MODULATED SEMI-INVARIANTS

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Abstract. We prove the basic properties of determinantal semi-invariants for presentation spaces over any finite dimensional hereditary algebra over any field. These include the virtual generic decomposition theorem, stability theorem and the c-vector theorem which says that the c-vectors of a cluster tilting object are, up to sign, the determinantal weights of the determinantal semi-invariants defined on the cluster tilting objects. Applications of these theorems are given in several concurrently written papers.

Introduction

There is a rich theory of semi-invariants for representations of quivers [S91], [Ki], [DW], [SW], [SvsB] and its relation to cluster categories and cluster algebras [IOTW09], [Ch], [BHIT]. In this paper, we show how this theory and its relation to cluster algebras can be extended to finite dimensional hereditary algebras over a field, including all modulated acyclic quivers over any field and the category of representations of such quivers. Furthermore, we prove the relationship between c-vectors and semi-invariants.

Over a fixed field K, a K*-modulated quiver* is a triple $(Q, \{F_i\}_{i\in Q_0}, \{M_{ij}\}_{i\to j\in Q_1})$ where Q is a quiver (directed graph) without oriented cycles, F_i is a finite dimensional division algebra for each vertex $i \in Q_0$ and M_{ij} is an F_i-F_j bimodule for every arrow $i \to j$ in Q_1. The standard modulation of a simply laced quiver Q is given by taking each $F_i = K$ and each $M_{ij} = K$. A representation V of this modulated quiver with *dimension vector* $\alpha = (\alpha_1, \cdots, \alpha_n)$ consists of F_i-modules V_i of dimension α_i at each vertex $i \in Q_0$ and an F_j-linear map $V_i \otimes_{F_i} M_{ij} \to V_j$ for every arrow $i \to j$ in Q_1.

We study representation and presentation spaces of modulated quivers. When Q is a simply laced quiver, the standard definition of the representation space of Q with dimension vector $\alpha \in \mathbb{N}^n$ is

$$Rep(Q, d) = \prod_{i\to j\in Q_1} \mathrm{Hom}_K(K^{\alpha_i}, K^{\alpha_j}).$$

When K is algebraically closed, any finite dimensional hereditary algebra is Morita equivalent to the path algebra KQ of a quiver Q. Choosing an element of $Rep(Q, \alpha)$ is equivalent to choosing a KQ-module structure on the KQ_0-module $\bigoplus_i K^{\alpha_i}$.

Over the modulated quiver $(Q, \{F_i\}_{i\in Q_0}, \{M_{ij}\}_{i\to j\in Q_1})$ the representation space for dimension vector $\alpha \in \mathbb{N}^n$ is

$$Rep(Q, \{F_i\}_{i\in Q_0}, \{M_{ij}\}_{i\to j\in Q_1}, \alpha) = \prod_{i\to j\in Q_1} \mathrm{Hom}_{F_j}(M_{ij}^{\alpha_i}, F_j^{\alpha_j}).$$

Each element of $Rep(Q, \{F_i\}_{i\in Q_0}, \{M_{ij}\}_{i\to j\in Q_1}, \alpha)$ gives the right $\prod_{i=1}^n F_i$-module $\bigoplus_{i=1}^n F_i^{\alpha_i}$ the structure of a right module over the tensor algebra of $(Q, \{F_i\}_{i\in Q_0}, \{M_{ij}\}_{i\to j\in Q_1})$. In all cases, the representation space is an affine space over K.

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If K is not perfect there may be finite dimensional hereditary algebras over K which are not Morita equivalent to the tensor algebras of modulated quivers. (Appendix A, Sec 5.)

For an arbitrary finite dimension hereditary algebra Λ we define the representation space $Rep(\Lambda, \alpha)$ to be a certain subspace of the space $\mathrm{Hom}_\Lambda(rad\, P(\alpha), P(\alpha))$, where $P(\alpha)$ denotes $\coprod_i P_i^{\alpha_i}$, which corresponds to $Rep(Q, \{F_i\}_{i\in Q_0}, \{M_{ij}\}_{i\to j\in Q_1})$ in the modulated case. We identify each element $f \in Rep(\Lambda, \alpha)$ with the Λ-module which is the cokernel of the homomorphism $f : rad\, P(\alpha) \to P(\alpha)$.

We consider $Rep(\Lambda, \alpha)$ as an affine space over K. At the beginning we assume that K is infinite so that nonempty open subsets of this space are dense. (We extend to arbitrary fields later.) The first theorem of this paper is Theorem 2.1.5: If there exists a Λ-module M which is *rigid* meaning $\mathrm{Ext}^1_\Lambda(M, M) = 0$ with $\underline{\dim}\, M = \alpha$, then the elements of $Rep(\Lambda, \alpha)$ which are isomorphic to M form an open dense subset. We call this the *generic module* of dimension α and denote it by M_α. If M_α is indecomposable then α is a *real Schur root* of Λ. This implies:

Theorem 0.0.1 (Generic Decomposition Theorem 2.1.6))**.** *: Let $\beta_1, \cdots, \beta_k$ be real Schur roots so that M_{β_i} do not extend each other. Then for any nonnegative integer linear combination $\gamma = \sum n_i\beta_i$, the general module with dimension vector γ is isomorphic to $\coprod_{i=1}^k M_{\beta_i}^{n_i}$.*

Representation spaces are defined for $\alpha \in \mathbb{N}^n$. Next, we generalize the construction to arbitrary integer vectors $\alpha \in \mathbb{Z}^n$ by constructing presentation spaces and considering their direct limit which we call *virtual representation space.* We choose vectors $\gamma_0, \gamma_1 \in \mathbb{N}^n$ so that $\underline{\dim}\, P(\gamma_0) - \underline{\dim}\, P(\gamma_1) = \alpha$. We call $\mathrm{Hom}_\Lambda(P(\gamma_1), P(\gamma_0))$ a *presentation space* and denote it by $Pres_\Lambda(\gamma_1, \gamma_0)$ and view it as a generalization of $Rep(\Lambda, \alpha)$. However, there are an infinite number of choices for γ_0, γ_1 for each $\alpha \in \mathbb{Z}^n$. To define a single space for each $\alpha \in \mathbb{Z}^n$ which contains all of these presentation spaces, we take their direct limit (colimit):

$$Vrep(\Lambda, \alpha) := colim\, Pres_\Lambda(\gamma_1, \gamma_0)$$

where the colimit is over all pairs γ_0, γ_1 so that $\underline{\dim}\, P(\gamma_0) - \underline{\dim}\, P(\gamma_1) = \alpha$. Then $Vrep(\Lambda, \alpha)$ is irreducible. The next theorem in our paper is:

Theorem 0.0.2 (Virtual Generic Decomposition Theorem 2.3.8)**.** *Let $\gamma = \sum r_i\beta_i \in \mathbb{Z}^n$ be an integer vector which is a nonnegative rational linear combination of the elements β_i of a partial cluster tilting set (Definition 2.3.7). Then the r_i are all integers, call them n_i, and the general presentation with dimension vector γ is isomorphic to $\coprod M_i^{n_i}$ where M_i are rigid with $\underline{\dim}\, M_i = \beta_i$. In other words, the set of all elements of $Vrep(\Lambda, \gamma)$ isomorphic to $\coprod M_i^{n_i}$ is open and dense.*

Since the groups $\mathrm{Aut}_\Lambda(P(\gamma_0))$, $\mathrm{Aut}_\Lambda(P(\gamma_1))$ act on presentation space $Pres_\Lambda(\gamma_1, \gamma_0)$, semi-invariants are defined. A *semi-invariant* is a polynomial function $\sigma : Pres_\Lambda(\gamma_1, \gamma_0) \to K$ so that, for any $(g_0, g_1) \in \mathrm{Aut}_\Lambda(P(\gamma_0)) \times \mathrm{Aut}_\Lambda(P(\gamma_1))^{op}$ and $f : P_\Lambda(\gamma_1) \to P(\gamma_0)$ we have $\sigma(g_0 f g_1) = \chi_0(g_0)\sigma(f)\chi_1(g_1)$ where χ_0, χ_1 are characters $\mathrm{Aut}_\Lambda(P(\gamma_s)) \to K^*$ for $s = 0, 1$ where by *character* we mean a regular (polynomial) function which is a homomorphism of groups. Every group homomorphism $\mathrm{Aut}_\Lambda(P(\alpha)) \to K^*$ factors through the group $\prod_{i=1}^n \mathrm{Aut}_\Lambda(P_i^{\alpha_i}) = \prod_{i=1}^n GL(\alpha_i, F_i)$. Since σ is defined on the affine space $Pres_\Lambda(\gamma_1, \gamma_0)$, these characters extend to the endomorphism rings of $P(\gamma_0)$, $P(\gamma_1)$ (by $g \mapsto \sigma(gf)/\sigma(f)$ for a fixed $f \in Pres_\Lambda(\gamma_1, \gamma_0)$ on which $\sigma(f) \neq 0$.) In Appendix B Theorem 6.0.9 we show that every character $\mathrm{End}_F(F^m) \to K$ is a power of the "reduced norm" (and thus a fractional power of the *determinantal character* given by taking the determinant of an F-endomorphism of F^m considered as a linear map over K. See Definition 6.0.8). Therefore, the characters associated to any semi-invariant on presentation space $Pres_\Lambda(\gamma_1, \gamma_0)$ are nonnegative integer powers of the reduced norm for each division algebra F_i. This gives a vector weight in $\mathbb{N}^n$. The weights coming from $P(\gamma_0)$ and $P(\gamma_1)$ are equal when defined.

In this paper we do not use the reduced norm weights. We use *determinantal (det-) weights.* The coefficients of the det-weights are in general fractions. They are integers if and only if the characters are powers of the determinantal character. We also consider only certain semi-invariants: the *determinantal semi-invariants* σ_β which have *determinantal weight* $\beta \in \mathbb{N}^n$. These semi-invariants are given on any presentation $f : P(\gamma_1) \to P(\gamma_0)$ by $\sigma_\beta(f) = \det_K \operatorname{Hom}_\Lambda(f, M_\beta)$. These semi-invariants are clearly compatible with stabilization and therefore define semi-invariants on $Vrep(\alpha)$ in the case when $\operatorname{Hom}_\Lambda(f, M_\beta)$ is an isomorphism. We call the set of such $\alpha \in \mathbb{Z}^n$ the (integral) *domain of the semi-invariant of det-weight* β and denote it by $D_\mathbb{Z}(\beta)$.

In Section 3 we prove the virtual stability theorem which states that these domains of semi-invariants are given by "stability conditions".

Theorem 0.0.3 (Virtual Stability Theorem 3.1.1)**.** *Let Λ be a finite dimensional hereditary algebra over a field with n simple modules. Let $\alpha \in \mathbb{Z}^n$ and β a real Schur root. Then, the following are equivalent:*

(1) *There exists a morphism of projective modules $f : P \to Q$ so that $\underline{\dim}\, Q - \underline{\dim}\, P = \alpha$ and f induces an isomorphism*

$$f^* : \operatorname{Hom}_\Lambda(Q, M_\beta) \xrightarrow{\approx} \operatorname{Hom}_\Lambda(P, M_\beta).$$

(2) *Stability conditions for α and β hold: $\langle \alpha, \beta\rangle = 0$ and $\langle \alpha, \beta'\rangle \leq 0$ for all real Schur subroots $\beta' \subseteq \beta$ where $\langle\cdot,\cdot\rangle$ is the Euler-Ringel form defined in Proposition 1.2.2.*

(3) *There is a semi-invariant of det-weight β on the presentation space* $\operatorname{Hom}_\Lambda(P, Q)$.

