3 \times 0$, start in the unbounded region and go to the center. The longest such green path, of length 6, goes up to the center from below. Now, take the four stairs going up to the top floor $0 \times A_3$. Equivalently, move in the vertical fan from the center out to the unbounded shaded region going through the shaded regions in FIgure 1. Then move to the center of the top floor. If we take the green stairs (dashed) in Figure 1, this m-MGS has length 16. But the maximum number of steps is $3 \times 6 = 18$. This is achieved by taking the longest path in the vertical fan from middle to outside.

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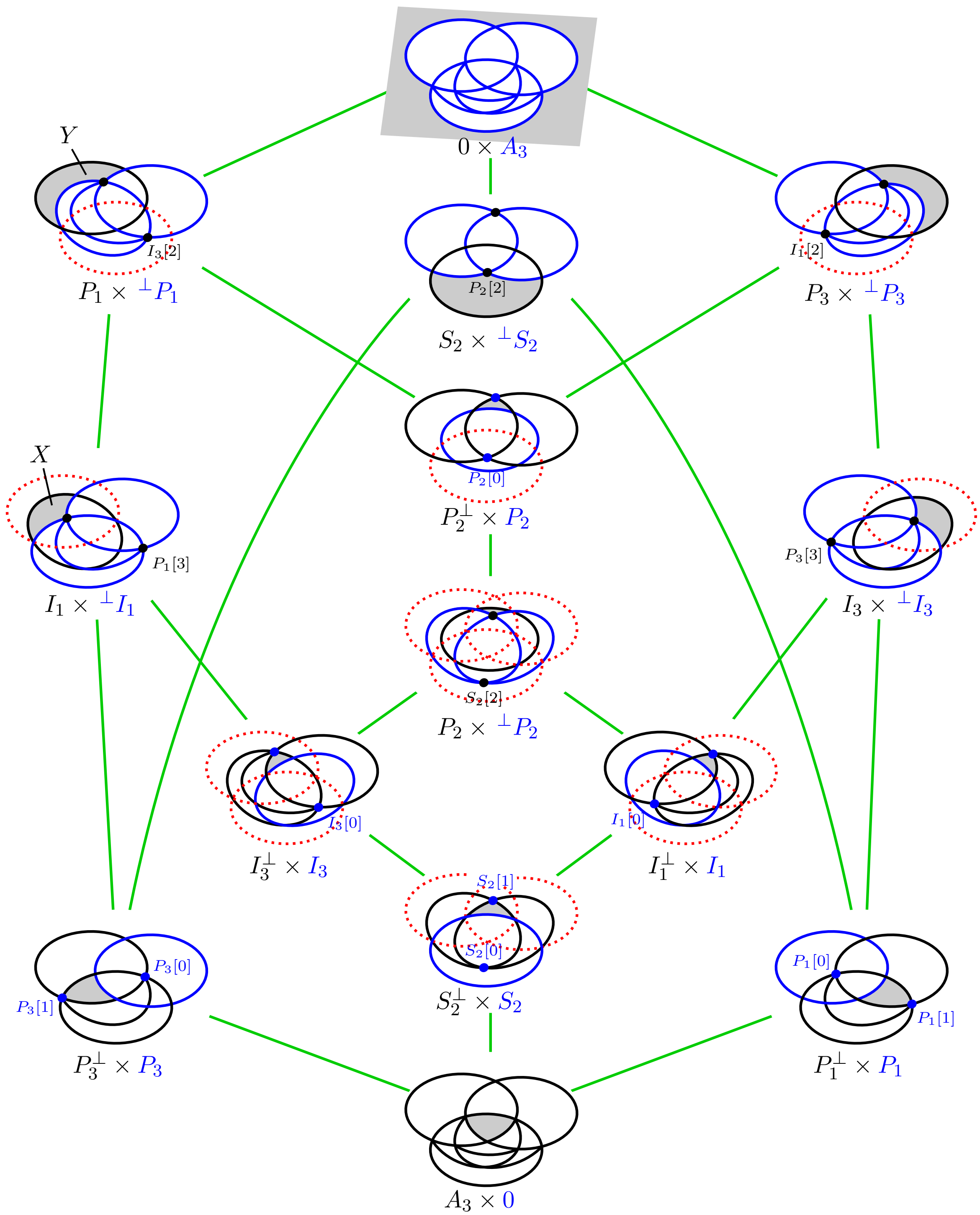

FIGURE 10. The 14 horizontal mutation fans for the A_3 quiver $1 \leftarrow 2 \rightarrow 3$ with $m = 3$. H_0 walls are black, H_1 walls are blue. The shaded regions are the 14 chambers of the vertical mutation fan for $0 \times A_3 \times 0$ which correspond to the torsion classes for $\Lambda = A_3$. The green lines show the partial ordering of these chambers (cut and pasted from Figure 7).

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