We prove this first in the case when β is sincere and K is infinite (subsection 3.4), then for any β (subsection 3.5), then for any field K (subsection 3.6).

In Section 4 we prove the c-vector theorem below which states that, up to a precisely given sign, the det-weights β_i associated to a cluster tilting object are equal to the c-vectors associated to the cluster tilting object.

Theorem 4.1.6 (c-vector theorem) *Let $T = \coprod_{i=1}^n T_i$ be a cluster tilting object for Λ and let $f_i = \dim_K \operatorname{End}_\Lambda(T_i)$.*

(1) *There exist unique real Schur roots $\beta_1, \cdots, \beta_n$ so that $\underline{\dim}\, T_i \in D(\beta_j)$ for $i \neq j$.*

(2) *The c-vectors associated to the cluster tilting object are equal to β_i up to sign: $c_i = \pm\beta_i$. More precisely, $c_i = (-\langle \underline{\dim}\, T_i, \beta_i\rangle / f_i)\beta_i$.*

(3) *$\langle \underline{\dim}\, T_i, c_i\rangle = -f_i$ for each $i = 1, \ldots, n$.*

The c-vector theorem implies the sign coherence of c-vectors since weight vectors always lie in $\mathbb{N}^n$. Sign coherence of c-vectors has been shown in many cases [DWZ], [P] and in general in [GHKK]. We end with an example of a semi-invariant picture (Figure 4) illustrating some of our theorems and the important properties of the picture used in other papers [BHIT], [IOTW4], [IT16a], [IT16b].

There are also two appendices. Appendix A (Sec 5) discusses when a hereditary algebra is Morita equivalent to the tensor algebra of a modulated quiver and gives an example when this is not true. Appendix B (Sec 6) reviews the basic properties of reduced norm and shows that every character $M_k(D) \to K$ is a power of the reduced norm. Thus every semi-invariant on presentation space has a weight vector $w \in \mathbb{N}^n$ so that, under automorphisms of $P(\gamma_0), P(\gamma_1)$, the semi-invariant changes by the product of $\overline{n}_i^{w_i}$ where $\overline{n}_i$ are the reduced norms of the $GL(F_i)$ blocks of the automorphisms. We call w the *reduced weight.* We define the *reduced norm semi-invariants* $\overline{\sigma}_\beta$ and show that their reduced weights $\overline{\beta}$ are the c-vectors associated to a *reduced exchange matrix* $\overline{B}_\Lambda = ZB_\Lambda Z^{-1}$.

1. Basic definitions

In this paper Λ will be a basic finite dimensional hereditary algebra over any field K. Basic means that, as a right module over itself, the summands of Λ are pairwise nonisomorphic. Finite dimensional hereditary algebras share many important properties with the tensor algebra of their associated modulated quiver. For example they have the same Euler matrix, the same real Schur roots, the same semi-invariant domains and the same c-vectors, which are the topics we study in this paper. So, we begin with modulated quivers which are slightly easier to understand than the general case. Then we extend the definitions of presentation spaces and semi-invariants on presentation spaces to general finite dimensional hereditary algebras.

1.1. Modulated quivers.

By a *modulated quiver* $(Q, \mathcal{M})$ over K we mean a finite quiver Q without oriented cycles together with

(1) a finite dimensional division algebra F_i over K at each vertex i of Q and
(2) a finite dimensional F_i-F_j bimodule M_{ij} for every arrow $i \to j$ in Q.

The absence of multiple arrows is not a restriction. If we have a quiver with more than one arrow $i \to j$ then these are combined into one arrow with the associated bimodule being the direct sum of the bimodules on the original arrows. For example, the quiver $1 \rightrightarrows 2$ is equivalent to $1 \to 2$ with bimodule K^2 on the arrow.

Definition 1.1.1. The *valuation* on $Q = (Q_0, Q_1)$ given by the modulation $\mathcal{M}$ is defined to be the sequence of positive integers $f_i, i \in Q_0$ and d_{ij}, d_{ji} for $i \to j$ in Q_1 given as follows.

(1) $f_i = \dim_K F_i$ for each $i \in Q_0$.
(2) $d_{ij} = \dim_{F_j} M_{ij}$, $d_{ji} = \dim_{F_i} M_{ij}$ for each $i \to j$ in Q_1.

Proposition 1.1.2. [DR] *For any sequence of positive integers $f_i, i \in Q_0$ and pairs of positive integers (d_{ij}, d_{ji}) for every arrow $i \to j$ in Q_1 there exists a modulation of Q having these numbers as valuation if and only if $d_{ij} f_j = f_i d_{ji}$ for all i, j.*

Proof. Let K be any finite field, $K = \mathbb{F}_q$. For each i let F_i be the field with q^{f_i} elements. For each arrow $i \to j$, let M_{ij} be the field with $q^{d_{ij} f_j}$ elements. □

A *representation* V of a modulated quiver Q is given by

(1) a finite dimensional F_i-vector space V_i at each vertex i in Q_0 and
(2) an F_j linear map $V_i \otimes_{F_i} M_{ij} \to V_j$ for every arrow $i \to j$ in Q_1.

A representation of a modulated quiver is the same as a finite dimensional module over the *tensor algebra* $T(Q, \mathcal{M})$ of $(Q, \mathcal{M})$ which is defined to be the direct sum of all *tensor paths*:

$$T(Q, \mathcal{M}) := \coprod M_{j_0, j_1} \otimes_{F_{j_1}} M_{j_1, j_2} \otimes_{F_{j_2}} \cdots \otimes_{F_{j_{r-1}}} M_{j_{r-1}, j_r}$$

including paths of length zero (F_j) with multiplication given by concatenation of paths. Since the quiver Q has no oriented cycles this algebra is a finite dimensional hereditary algebra over K.

Definition 1.1.3. Given a finite dimensional hereditary algebra Λ over a field K, the *associated modulated quiver* $(Q, \mathcal{M})$ is given as follows. Fix an ordering of the simple Λ-modules $S_1, \cdots, S_n$. Let P_i be the projective cover of S_i.

(1) Let Q be the quiver with $Q_0 = \{1, \cdots, n\}$ and arrows $i \to j$ when $\mathrm{Ext}^1_\Lambda(S_i, S_j) \neq 0$.
(2) Let $F_i = \mathrm{End}_\Lambda(S_i)$ for each $i \in Q_0$.
(3) For each $i \to j$ in Q_1 let $M_{ij} = \mathrm{Hom}_\Lambda(P_j, rP_i / r^2 P_i)$.

There are examples of hereditary algebras which are not equivalent to their associated modulated quiver. We discuss these pathologies in Appendix A (Sec 5). Results in the paper are done in general, hence including these pathological cases.

1.2. **Euler matrix.** The underlying valued quiver of a hereditary algebra Λ is the valued quiver of its associated modulated quiver. However, it is useful to go directly from Λ to its underlying valued quiver, i.e., $f_i = \dim_K F_i$ where $F_i = \mathrm{End}_\Lambda(S_i)$ for each $i \in Q_0$, $d_{ij} = \dim_{F_j} \mathrm{Hom}_\Lambda(P_j, rP_i/r^2P_i)$, $d_{ji} = \dim_{F_i} \mathrm{Hom}_\Lambda(P_j, rP_i/r^2P_i)$ for each $i \to j$ in Q_1.

The *dimension vector* $\underline{\dim}\, V$ of V is defined to be the vector whose i-th coordinate is $\dim_{F_i} V_i$. We also have the dimension vector over K given by

$$\underline{\dim}_K V = D\, \underline{\dim}\, V$$

where D is the diagonal matrix with diagonal entries $f_i = \dim_K F_i$. Let E, L, R be the $n \times n$ matrices with ij entries

$$E_{ij} = \dim_K \mathrm{Hom}_\Lambda(S_i, S_j) - \dim_K \mathrm{Ext}^1_\Lambda(S_i, S_j)$$
$$L_{ij} = \dim_{F_j} \mathrm{Hom}_\Lambda(S_i, S_j) - \dim_{F_j} \mathrm{Ext}^1_\Lambda(S_i, S_j)$$
$$R_{ij} = \dim_{F_i} \mathrm{Hom}_\Lambda(S_i, S_j) - \dim_{F_i} \mathrm{Ext}^1_\Lambda(S_i, S_j)$$

Then $E_{ij} = L_{ij} f_j = f_i R_{ij}$ or, equivalently,

$$E = LD = DR\,.$$

We call E the *Euler matrix* of Λ, L the *left Euler matrix* of Λ and R the *right Euler matrix* of Λ. The underlying valued quiver of Λ has vertices $1 \le i \le n$ corresponding to the simple modules S_i and an arrow $i \to j$ if $E_{ij} < 0$ with valuations f_i on vertex i and $(d_{ij}, d_{ji}) = (-L_{ij}, -R_{ij})$ for every arrow $i \to j$.

Example 1.2.1. For example, for the valued quiver $_{f_2=3}\bullet \xrightarrow{(d_{21},d_{12})=(3,2)} \bullet_{f_1=2}$ we have:

$$LD = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = E = \begin{bmatrix} 2 & 0 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = DR.$$

The matrices L, R are always unimodular and $\det E = \det D$ is always the product of the dimensions f_i of $F_i = \mathrm{End}_\Lambda(P_i) = \mathrm{End}_\Lambda(S_i)$.

We also use, in the subsection on c-vectors (sec 4.2), the *exchange matrix* $B = L^t - R$. Since $DB = DL^t - DR = E^t - E$, DB is always skew symmetric. In the example this is:

$$B = L^t - R = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix},\ DB = E^t - E = \begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix}.$$

Proposition 1.2.2. *Let $\langle \cdot, \cdot \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be the Euler-Ringel pairing given by $\langle x, y \rangle = x^t E y$. Then, for any two Λ-modules M, N we have:*

$$\langle \underline{\dim}\, M, \underline{\dim}\, N \rangle = \dim \mathrm{Hom}_\Lambda(M, N) - \dim \mathrm{Ext}^1_\Lambda(M, N)\,.$$

For example, pairing $(P_1, \cdots, P_n)$ with $(S_1, \cdots, S_n)$ gives

$$\langle \underline{\dim}\, P_i, \underline{\dim}\, S_j \rangle = \dim \mathrm{Hom}_\Lambda(P_i, S_j) = f_i \delta_{ij}.$$

This equation can be written as $XEI_n = D$ where the i-th row of the matrix X is $\underline{\dim}\, P_i$. Furthermore $XE = D$ and $E = LD$ imply that $X = L^{-1}$.

Proposition 1.2.3. *Suppose that* $\mathrm{End}_\Lambda(M)$ *is a division algebra. Then* $\langle \underline{\dim}\, M, \underline{\dim}\, N \rangle$ *and* $\langle \underline{\dim}\, N, \underline{\dim}\, M \rangle$ *are divisible by* $f_M = \dim_K \mathrm{End}_\Lambda(M)$.

Proof. $\langle \underline{\dim}\, M, \underline{\dim}\, N \rangle = \dim_K \mathrm{Hom}_\Lambda(M, N) - \dim_K \mathrm{Ext}^1_\Lambda(M, N)$ which is divisible by f_M since $\mathrm{Hom}_\Lambda(M, N)$ and $\mathrm{Ext}^1_\Lambda(M, N)$ are vector spaces over $\mathrm{End}_\Lambda(M)$. □

For any $\gamma = (\gamma_i) \in \mathbb{N}^n$ we will use the notation $P(\gamma) = \coprod_{i=1}^n P_i^{\gamma_i}$. With this notation, another useful formula is the following. Suppose $\underline{\dim}\, M = \beta$ and $\gamma \in \mathbb{N}^n$. Then

$$\langle \underline{\dim}\, P(\gamma), \underline{\dim}\, M \rangle = \sum_{i=1}^n \gamma_i \dim_K \mathrm{Hom}_\Lambda(P_i, M) = \sum_{i=1}^n \gamma_i f_i \beta_i\,.$$

1.3. **Exceptional sequences.** We review the definition and basic properties of exceptional sequences. See [CB93], [Rin94] for details.

Definition 1.3.1. Let Λ be a finite dimensional hereditary algebra over any field K. Then a Λ-module M is called *exceptional* if $\mathrm{Ext}^1_\Lambda(M,M)=0$ and $\mathrm{End}_\Lambda(M)$ is a division algebra. In particular M is indecomposable. A sequence of modules $(X_1,\cdots,X_k)$ is called an *exceptional sequence* if all objects are exceptional and

$$\mathrm{Hom}_\Lambda(X_j,X_i)=\mathrm{Ext}^1_\Lambda(X_j,X_i)=0$$

for all $j>i$. An exceptional sequence is called *complete* if it is of maximal length. By 1.3.3(1) below, the maximal length is equal to the number of nonisomorphic simple modules.

The following are standard examples of complete exceptional sequences.

Proposition 1.3.2. *Let Λ be a finite dimensional hereditary algebra with admissible order given by* $\mathrm{Hom}_\Lambda(P_j,P_i)=0$ *for all $j>i$. Then:*

(1) *The simple modules $(S_n,S_{n-1},\cdots,S_1)$ form an exceptional sequence.*
(2) *The projective modules $(P_1,P_2,\cdots,P_n)$ form an exceptional sequence.*
(3) *The injective modules $(I_1,I_2,\cdots,I_n)$ form an exceptional sequence where I_i is the injective envelope of the simple module S_i for $i=1,\cdots,n$.*

Exceptional sequences have many nice properties. We list here only those properties that we use to prove the main theorems in this paper.

Proposition 1.3.3. *Let n be the number of simple Λ modules.*

(1) *Exceptional sequences are complete if and only if they have n objects.*
(2) *Every exceptional sequence can be extended to a complete exceptional sequence.*
(3) *If $(X_1,\cdots,X_n)$ is an exceptional sequence, $\underline{\dim}\,X_i$ generate $\mathbb{Z}^n$ as a $\mathbb{Z}$-module.*
(4) *Given an exceptional sequence $(X_1,\cdots,X_{n-1})$ of length $n-1$ and any $j=1,\cdots,n$, there are modules Y_j, unique up to isomorphism, so that*

$$(X_1,\cdots,X_{j-1},Y_j,X_j,\cdots,X_{n-1})$$

is a (complete) exceptional sequence.
(5) $\mathrm{End}_\Lambda(Y_j)\cong\mathrm{End}_\Lambda(Y_{j'})$ *for all j,j' in (4) above.*
(6) *Let $(X_1,\cdots,X_n)$ be an exceptional sequence. Then:*
If X_n is non-projective, then $(\tau X_n,X_1,\cdots,X_{n-1})$ is an exceptional sequence.
If $X_n=P_k$ is projective, then $(I_k,X_1,\cdots,X_{n-1})$ is an exceptional sequence.

Condition (4) implies that there is an action of the braid group on n strands on the set of (isomorphism classes of) complete exceptional sequences. For example, the braid generator σ_i which moves the i-th strand over the $(i+1)$-st strand acts by:

$$\sigma_i(X_1,\cdots,X_n)=(X_1,\cdots,X_{i-1},X^*_{i+1},X_i,X_{i+2},\cdots,X_n) \tag{1.1}$$

where, by property (4), X^*_{i+1} is the unique exceptional module which fits in the indicated location in the exceptional sequence given the other objects. One of the very important theorems about exceptional sequences used in this paper is the following result proved in [CB93] in the algebraically closed case and [Rin94] in general.

Theorem 1.3.4 (Crawley-Boevey, Ringel)**.** *The braid group on n strands acts transitively on the set of all complete exceptional sequences.*

In the case when K is algebraically closed, or, more generally when $\Lambda=KQ$ is the path algebra of a simply laced quiver without oriented cycles, it follows that $\mathrm{End}_\Lambda(M)=K$ for every exceptional Λ-module M. In general, the endomorphism rings of the X_i are division algebras which remain the same after any braid move by Proposition 1.3.3(5). So, Theorem 1.3.4 implies the following.

Corollary 1.3.5. *For any exceptional sequence $(X_1, \cdots, X_n)$, there is a permutation π of n so that $\mathrm{End}_\Lambda(X_i) \cong \mathrm{End}_\Lambda(S_{\pi(i)})$ for all i.*

Another important consequence of this theorem is the following.

Proposition 1.3.6. *Suppose that $(\beta_1, \cdots, \beta_n)$ is the set of dimension vectors of a complete exceptional sequence. Then each vector β_j is uniquely detemined by the other vectors together with the requirements that*

(1) *$\langle \beta_k, \beta_i \rangle = 0$ for $k > i$.*
(2) *The vectors β_i additively generate $\mathbb{Z}^n$.*

Note that these conditions depend only on n and the Euler form $\langle \cdot, \cdot \rangle$. This implies that there is an action of the braid group on n-strands on the set of dimension vectors of exceptional sequences which is given by

$$\sigma_i(\beta_1, \cdots, \beta_n) = (\beta_1, \cdots, \beta_{i-1}, \beta_{i+1}^*, \beta_i, \beta_{i+2}, \cdots, \beta_n)$$

where β_{i+1}^* is the unique vector in $\mathbb{N}^n$ satisfying the conditions of the proposition above.

Corollary 1.3.7. *An exceptional Λ-module is uniquely determined up to isomorphism by its dimension vector. Furthermore, a vector $\beta \in \mathbb{N}^n$ is the dimension vector of an exceptional module if and only if it appears in $\sigma(\alpha_1, \cdots, \alpha_n)$ for some element σ of the braid group on n strands where $\alpha_i = \underline{\dim}\, S_i$ are the unit vectors in $\mathbb{Z}^n$. In particular, the set of such dimension vectors depends only on the underlying valued quiver of Λ.*

This is a restatement of another important theorem of Schofield [S91]: The dimension vectors of the exceptional Λ-modules are the real Schur roots of Λ. In this paper we will not need the original definition of a real Schur root [Ka]. Following [S91], [S92], we use the characterizing property of real Schur roots given by the above corollary as the definition.

Definition 1.3.8. [S92] A *real Schur root* of Λ is a vector $\beta \in \mathbb{N}^n$ with the property that there exists an exceptional module M_β with $\underline{\dim}\, M_\beta = \beta$.

As a special case of Corollary 1.3.7 above we have the following.

Corollary 1.3.9. *The real Schur roots of a hereditary algebra are the same as those of the associated modulated quiver.*

1.4. **Extension to arbitrary fields.** The main results of this paper hold for arbitrary fields. The proofs are first done for infinite fields and they are extended to all fields using the following arguments.

Recall that if K is a finite field and F is a finite field extension of K then

$$F \otimes_K K(t) \cong F(t)$$

is a finite field extension of $K(t)$. For any K-algebra Λ we will use the notation $\Lambda(t)$ to denote $\Lambda \otimes_K K(t)$. This is a finite dimensional hereditary algebra over $K(t)$. For any Λ-module M, let $M(t)$ denote the $\Lambda(t)$-module $M \otimes_K K(t)$. Recall that the dimension vector of $M(t)$ as a $\Lambda(t)$-module is the vector whose i-th coordinate is $\dim_{F_i(t)} \mathrm{Hom}_{\Lambda(t)}(P_i(t), M(t))$.

Theorem 1.4.1. *Let Λ be a finite dimensional hereditary algebra over a finite field K and let M be an exceptional Λ-module with $\underline{\dim}\, M = \beta$. Then, $\Lambda(t)$ is a finite dimensional hereditary algebra over $K(t)$ and $M(t)$ is an exceptional $\Lambda(t)$-module with the same dimension vector β. Furthermore, every exceptional $\Lambda(t)$ module is isomorphic to $M(t)$ for a unique exceptional Λ-module M.*

Proof. Since tensor product over K with $K(t)$ is exact we get:

$$\mathrm{Hom}_\Lambda(X, Y) \otimes_K K(t) \cong \mathrm{Hom}_{\Lambda(t)}(X(t), Y(t))$$

$$\mathrm{Ext}^1_\Lambda(X,Y)\otimes_K K(t)\cong \mathrm{Ext}^1_{\Lambda(t)}(X(t),Y(t))$$

for any two Λ-modules X, Y. Therefore, a Λ-module M is exceptional if and only if $M(t) = M\otimes_K K(t)$ is an exceptional $\Lambda(t)$-module. The rest follows from Corollary 1.3.7. □

2. Virtual representations and semi-invariants

Throughout this section we consider representations of a finite dimensional hereditary algebra Λ over an infinite field K.

2.1. **Generic decomposition theorem.** We first recall the Happel-Ringel Lemma [HR].

Lemma 2.1.1 (Happel-Ringel)**.** *Suppose that T_1, T_2 are indecomposable modules over a hereditary algebra Λ so that $\mathrm{Ext}^1_\Lambda(T_1,T_2)=0$. Then any nonzero morphism $T_2\to T_1$ is either a monomorphism or an epimorphism.*

An important consequence of this lemma is the following observation of Schofield.

Lemma 2.1.2 (Schofield)**.** *Suppose that $\{M_i\}$ is a set of nonisomorphic indecomposable modules so that $\mathrm{Ext}^1_\Lambda(M_i,M_j)=0$ for all i,j. Then $\{M_i\}$ can be renumbered so that $\mathrm{Hom}_\Lambda(M_j,M_i)=0$ for $j>i$, i.e., so that it forms an exceptional sequence.*

Proof. If not, there is an oriented cycle of nonzero morphisms between the M_i which are monomorphisms or epimorphisms. In this oriented cycle there must be an epimorphism followed by a monomorphism, hence the composition is neither a monomorphism nor an epimorphism which contradicts the Happel-Ringel Lemma. □

We give a definition of the representation space $Rep(\Lambda,\alpha)$ for any finite dimensional hereditary algebra Λ and $\alpha\in\mathbb{N}^n$.

Definition 2.1.3. Choose a decomposition of the radical of each projective Λ-module P_i:

$$\varphi_i : radP_i \cong \coprod P_j^{b_{ij}} \tag{2.1}$$

Note that $b_{ij}=\dim_{F_j}\mathrm{Ext}^1_\Lambda(S_i,S_j)=-L_{ij}$ for $i\neq j$ and $b_{ii}=0$ where L is the left Euler matrix. For every $\alpha\in\mathbb{N}^n$ we define the *representation space* $Rep(\Lambda,\alpha)$ to be

$$Rep(\Lambda,\alpha)=\prod_{i\to j}\mathrm{Hom}_\Lambda(P_j^{b_{ij}\alpha_i},P_j^{\alpha_j})\cong\prod_{i\to j}F_j^{b_{ij}\alpha_i\alpha_j}.$$

This is a vector space and therefore an affine space over K. The Λ-module corresponding to an element $f\in Rep(\Lambda,\alpha)$ is the cokernel of the map

$$(f\circ\varphi - inc):\coprod radP_i^{\alpha_i}\to\coprod P_j^{\alpha_j}$$

where $\varphi=\prod\varphi_i^{\alpha_i}$ is a product of the φ_i chosen in (2.1) and inc is the inclusion map. We view elements of $Rep(\Lambda,\alpha)$ as modules with specified presentations $radP(\alpha)\to P(\alpha)$ where we use the notation

$$P(\alpha)=\coprod P_i^{\alpha_i}.$$

It is easy to see that every Λ-module with dimension vector α has a presentation of the above form. When $\Lambda = KQ$ is the path algebra of a quiver, this agrees with the classical definition of $Rep(\Lambda,\alpha)$. The following theorem was originally proved by Kac [Ka] for quivers over an algebraically closed field K. However, the following proof works over any infinite field.

Theorem 2.1.4. *Let Λ be a hereditary algebra over an infinite field. Suppose that $\alpha\in\mathbb{N}^n$ and M is a rigid module with $\underline{\dim}\,M=\alpha$. Then the set of all $N\in Rep(\Lambda,\alpha)$ which are isomorphic to M forms an open subset of $Rep(\Lambda,\alpha)$.*

Proof. Consider the set U of all $N \in Rep(\Lambda, \alpha)$ so that $\mathrm{Ext}^1_\Lambda(M, N) = 0 = \mathrm{Ext}^1_\Lambda(N, M)$. This is an open set containing all modules isomorphic to M since, e.g., $\mathrm{Ext}^1_\Lambda(M, \mathrm{coker}\, f - 1) = 0$ is equivalent to the condition that $\mathrm{Ext}^1(M, f-1) : \mathrm{Ext}^1(M, rP(\alpha)) \to \mathrm{Ext}^1(M, P(\alpha))$ is surjective and this is an open condition on $f \in Rep(\Lambda, \alpha)$. We will show that any $N \in U$ is isomorphic to M. This will imply that $\{N \in Rep(\Lambda, \alpha) : N \cong M\} = U$ is open.

Let $\{N_j\}$ be the components of one such N. Let $\{M_i\}$ be the components of M. Then $\{M_i, N_j\}$ form a collection of indecomposable modules which do not extend each other. So, by Schofield's observation, we can arrange them into an exceptional sequence, possibly with repetitions. Take the last object in the sequence. By symmetry, suppose it is N_j. Then

$$\dim \mathrm{Hom}_\Lambda(N_j, M) = \langle \underline{\dim}\, N_j, \alpha \rangle = \dim \mathrm{Hom}_\Lambda(N_j, N) \neq 0.$$

So, there is a nonzero morphism $N_j \to M_i$ for some i. Since N_j is last in the exceptional sequence, this can happen only if $N_j \cong M_i$. Then $N/N_j, M/M_i$ are rigid modules of the same dimension vector. So, $N/N_j \cong M/M_i$ by induction on dimension. We conclude that $N \cong N_j \oplus N/N_j \cong M_i \oplus M/M_i \cong M$ as claimed. □

Since exceptional modules are rigid, we have the following immediate consequence.

Corollary 2.1.5. *Suppose that M_α is an exceptional Λ-module with $\underline{\dim}\, M_\alpha = \alpha$. Then the set of all $N \in Rep(\Lambda, \alpha)$ which are isomorphic to M_α forms an open and thus dense subset of $Rep(\Lambda, \alpha)$. In particular M_α is uniquely determined up to isomorphism by α.*

Another important consequence of Theorem 2.1.4 is the following.

Corollary 2.1.6 (Generic Decomposition Theorem for rigid modules in modulated case)**.** *Suppose that $\alpha_1, \cdots, \alpha_k$ are real Schur roots so that $\mathrm{Ext}^1_\Lambda(M_{\alpha_i}, M_{\alpha_j}) = 0$ for all i, j. Let $\gamma = \sum_{i=1}^k n_i \alpha_i$ be a nonnegative integer linear combination of these roots. Then the general representation with dimension vector γ is isomorphic to $\coprod_{i=1}^k M_{\alpha_i}^{n_i}$.*

Proof. Apply Theorem 2.1.4 to the module $M = \coprod_{i=1}^k M_{\alpha_i}^{n_i}$ with $\underline{\dim}\, M = \gamma$. □

2.2. **Presentation Space and Semi-invariants.** Let γ_0, γ_1 to be vectors in $\mathbb{N}^n$. We define the *presentation space* $Pres_\Lambda(\gamma_1, \gamma_0)$ to be

$$Pres_\Lambda(\gamma_1, \gamma_0) := \mathrm{Hom}_\Lambda(P(\gamma_1), P(\gamma_0))$$

where we use the notation $P(\alpha) = \coprod P_i^{\alpha_i}$. For example, $\Lambda = P(1, 1, \cdots, 1)$. Presentation spaces are affine spaces over K. They are related to representation spaces as follows. Suppose that $\alpha \in \mathbb{N}^n$. Then $radP(\alpha) = P(\gamma)$ for $\gamma \in \mathbb{N}^n$ and we have the K-linear embedding:

$$Rep(\Lambda, \alpha) \hookrightarrow Pres_\Lambda(\gamma, \alpha)$$

sending $f : P(\gamma) \to P(\alpha)$ to $f - inc$. The elements of $Rep(\Lambda, \alpha)$ and their images in $Pres_\Lambda(\gamma, \alpha)$ give presentations of the same module with dimension vector α. The algebraic group $Aut(P(\gamma_1))^{op} \times Aut(P(\gamma_0))$ acts on presentation space by composition: $(a, b)f = bfa$. In the algebraically closed case there is a corresponding group action on $Rep(\Lambda, \alpha)$.

Definition 2.2.1. A *semi-invariant* on $Pres_\Lambda(\gamma_1, \gamma_0) = \mathrm{Hom}_\Lambda(P(\gamma_1), P(\gamma_0))$ is defined to be a regular function

$$\sigma : Pres_\Lambda(\gamma_1, \gamma_0) \to K$$

for which there exist characters $\eta_s : \mathrm{Aut}(P(\gamma_s)) \to K^*$, $s = 0, 1$, so that, for all $(g_0, g_1) \in \mathrm{Aut}(P(\gamma_0)) \times \mathrm{Aut}(P(\gamma_1))$ and $f \in Pres_\Lambda(\gamma_1, \gamma_0)$, we $\sigma(g_0 f g_1) = \sigma(f)\eta_0(g_0)\eta_1(g_1)$. The pair of characters (η_0, η_1) is called the *weight* of σ.

The following lemma shows that such characters are products of character on $GL(\alpha_i, F_i)$.

Lemma 2.2.2. *Every group homomorphism $\mathrm{Aut}_\Lambda(P(\alpha)) \to K^*$ factors through the group $\prod_i \mathrm{Aut}_\Lambda(P_i^{\alpha_i})$.*

Proof. When K has only two elements, the lemma holds trivially. So, we may assume K has at least three elements. Then every element of K can be written as $a-b$ where $a, b \neq 0$. So, the elementary matrix

$$\begin{bmatrix} 1 & 0 \\ a-b & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ -ab & b \end{bmatrix} \begin{bmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix} \begin{bmatrix} a^{-1} & 0 \\ 1 & b^{-1} \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

is a commutator. We write automorphisms of $P(\alpha)$ in block matrix form. Since $\mathrm{Hom}_\Lambda(P_j, P_i) = 0$ for $i < j$, the matrix is lower triangular with diagonal blocks in $\mathrm{Aut}_\Lambda(P_i^{\alpha_i})$. So, every element in the kernel of the homomorphism $\pi : \mathrm{Aut}_\Lambda(P(\alpha)) \twoheadrightarrow \prod \mathrm{Aut}_\Lambda(P_i(\alpha_i))$, when written in matrix form, is lower triangular with 1s on the diagonal. But all such matrices are products of elementary matrices such as the one above. So $\ker \pi$ lies in the commutator subgroup of $\mathrm{Aut}_\Lambda(P(\alpha))$. Since K^* is abelian, any homomorphism $\varphi : \mathrm{Aut}_\Lambda(P(\alpha)) \to K^*$ is trivial on commutators. Therefore $\ker \pi \subseteq \ker \varphi$ which implies that φ factors through π proving the lemma. □

Since $\mathrm{End}_\Lambda(P_i) = F_i$ is a division algebra, we have $\mathrm{Aut}_\Lambda(P_i^{n_i}) \cong GL(n_i, F_i)$. This has a character

$$det_K : \mathrm{Aut}_\Lambda(P_i^{n_i}) \to K^*$$

given by taking determinant of an automorphism $P_i^{n_i} \to P_i^{n_i}$ by considering it as a K-linear automorphism of $P_i^{n_i}$. When $F_i = \mathrm{End}_\Lambda(P_i)$ is a separable commutative field extension of K, it is well-known that every polynomial character $\mathrm{Aut}_\Lambda(P_i^{n_i}) \to K^*$ is a power of det_K. In general, we will consider only those characters which are integer powers of det_K. (There may be other characters called "reduced norms" as explained in Appendix B, Sec 6.) Since nonzero endomorphisms of P_i do not factor through any P_j, we have n well-defined characters

$$\chi_i : \mathrm{Aut}_\Lambda(\coprod P_j^{n_j}) \twoheadrightarrow \mathrm{Aut}_\Lambda(P_i^{n_i}) \to K^*.$$

We call a character $\chi : \mathrm{Aut}_\Lambda(\coprod P_j^{n_j}) \to K^*$ *determinantal* if there exists a vector $\alpha \in \mathbb{Z}^n$ so that $\chi = \prod_i \chi_i^{\alpha_i}$. The coordinate α_i is uniquely determined by χ if and only if $n_i \neq 0$ (and K is infinite).

The following proposition is analogous to Proposition 3.3.3 from [IOTW09] which was proved for simply laced quivers. But the same proof works in general.

Proposition 2.2.3. *Let Λ be a finite dimensional hereditary algebra over an infinite field K. Suppose that σ is a nonzero semi-invariant on $Pres_\Lambda(\gamma_1, \gamma_0)$ with weights η_0, η_1 which are determinantal characters given by $\eta_0 = \prod_i \chi_i^{\alpha_i}$, $\eta_1 = \prod_i \chi_i^{\beta_i}$ where $\alpha, \beta \in \mathbb{Z}^n$. Then $\alpha_i = \beta_i$ whenever they are both well-defined, i.e., when the i-th coordinates of γ_0, γ_1 are both nonzero.* □

Definition 2.2.4. We say that a semi-invariant σ on $Pres_\Lambda(\gamma_1, \gamma_0)$ has *determinantal weight vector* (det-weight) $\beta \in \mathbb{Z}^n$ if both of its weights can be written as $\chi_i^{\beta_i}$. In other words, for any $f : P(\gamma_1) \to P(\gamma_0), h \in \mathrm{Aut}(P(\gamma_1)), g \in \mathrm{Aut}(P(\gamma_0))$ we have:

$$\sigma(gfh) = \sigma(f) \prod \chi_i(g)^{\beta_i} \chi_i(h)^{\beta_i} \tag{2.2}$$

We also say that $f : P(\gamma_1) \to P(\gamma_0)$ *admits a semi-invariant* of det-weight β if there exists a semi-invariant σ of det-weight β on $Pres_\Lambda(\gamma_1, \gamma_0)$ so that $\sigma(f) \neq 0$.

As an example, let $\gamma_0 = \gamma_1$ be the unit vector $e_1 = (1, 0, 0, \cdots, 0)$. Then $P(\gamma_0) = P_1 = P(\gamma_1)$ and any nonzero homomorphism $f \in Pres_\Lambda(e_1, e_1)$ admits a semi-invariant of det-weight β for any $\beta \in \mathbb{Z}^n$ whose first coordinate is 1. The semi-invariant on $Pres_\Lambda(e_1, e_1) = F_1$ is the determinantal character $\chi_1 : F_1 \to K$. The group which acts on this presentation space is $(GL(1, F_1) \times GL(0, F_2) \times \cdots \times GL(0, F_n))^2$. Since $GL(0, F_i)$ is a trivial group, it has only one character which is equal to any power of itself, including negative powers. So, the det-weight β is $(1, *, *, \cdots, *)$.

We now show that the coordinates of β are nonnegative when they are well-defined.

Proposition 2.2.5. *If a semi-invariant on $Pres_\Lambda(\gamma_1, \gamma_0)$ has well-defined det-weight β, i.e., when γ_0, γ_1 are sincere and K is infinite, then $\beta \in \mathbb{N}^n$.*

Proof. For fixed f, h, $\sigma(gfh)$ is a regular function on $g \in \mathrm{End}_\Lambda(\coprod P_i^{n_i})$. But $\chi_i(g)^{-1}$ does not extend to a regular function on $\mathrm{End}_\Lambda(\coprod P_i^{n_i})$ since $0 \in \mathrm{End}_\Lambda(\coprod P_i^{n_i})$. □

The following proposition is one of the motivations for the uniform notation *Vrep*(Λ, α) introduced in the next section in Definition 2.3.1.

Proposition 2.2.6. *Suppose that $Pres_\Lambda(\gamma_1, \gamma_0)$ has a semi-invariant of det-weight β and $(L^t)^{-1}(\gamma_0 - \gamma_1) = \alpha$ where L is the left Euler matrix and K is infinite. Then $\langle \alpha, \beta \rangle = 0$.*

Proof. Consider multiplication by $\lambda \in K^*$. The character of this automorphism of $P_i^{\gamma_{1,i}}$ is $\chi_i(\lambda) = \det(\lambda^{\gamma_{1,i}}) = \lambda^{\gamma_{1,i} f_i}$. Since $\lambda f = f \lambda$ for all $f \in Pres_\Lambda(\gamma_1, \gamma_0)$ we conclude that

$$\lambda^{\sum \gamma_{1,i} f_i \beta_i} = \lambda^{\sum \gamma_{0,i} f_i \beta_i}$$

where $P(\gamma_1) = \coprod_i P_i^{\gamma_{1,i}}$ and $P(\gamma_0) = \coprod_i P_i^{\gamma_{0,i}}$. Since this polynomial equation holds for all $\lambda \in K^*$ which is infinite, we conclude that $\sum \gamma_{1,i} f_i \beta_i = \sum \gamma_{0,i} f_i \beta_i$. So,

$$0 = \sum (\gamma_{0,i} - \gamma_{1,i}) f_i \beta_i = \langle \alpha, \beta \rangle$$

since $(L^t)^{-1}(\gamma_0 - \gamma_1) = \alpha$ and $\langle \alpha, \beta \rangle = \alpha^t E \beta = \alpha^t L D \beta = (\gamma_0 - \gamma_1) D \beta$. □

As a corollary of this proof we have the following.

Corollary 2.2.7. *A semi-invariant σ on $Pres_\Lambda(\gamma_1, \gamma_0)$ with det-weight β is a homogeneous polynomial function of degree $\sum_i \gamma_{1,i} f_i \beta_i$ which is also equal to $\sum_i \gamma_{0,i} f_i \beta_i$ assuming K is infinite. In particular, $\beta = 0$ if and only if σ is constant.* □

When $f \in Pres_\Lambda(\gamma_1, \gamma_0)$ is a monomorphism $P(\gamma_1) \hookrightarrow P(\gamma_0)$, the dimension vector of the cokernel is

$$\underline{\dim}\, \mathrm{coker}\, f = \underline{\dim}\, P(\gamma_0) - \underline{\dim}\, P(\gamma_1) = (L^t)^{-1}(\gamma_0 - \gamma_1)$$

which is α in the proposition above. We view different presentations of the same module as being equivalent. This leads to the following definitions.

2.3. **Virtual representations.** "Vitual representations" will be given by "stablilizing" presentation $f : P(\gamma_1) \to P(\gamma_0)$. These will form the objects of the "virtual representation category" and the elements of the "virtual representation space." First, note that

$$P(\gamma + \delta) = P(\gamma) \oplus' P(\delta)$$

where $\oplus'$ denotes the "shuffle sum" given by collecting isomorphic summands together. We use this to make the equality strict. For example $(P_1 \oplus P_2) \oplus' P_1$ denotes $P_1 \oplus P_1 \oplus P_2$. Given any three dimension vectors $\gamma_0, \gamma_1, \delta \in \mathbb{N}^n$, consider the linear monomorphism

$$Pres_\Lambda(\gamma_1, \gamma_0) \hookrightarrow Pres_\Lambda(\gamma_1 + \delta, \gamma_0 + \delta)$$

given by sending $f : P(\gamma_1) \to P(\gamma_0)$ to $f \oplus' 1_{P(\delta)} : P(\gamma_1) \oplus' P(\delta) \to P(\gamma_0) \oplus' P(\delta)$. We call this map *stabilization.* This gives a directed system whose objects are all presentation spaces $Pres_\Lambda(\delta_1, \delta_0)$ having the property that $\delta_0 - \delta_1 = \gamma_0 - \gamma_1$. This implies $\underline{\dim}\, P(\delta_0) - \underline{\dim}\, P(\delta_1) = \alpha = \underline{\dim}\, P(\gamma_0) - \underline{\dim}\, P(\gamma_1) \in \mathbb{Z}^n$. Equivalently, $\gamma_0 - \gamma_1 = L^t \alpha$.

Definition 2.3.1. For any $\alpha \in \mathbb{Z}^n$ we define the *virtual representation space* *Vrep*(Λ, α) to be the direct limit (colimit):

$$Vrep(\Lambda, \alpha) := colim\, Pres_\Lambda(\gamma_1, \gamma_0) = colim\, \mathrm{Hom}_\Lambda(P(\gamma_1), P(\gamma_0))$$

where the colimit is taken over all pairs $\gamma_0, \gamma_1 \in \mathbb{N}^n$ so that $\gamma_0 - \gamma_1 = L^t \alpha$. Elements of *Vrep*$(\Lambda, x)$ will be called *virtual representations* of Λ of *dimension vector* $\alpha \in \mathbb{Z}^n$. We take the direct limit topology on $Vrep(\Lambda, \alpha)$. Since each presentation space is irreducible, it follows that $Vrep(\Lambda, \alpha)$ is irreducible, i.e., any nonempty open subset is dense.

The main purpose of the virtual representation space is to make the weights of semi-invariants well-defined. See Definition 2.4.6 below.

We now construct the category $Vrep(\Lambda)$ of all virtual representations of Λ. The object set of this category will be the disjoint union

$$\mathcal{O}b(Vrep(\Lambda)) := \coprod_{\alpha \in \mathbb{Z}^n} Vrep(\Lambda, \alpha).$$

A morphism $f : X \to Y$ is given as follows. Choose representatives $P(\xi_*)$, $P(\eta_*)$ for X, Y and consider commuting diagrams

$$\begin{array}{ccc} P(\xi_1) & \xrightarrow{f_1} & P(\eta_1) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ P(\xi_0) & \xrightarrow{f_0} & P(\eta_0). \end{array}$$

In other words, (f_0, f_1) gives a chain map $P(\xi_*) \to P(\eta_*)$. Two such chain maps are equivalent $(f_0, f_1) \sim (f_0', f_1')$ if they are homotopic, i.e., if there is a map $h : P(\xi_0) \to P(\eta_1)$ so that $f_1' = f_1 + hp$ and $f_0' = f_0 + qh$. We define a morphism $X \to Y$ to be an equivalence class of such chain maps under the equivalence relation generated by homotopy as explained above and stabilization which means $(f_0, f_1) \sim (f_0 \oplus' 1_P, f_1 \oplus' 1_P)$ for any projective module $P = P(\zeta)$.

There is another model for $Vrep(\Lambda)$ which may be easier to deal with. This is the homotopy category $Pres(\Lambda)$ whose objects are all chain complexes of finitely generated projective modules in degrees 0 and 1: $P(\gamma_*) = (p : P(\gamma_1) \to P(\gamma_0))$ and whose morphisms are homotopy classes of degree 0 chain maps. Objects of $Pres(\Lambda)$ will be called *presentations*.

Proposition 2.3.2. *The stabilization map $P(\gamma_*) \mapsto \langle P(\gamma_*) \rangle$ gives an equivalence of categories*

$$Pres(\Lambda) \cong Vrep(\Lambda).$$

Proof. As a chain complex, every presentation is homotopy equivalent to each of its stabilizations. Therefore, any two representatives of the same virtual representation are canonically isomorphic as objects of $Pres(\Lambda)$. Given any two objects X, Y in $Vrep(\Lambda)$, a morphism $f : X \to Y$ is represented by a morphism $\tilde{f} = (f_0, f_1) : P(\xi_*) \to P(\eta_*)$ in $Pres(\Lambda)$. Since these representatives are unique up to canonical isomorphism in $Pres(\Lambda)$, $\tilde{f}$ is unique. So, $\mathrm{Hom}_{Vrep(\Lambda)}(X, Y) \cong \mathrm{Hom}_{Pres(\Lambda)}(P(\xi_*), P(\eta_*))$. In other words the stabilization functor is full, faithful and dense. So, it is an equivalence. □

Since direct sum does not commute with stabilization, we use the above proposition to define direct sums of virtual representations as those isomorphic to direct sums of corresponding objects of $Pres(\Lambda)$.

Recall that homotopic maps of chain complexes induce the same map in homology. Therefore, we have well defined functors

$$H_0, H_1 : Pres(\Lambda) \to mod\text{-}\Lambda.$$

By Proposition 2.3.2, these extend uniquely to $Vrep(\Lambda)$. Since every submodule of $P(\gamma_1)$ is projective, $H_1(p : P(\gamma_1) \to P(\gamma_0)) = \ker p$ is projective. When $\ker p = 0$, we have a short exact sequence

$$0 \to P(\gamma_1) \xrightarrow{p} P(\gamma_0) \to H_1(P(\gamma_*)) \to 0.$$

Thus p is a presentation in the usual sense of the module $M = H_0(P(\gamma_*)) = \mathrm{coker}\, p$. When $\gamma_0 = 0$, the presentation $P(\gamma_1) \to 0$ is denoted $P(\gamma_1)[1]$ and called the *shift* of $P(\gamma_1)$. More generally, the *shift* of $P(\gamma_*) = (p : P(\gamma_1) \to P(\gamma_0))$ is the chain complex (C_*, d) where $C_2 = P(\gamma_1), C_1 = P(\gamma_0)$, $C_i = 0$ for $i \neq 1, 2$ and the boundary map is $d = -p : C_2 \to C_1$.

Since the kernel of $p : P(\gamma_1) \to P(\gamma_0)$ splits off of $P(\gamma_1)$, we get the following.

Proposition 2.3.3. *The indecomposable objects of $Pres(\Lambda)$ and $Vrep(\Lambda)$ are*

(1) *projective presentations of indecomposable Λ-modules and*
(2) *shifted indecomposable projective Λ-modules $P[1]$.*

For two objects $P(\xi_*), P(\eta_*)$ of $Pres(\Lambda)$ (or $Vrep(\Lambda)$) we define $\mathrm{Ext}^1_{Pres(\Lambda)}(P(\xi_*), P(\eta_*))$ in the usual way as the space of homotopy classes of chain maps $P(\xi_*) \to P(\eta_*)[1]$.

Corollary 2.3.4. *$Pres(\Lambda)$ is equivalent to the full subcategory of the bounded derived category of mod-Λ with objects all X so that $\mathrm{Hom}_{\mathcal{D}^b}(X, Y[k]) = 0$ for all $Y \in$ mod-Λ and all $k \neq 0, 1$. Furthermore, $\mathrm{Ext}^1_{Pres(\Lambda)}(X, Y) = \mathrm{Ext}^1_{\mathcal{D}^b}(X, Y)$ for all $X, Y \in Pres(\Lambda)$.*

Proof. It is clear that all $X \in Pres(\Lambda)$ satisfy this condition. Conversely, suppose that X satisfies the condition. Then $X \in$ *mod*-Λ or $X = Z[1]$ where $Z \in$ *mod*-Λ. In the second case we have $\mathrm{Hom}_{\mathcal{D}^b}(Z[1], Y[2]) = 0$ for all modules Y. This implies that Z is projective. □

Recall that the *cluster category* $\mathcal{C}_\Lambda$ of Λ of the orbit category of the bounded derived category $\mathcal{D}^b(mod\text{-}\Lambda)$ under the functor $F = \tau^{-1}[1]$ (see [BMRRT]). The fundamental domain of this category consists of Λ-modules and shifted projective modules. Therefore we get the following.

Corollary 2.3.5. *The functor $\Psi : Pres(\Lambda) \to \mathcal{C}_\Lambda$ which sends each object to its F-orbit is a faithful functor which induces a bijection between isomorphism classes of objects. Furthermore, $V, W \in Pres(\Lambda)$ are* Ext*-orthogonal in the sense that $\mathrm{Ext}^1_{\mathcal{D}^b}(X, Y) = 0 = \mathrm{Ext}^1_{\mathcal{D}^b}(Y, X)$ if and only if $\mathrm{Ext}^1_{\mathcal{C}_\Lambda}(\Psi X, \Psi Y) = 0 = \mathrm{Ext}^1_{\mathcal{C}_\Lambda}(\Psi Y, \Psi X)$.*

The *dimension vector* of $P(\gamma_*)$ is defined by $\underline{\dim}\, P(\gamma_0) - \underline{\dim}\, P(\gamma_1)$. It is easy to see that this is the unique integer vector $\alpha \in \mathbb{Z}^n$ satisfying $L^t\alpha = \gamma_0 - \gamma_1$ where L is the left Euler matrix of Λ.

Theorem 2.3.6. *Suppose that $V = (P(\gamma_1) \xrightarrow{p} P(\gamma_0))$ is a presentation with dimension vector $\underline{\dim}\, V = \alpha$ so that $\mathrm{Ext}^1_{Pres(\Lambda)}(V, V) = 0$. Then the set of all presentations isomorphic to V is an open dense subset of the K-affine space $Pres_\Lambda(\gamma_1, \gamma_0)$.*

Proof. Let $P = \ker p$. Then $V = P[1] \coprod W$ where $W = (P(\gamma_1') \xrightarrow{q} P(\gamma_0))$ is a projective presentation of a Λ-module M with $\mathrm{Ext}^1_\Lambda(M, M) = 0$ and $\underline{\dim}\, M = \beta$. By assumption, $0 = \mathrm{Ext}^1_{Pres(\Lambda)}(P[1], W) = \mathrm{Ext}^1_{\mathcal{D}^b(\Lambda)}(P[1], W) = \mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(P, W) = \mathrm{Hom}_\Lambda(P, M)$. So,

$$q_* : \mathrm{Hom}_\Lambda(P, P(\gamma_1')) \to \mathrm{Hom}_\Lambda(P, P(\gamma_0))$$

is surjective. Let $f : P(\gamma_1) \to P(\gamma_0)$ be a general morphism. Restrict f to the components of $P(\gamma_1)$ to get $f_1 : P \to P(\gamma_0)$ and $f_2 : P(\gamma_1') \to P(\gamma_0)$. Since q is a monomorphism f_2, being a general map, will also be a monomorphism. So, the cokernel of f_2 will be a module N with $\underline{\dim}\, N = \beta$. By Theorem 2.1.4, $N \cong M$ for general f_2. So, f_2 is equivalent to $q : P(\gamma_1') \to P(\gamma_0)$. Since $\mathrm{Hom}_\Lambda(P, M) = 0$ this implies that the image of f_1 lies in the image of f_2, say $f_1 = f_2 \circ s$. Replacing $f = (f_1, f_2)$ with $(f_1 - f_2 \circ s, f_2)$ gives a chain complex isomorphic to $f : P(\gamma_1) \to P(\gamma_0)$. But the new chain complex is isomorphic to $P[1] \coprod W$. Thus the general morphism $f : P(\gamma_1) \to P(\gamma_0)$ is equivalent to the given morphism p as claimed proving the theorem. □

Definition 2.3.7. Recall that a *partial cluster tilting set* is a set $\{\beta_i\}$ of distinct real Schur roots and shifted projective roots so that the general presentations M_i with $\underline{\dim}\, M_i = \beta_i$ do not extend each other (in the derived category). If the partial cluster tilting set has exactly n elements it is called a *cluster tilting set*. We call the direct sum $\coprod_i M_i$ of the corresponding objects of $Pres(\Lambda)$ a *(partial) cluster tilting object*. This terminology is justified by Corollary 2.3.5.

Corollary 2.3.8 (Virtual Generic Decomposition Theorem)**.** *Let $\gamma = \sum_{i=1}^k r_i\beta_i \in \mathbb{Z}^n$ be an integer vector which is a nonnegative rational linear combination of the elements β_i of a partial cluster tilting set. Then the r_i are all integers, call them n_i, and the general presentation with dimension vector γ is isomorphic to $\coprod_{i=1}^k M_i^{n_i}$ where M_i are rigid with $\underline{\dim}\, M_i = \beta_i$. In other words, the set of all elements of $Vrep(\Lambda,\gamma)$ isomorphic to $\coprod_{i=1}^k M_i^{n_i}$ is open and dense.*

Proof. The underlying modules $|M_i|$ form an exceptional sequence which can be extended to a complete exceptional sequence, say $|M_j|$, $j = 1,\cdots,n$. (See section 1.3.) The dimension vectors $\underline{\dim}\, M_j$ generate $\mathbb{Z}^n$ by Proposition 1.3.3 (3). Therefore, the integer vectors in the $\mathbb{Q}$ span of the vectors in the subset $\underline{\dim}\, M_i$ lie in the $\mathbb{Z}$-span of these vectors. By Theorem 2.3.6, the virtual representations isomorphic to $\coprod_{i=1}^k M_i^{n_i}$ form an open dense subset of each presentation space and therefore of the colimit $Vrep(\Lambda,\gamma)$. □

Corollary 2.3.9. *Every partial cluster tilting set has at most n elements.*

Proof. If a set of roots in $\mathbb{Z}^n$ has more than n elements then it will be linearly dependent over $\mathbb{Z}$ and any such linear dependence can be written in the form $\sum n_i\beta_i = \sum m_j\beta_j$ where $n_i, m_j \geq 0$. By the above theorem, this would give two incompatible descriptions of the general presentation of this dimension. □

2.4. **Virtual semi-invariants.** We return to the discussion of semi-invariants. We consider direct sums of presentations.

Lemma 2.4.1. *Let $f : P(\gamma_1+\delta_1) \to P(\gamma_0+\delta_0)$ be a direct sum of two projective presentations $f = f_1 \coprod f_2$ where $f_1 : P(\gamma_1) \to P(\gamma_0)$ and $f_2 : P(\delta_1) \to P(\delta_0)$. If f admits a semi-invariant of det-weight β then so does each f_i.*

Proof. Consider the composition:

$$Pres_\Lambda(\gamma_1,\gamma_0) \times Pres_\Lambda(\delta_1,\delta_0) \xrightarrow{\iota} Pres_\Lambda(\gamma_1+\delta_1,\gamma_0+\delta_0) \xrightarrow{\sigma} K$$

where σ is a semi-invariant of det-weight β on $Pres_\Lambda(\gamma_1+\delta_1,\gamma_0+\delta_0)$ so that $\sigma(\iota(f_1,f_2)) \neq 0$. Then, semi-invariants on $Pres_\Lambda(\gamma_1,\gamma_0)$ and $Pres_\Lambda(\delta_1,\delta_0)$ can be defined by $\sigma(\iota(-,f_2)) : Pres_\Lambda(\gamma_1,\gamma_0) \to K$ and analogously for $Pres_\Lambda(\delta_1,\delta_0)$. It is easy to see that these are regular functions and they are semi-invariants of det-weight β. Indeed, suppose that g_1, g_2, h_1, h_2 are automorphisms of $P(\gamma_1), P(\delta_1), P(\gamma_0), P(\delta_0)$. Then $g = g_1 \coprod g_2$ and $h = h_1 \coprod h_2$ are automorphisms of $P(\gamma_0) \coprod P(\delta_0)$ and $P(\gamma_1) \coprod P(\delta_1)$ respectively so that $\chi_i(g) = \chi_i(g_1)\chi_i(g_2)$ and $\chi_i(h) = \chi_i(h_1)\chi_i(h_2)$. Therefore

$$\sigma(g_1f_1h_1, g_2f_2h_2) = \sigma(gfh) = \sigma(\iota(f_1,f_2)) \prod \chi_i(g_1)^{\beta_i}\chi_i(g_2)^{\beta_i}\chi_i(h_1)^{\beta_i}\chi_i(h_2)^{\beta_i}$$

So, $\sigma(\iota(-,f_2))$ is a semi-invariant on $Pres_\Lambda(\gamma_1,\gamma_0)$ of det-weight β which is nonzero on f_1 and similarly with $\sigma(\iota(f_1,-))$. □

Let $Pres(\Lambda,\alpha) = \bigsqcup Pres_\Lambda(\gamma_1,\gamma_0)$ denote the disjoint union of presentation spaces $Pres_\Lambda(\gamma_1,\gamma_0)$ over all pairs $\gamma_0,\gamma_1 \in \mathbb{N}^n$ so that $L^t\alpha = \gamma_0 - \gamma_1$.

Proposition 2.4.2. *Suppose that $\alpha_1,\cdots,\alpha_n \in \mathbb{Z}^n$ are linearly independent. Suppose that $f_i \in Pres(\Lambda,\alpha_i)$. Then $\coprod f_i \in Pres(\Lambda, \sum \alpha_i)$ does not admit a semi-invariant with nonzero det-weight.*

Proof. If $f = \coprod f_i$ admits a semi-invariant of det-weight β then, by Lemma 2.4.1, so does every f_i. By Proposition 2.2.6, we conclude that $\langle \alpha_i, \beta\rangle = 0$ for all i. But this implies $\beta = 0$ which is not allowed. □

Remark 2.4.3. For any semi-invariant σ, there is a power of σ which has determinantal weight. This follows from the fact that det_K is a power of the reduced norm $\overline{n} : M_k(F_i) \to K$. We refer the reader to Appendix B, Sec 6 for the definition of reduced norm and the proof of the theorem (Theorem 6.0.9) that all characters are powers of the reduced norm. Consequently, in Proposition 2.4.2 above, $\coprod f_i$ does not admit a semi-invariant of any weight since, if it did, then some power of that semi-invariant would be a semi-invariant with nonzero determinantal weight.

Definition 2.4.4. By a semi-invariant on $Pres(\Lambda, \alpha)$ we mean a semi-invariant on one of the presentation spaces $Pres_\Lambda(\gamma_1, \gamma_0)$ in the disjoint union. Such a semi-invariant σ will be called *determinantal* if there is a module M so that, for all $f : P(\gamma_1) \to P(\gamma_0)$ in $Pres_\Lambda(\gamma_1, \gamma_0)$, $\sigma(f)$ is the determinant of the induced map

$$\mathrm{Hom}_\Lambda(f, M) : \mathrm{Hom}_\Lambda(P(\gamma_0), M) \to \mathrm{Hom}_\Lambda(P(\gamma_1), M).$$

We denote this by σ_M. It is easy to see that σ_M is a semi-invariant of det-weight $\underline{\dim}\, M$. In case the det-weight of σ_M is not well-defined, we take it to be $\underline{\dim}\, M$ by definition.

When the ground field K is algebraically closed then Schofield [S91] showed that the determinantal semi-invariants generate the ring of all semi-invariants in the Dynkin case and this theorem was extended in [DW] to all quivers over an algebraically closed field.

Corollary 2.4.5. *Let $\alpha = \sum n_i \beta_i$ be an integer linear combination of the vectors β_i in a cluster tilting set. If $n_i > 0$ for all $1 \leq i \leq n$, then $Pres(\Lambda, \alpha)$ has no semi-invariant with nonzero determinantal weight.*

Proof. If there is a nonzero semi-invariant σ on $\mathrm{Hom}_\Lambda(P, Q)$ with $\dim(Q) - \dim(P) = \alpha$ then σ will be nonzero on the generic element of $\mathrm{Hom}_\Lambda(P, Q)$. By Corollary 2.3.8, the generic element splits as a direct sum of n objects with linearly independent dimension vectors. But this contradicts Proposition 2.4.2. Our Corollary follows. □

Definition 2.4.6. A *virtual semi-invariant* of det-weight β on $Vrep(\Lambda, \alpha)$ is defined to be a mapping

$$\sigma : Vrep(\Lambda, \alpha) \to K$$

with the property that the restriction of σ to each term $Pres_\Lambda(\gamma_1, \gamma_0)$ in the directed system is a semi-invariant of det-weight β.

By definition of direct limit, a virtual semi-invariant on $Vrep(\Lambda, \alpha)$ is the same as a system of semi-invariants one on each $Pres_\Lambda(\gamma_1, \gamma_0)$ which are compatible with stabilization. One example is the determinantal semi-invariant σ_M defined above. Since each coordinate of γ_0 and γ_1 become arbitrarily large, the weight of a virtual semi-invariant is well-defined when K is infinite.

3. Virtual stability theorem

In this section we will prove the virtual stability theorem (3.1.1) which states that the domain $D_{\mathbb{Z}}(\beta)$ of the semi-invariant with det-weight β defined in 3.1.3 is the subset of $\mathbb{Z}^n$ given by the stability conditions of 3.1.1(2). We also give a description of all elements of this set (Theorem 3.5.2).

3.1. Statements of the theorem.

Let β, β' be real Schur roots. We say that β' is a real Schur subroot of β if M_β contains a submodule isomorphic to $M_{\beta'}$.

Theorem 3.1.1 (Virtual Stability Theorem)**.** *Let K be any field. Let Λ be a finite dimensional hereditary K-algebra with n non-isomorphic simple modules. Let $x \in \mathbb{Z}^n$ and β a real Schur root. Then, the following are equivalent:*

(1) *There exists a virtual representation $f : P \to Q$ so that $\underline{\dim} Q - \underline{\dim} P = x$ and f induces an isomorphism*

$$f^* : \mathrm{Hom}_\Lambda(Q, M_\beta) \xrightarrow{\approx} \mathrm{Hom}_\Lambda(P, M_\beta).$$

(2) *Stability conditions for x and β hold: $\langle x, \beta\rangle = 0$ and $\langle x, \beta'\rangle \le 0$ for all real Schur subroots $\beta' \subseteq \beta$.*

(3) *There is a determinantal semi-invariant of det-weight β on the virtual representation space $Vrep(\Lambda, x)$.*

Remark 3.1.2. In the previous paper [IOTW09], the authors proved the Virtual Stability Theorem for hereditary algebras over an algebraically closed field and vectors $x \in \mathbb{Z}^n$, which may have negative coordinates. This was an extension of the results of [DW] and [Ki] from $x \in \mathbb{N}^n$ to $x \in \mathbb{Z}^n$. Here we extend this theorem to hereditary algebras over any field. We also note that Condition (2) is weaker and thus the theorem is stronger than the original theorems of [DW] and [Ki] since the condition $\langle x, \beta'\rangle \le 0$ is only required for real Schur subroots β' of β and not for all subroots of β.

We now restate the theorem in terms of the following sets, usually referred to as various "domains of virtual semi-invariants" or "supports of virtual semi-invariants".

Definition 3.1.3. Let β be a real Schur root. We define the following:

$$D_\mathbb{Z}(\beta) := \{x \in \mathbb{Z}^n : \text{Condition(1) holds}\} = \textit{integral support} \text{ of det-weight } \beta,$$

$$D(\beta) := \text{convex hull of } D_\mathbb{Z}(\beta) \text{ in } \mathbb{R}^n = \text{real support of det-weight } \beta,$$

$$D^{ss}(\beta) := \{x \in \mathbb{R}^n : \langle x, \beta\rangle = 0 \text{ and } \langle x, \beta'\rangle \le 0 \text{ for all real Schur subroots } \beta' \subseteq \beta\}$$
$$= \textit{support} \text{ of real semi-stability conditions},$$

$$D^{ss}_\mathbb{Z}(\beta) := \{x \in \mathbb{Z}^n : \langle x, \beta\rangle = 0 \text{ and } \langle x, \beta'\rangle \le 0 \text{ for all real Schur subroots } \beta' \subseteq \beta\}$$
$$= D^{ss}(\beta) \cap \mathbb{Z}^n = \textit{support} \text{ of integral semi-stability conditions}.$$

Note that $D_\mathbb{Z}(\beta)$ is an additive monoid. It contains 0 and is closed under sum.

The Virtual Stability Theorem 3.1.1 can now be re-stated as:

Theorem 3.1.4 (Virtual Stability Theorem'). *Let K be any field, Λ a finite dimensional hereditary K-algebra with n simple modules, $x \in \mathbb{Z}^n$. Let β be a real Schur root. Then*

$$D^{ss}_\mathbb{Z}(\beta) = D_\mathbb{Z}(\beta).$$

Furthermore, this is equivalent to the statement: There is a determinantal semi-invariant of det-weight β on virtual representation space $Vrep(\Lambda, x)$.

The proof of the theorem occupies the rest of this section. We will first prove the theorem for infinite fields and in subsection 3.6 we will extend the proof to all fields. We start with the simple lemma showing the equivalence of conditions (1) and (3) in the Virtual Stability Theorem 3.1.1, hence reducing the proof to showing that $D^{ss}_\mathbb{Z}(\beta) = D_\mathbb{Z}(\beta)$, i.e. Virtual Stability Theorem' 3.1.4.

Lemma 3.1.5. *Let K be an infinite field, Λ a finite dimensional hereditary K-algebra with n simple modules, $x \in \mathbb{Z}^n$ and β a real Schur root. The following are equivalent:*

(1) *There exists a virtual representation $f : P \to Q$ with $\underline{\dim} Q - \underline{\dim} P = x$ so that f induces an isomorphism $f^* : \mathrm{Hom}_\Lambda(Q, M_\beta) \xrightarrow{\approx} \mathrm{Hom}_\Lambda(P, M_\beta)$.*

(2) *There is a determinantal semi-invariant of det-weight β on virtual representation space $Vrep(\Lambda, x)$.*

Proof. (1) $\Longrightarrow$ (2) It follows from (1) that $dim_K \operatorname{Hom}_\Lambda(Q, M_\beta) = dim_K \operatorname{Hom}_\Lambda(P, M_\beta)$ and therefore determinant of h^* is defined for all $h \in Pres(\Lambda, x)$, is non-zero for $h = f$ and is compatible with stabilization. Hence σ = determinant is a (determinantal) semi-invariant on $Vrep(\Lambda, x)$.

(2) $\Longrightarrow$ (1) Given a determinantal virtual semi-invariant of det-weight β, we have an isomorphism $f^* : \operatorname{Hom}_\Lambda(Q, M) \cong \operatorname{Hom}_\Lambda(P, M)$ for some M with $\underline{\dim}\, M = \beta$. Since being an isomorphism is an open condition, f^* must also be an isomorphism for $M = M_\beta$. $\square$

Lemma 3.1.6. *Let β be a real Schur root. Then $D_{\mathbb{Z}}(\beta) \subseteq D^{ss}_{\mathbb{Z}}(\beta)$.*

Proof. Let $x \in D_{\mathbb{Z}}(\beta)$. Then there is $f : P \to Q$ such that $f^* : \operatorname{Hom}(Q, M_\beta) \to \operatorname{Hom}(P, M_\beta)$ is an isomorphism. Since $\langle x, \beta\rangle = \dim_k \operatorname{Hom}_\Lambda(Q, M_\beta) - \dim_k \operatorname{Hom}_\Lambda(P, M_\beta)$ it follows that $\langle x, \beta\rangle = 0$. Furthermore, the induced map $\operatorname{Hom}(Q, M_{\beta'}) \to \operatorname{Hom}(P, M_{\beta'})$ is a monomorphism for for all real Schur subroots $\beta' \subseteq \beta$. Therefore $\langle x, \beta'\rangle \le 0$, i.e. stability condition (2) holds and $x \in D^{ss}_{\mathbb{Z}}(\beta)$. $\square$

3.2. **Perpendicular categories of M_β and associated exceptional sequences.** To a real Schur root β we associate an exceptional sequence $(M_\beta, E_1, \cdots, E_{n-1})$ which will play a crucial role in the proof of the theorem.

Definition 3.2.1. Let ${}^{\perp_V}M_\beta$ be the left $\operatorname{Hom}_{Vrep(\Lambda)}$-, $\operatorname{Ext}^1_{Vrep(\Lambda)}$-perpendicular category of M_β in $Vrep(\Lambda)$, i.e., ${}^{\perp_V}M_\beta$ is the full subcategory of $Vrep(\Lambda)$ with objects X so that $\operatorname{Hom}_{Vrep(\Lambda)}(X, M_\beta) = 0 = \operatorname{Ext}^1_{Vrep(\Lambda)}(X, M_\beta)$.

The following lemma will relate perpendicular categories in $Vrep(\Lambda)$ and in mod-Λ allowing us to use some well known theorems for module categories.

Lemma 3.2.2. *Let ${}^{\perp_V}M_\beta$ be the perpendicular category in $Vrep(\Lambda)$ and let ${}^{\perp}M_\beta$ be the left $\operatorname{Hom}_\Lambda$-, $\operatorname{Ext}^1_\Lambda$-perpendicular category of M_β in mod-Λ. Then: ${}^{\perp_V}M_\beta \cap mod\text{-}\Lambda = {}^{\perp}M_\beta$.*

It is well-known (see, e.g., [Rin94]) that ${}^{\perp}M_\beta$ is a hereditary abelian category exactly embedded as an extension closed full subcategory of mod-Λ. Such subcategories are called *wide subcategories* [InTh]. We say that a wide subcategory has *rank k* if it isomorphic to the module category of a hereditary algebra with k simple modules. In our case ${}^{\perp}M_\beta$ is a wide subcategory of rank $n-1$ since M_β is indecomposable. Let $E_1, \cdots, E_{n-1}$ be the simple objects of ${}^{\perp}M_\beta$. These objects are exceptional and using Proposition 1.3.2(1) can be ordered in such a way that the following sequence is an exceptional sequence in mod-Λ:

$$(M_\beta, E_1, \cdots, E_{n-1}). \tag{3.1}$$

Using similar observations about the right perpendicular categories, we notice:

$$(E_1 \oplus \cdots \oplus \widehat{E_k} \oplus \cdots \oplus E_{n-1})^{\perp_V} \cap mod\text{-}\Lambda = (E_1 \oplus \cdots \oplus \widehat{E_k} \oplus \cdots \oplus E_{n-1})^{\perp},$$

and use it to define for each $k = 1, \ldots, n-1$ the following subcategories of mod-Λ ,

$$\mathcal{W}_k := (E_1 \oplus \cdots \oplus \widehat{E_k} \oplus \cdots \oplus E_{n-1})^{\perp} \subset mod\text{-}\Lambda. \tag{3.2}$$

These are wide subcategory of mod-Λ of rank 2 which contains M_β by definition of the E_is.

Lemma 3.2.3. *Let β be a real Schur root. Let $(E_1, \cdots, E_{n-1})$ be an exceptional sequence of simple objects of ${}^{\perp}M_\beta$ and let P'_k be the projective cover of E_k in ${}^{\perp}M_\beta \subset mod$-$\Lambda$. Then P'_k is a projective Λ-module if and only if M_β is a simple object in $\mathcal{W}_k$.*

Proof. Several exceptional sequences will be created out of $(E_1, \cdots, E_{n-1})$ and will be used in the proof. Since all E_i are simple objects and P'_k is projective in ${}^{\perp}M_\beta \subset mod$-$\Lambda$ it follows that $\operatorname{Hom}_{({}^{\perp}M_\beta)}(P'_k, E_i) = 0$ for $i \neq k$ and also $Ext^1_{({}^{\perp}M_\beta)}(P'_k, E_i) = 0$. Therefore:

(a) $(E_1, \cdots, \widehat{E_k}, \cdots, E_{n-1}, P'_k)$ is an exceptional sequence in ${}^{\perp}M_\beta$.

Since ${}^{\perp}M_\beta \hookrightarrow mod\text{-}\Lambda$ is exact embedding it follows that $\mathrm{Hom}_\Lambda(P'_k, E_i) = 0 = \mathrm{Ext}^1_\Lambda(P'_k, E_i)$ for all $i \neq k$. This together with the fact that $\{E_1, \cdots, \widehat{E_k}, \cdots, E_{n-1}, P'_k\} \subset^{\perp} M_\beta$ implies:
(b) $(M_\beta, E_1, \cdots, \widehat{E_k}, \cdots, E_{n-1}, P'_k)$ is an exceptional sequence in mod-Λ.
After applying Proposition 1.3.3(6) to the exceptional sequence (a), one obtains:
(c) $(I'_k, E_1, \cdots, \widehat{E_k}, \cdots, E_{n-1})$ is an exceptional sequence in ${}^{\perp}M_\beta$, and
(d) $(M_\beta, I'_k, E_1, \cdots, \widehat{E_k}, \cdots, E_{n-1})$ is an exceptional sequence in mod-Λ.
After applying Proposition 1.3.3(6) to the exceptional sequence (b), one obtains:
(e) $(X, M_\beta, E_1, \cdots, \widehat{E_k}, \cdots, E_{n-1})$ is an exceptional sequence in mod-Λ, where $X = \tau_\Lambda P'_k$ if and only if P'_k is not projective Λ-module, and X is the injective envelope of $P'_k/radP'_k$ if and only if P'_k is projective Λ-module.
(f) (M_β, I'_k) and (X, M_β) are exceptional sequences in $\mathcal{W}_k$. This follows from (d), (e) and definition of $\mathcal{W}_k$.
(g) There is a $\mathcal{W}_k$-irreducible map $M_\beta \to I'_k$ if and only if $M_\beta \oplus I'_k$ is not semi-simple, since rank$(\mathcal{W}_k) = 2$.
(h) There is a $\mathcal{W}_k$-irreducible map $X \to M_\beta$ if and only if $X \oplus M_\beta$ is not semi-simple, since rank$(\mathcal{W}_k) = 2$.

Claim 1: If M_β is not simple in $\mathcal{W}_k$ then P'_k is not a projective Λ-module.
Proof: Since M_β is not simple it follows from (g) and (h) that there is an almost split sequence $X \hookrightarrow M^m_\beta \twoheadrightarrow I'_k$ in $\mathcal{W}_k$. Since $P'_k \in {}^{\perp}M_\beta$ we have $\mathrm{Ext}^1_\Lambda(P'_k, X) \cong \mathrm{Hom}_\Lambda(P'_k, I'_k) \neq 0$. So, P'_k is not projective in mod-Λ.

Claim 2: If M_β is simple in $\mathcal{W}_k$ then P'_k is a projective Λ-module.
Proof: If M_β is simple in $\mathcal{W}_k$, either M_β is simple injective or simple projective.

Case 2a: If M_β is a simple injective object in $\mathcal{W}_k$, then there is no $\mathcal{W}_k$-irreducible map $M_\beta \to I'_k$. So it follows by (g) that I'_k is simple projective object in $\mathcal{W}_k$. Since $\mathcal{W}_k$ has rank=2, there are only two simple objects. Therefore X is not simple and by (h) there is a $\mathcal{W}_k$-irreducible map $X \to M_\beta$. Since M_β is simple injective in $\mathcal{W}_k$ it follows that X is injective envelope of the simple object I'_k in $\mathcal{W}_k$. If X is injective Λ-module, it follows by (e) that P'_k is a projective Λ-module. If X is not injective Λ-module, then $X = \tau_\Lambda P'_k$. But in this case there is a non-zero composition of $\mathcal{W}_k$, and therefore Λmaps $P'_k \to S_k = I'_k \hookrightarrow X = \tau_\Lambda P'_k$. However this would imply $\mathrm{Ext}^1_\Lambda(P'_k, P'_k) \neq 0$ giving a contradiction to the fact that P'_k is rigid. Hence, P'_k is a projective Λ-module.

Case 2b If M_β is simple projective in $\mathcal{W}_k$ then there is no $\mathcal{W}_k$-irreducible map $X \to M_\beta$ and therefore X must be simple $\mathcal{W}_k$ object by (h), hence simple injective. Since the rank of $\mathcal{W}_k$ is 2, it follows that I'_k is not simple, hence there is a $\mathcal{W}_k$ irreducible map $M_\beta \to I'_k$. Therefore I'_k is projective $\mathcal{W}_k$ object, projective cover of X. So, there is a short exact sequence in $\mathcal{W}_k$: $0 \to M^m_\beta \to I'_k \to X \to 0$ where $m \geq 1$. Since $\mathrm{Hom}_\Lambda(P'_k, M_\beta) = 0$ and $\mathrm{Hom}_\Lambda(P'_k, I'_k) \neq 0$ we have a nonzero Λ morphism $P'_k \to X = |\tau P'_k|$. Therefore, P'_k is a projective Λ-module by (e). This proves the proof of the lemma. □

3.3. **Subsets $\Delta^+(\beta) \subseteq \Delta(\beta) \subseteq D_{\mathbb{Z}}(\beta) \subseteq D^{ss}_{\mathbb{Z}}(\beta)$.** In this subsection we define two new subsets, which will be used in the proof of the Virtual Stability Theorem 3.1.1, i.e. we will prove $D_{\mathbb{Z}}(\beta) = D^{ss}_{\mathbb{Z}}(\beta)$.

Definition 3.3.1. Let β be a real Schur root and let $(M_\beta, E_1, \cdots, E_{n-1})$ be the exceptional sequence as defined in (3.1). Let:

$$\Delta^+(\beta) := \left\{ \sum_{1 \leq i \leq n-1} k_i \,\underline{\dim}\, E_i \ : \ k_i \in \mathbb{N} \right\}.$$

Lemma 3.3.2. *The set $\Delta^+(\beta)$ contains all integer points in its convex hull in $\mathbb{R}^n$.*

Proof. Since $(M_\beta, E_1, \cdots, E_{n-1})$ is an exceptional sequence, by Proposition 1.3.3, every vector in $\mathbb{Z}^n$ can be expressed uniquely as an integer linear combination of the vectors $\underline{\dim}\, E_i$ and β. Since all elements in the convex hull of $\Delta^+(\beta)$ can be written as nonnegative real linear combinations of the vectors $\underline{\dim}\, E_i$ it follows that when the integer points in this convex hull are written in this way, the nonnegative coefficients are necessarily integers. So, they are nonnegative integers. □

Let J_β be the set of all integers j so that the j-th coordinate of β is zero. When $j \in J_\beta$, the j-th projective module P_j is in ${}^\perp M_\beta$. Also, the virtual representation $P_j[1] = (P_j \to 0)$ is in ${}^\perp M_\beta$. This implies that the following set of vectors is contained in $D_{\mathbb{Z}}(\beta)$.

$$\Delta(\beta) := \left\{ \sum_{1\le i\le n-1} k_i\, \underline{\dim}\, E_i + \sum_{j\in J_\beta} \ell_j\, \underline{\dim}\, P_j \; : \; k_i \in \mathbb{N}, \ell_j \in \mathbb{Z} \right\}. \tag{3.3}$$

We have the following easy observation.

$$\Delta^+(\beta) \subseteq \Delta(\beta) \subseteq D_{\mathbb{Z}}(\beta) \subseteq D^{ss}_{\mathbb{Z}}(\beta). \tag{3.4}$$

To prove the stability theorem we will show that $D^{ss}_{\mathbb{Z}}(\beta) \subseteq \Delta(\beta)$ and therefore the last three sets in (3.4) are equal. To do this we first consider the case when J_β is empty, i.e., when β is sincere.

3.4. **Sincere case.** When β is sincere we have $\Delta^+(\beta) = \Delta(\beta)$. Thus we are reduced to showing that $D^{ss}_{\mathbb{Z}}(\beta) \subseteq \Delta^+(\beta)$. Note that ${}^\perp M_\beta \subseteq$ *mod*-Λ when β is sincere.

Lemma 3.4.1. *If β is sincere then $D^{ss}_{\mathbb{Z}}(\beta) = \Delta^+(\beta)$ and therefore $D^{ss}_{\mathbb{Z}}(\beta) = D_{\mathbb{Z}}(\beta)$.*

Proof. Since β is sincere, it follows that there are no projective Λ-modules in ${}^\perp M_\beta$ and therefore none of the P_k' are projective Λ-modules. This implies that M_β is not a simple object in $\mathcal{W}_k$ (as defined in equation 3.2) for each $k = 1, \ldots, n-1$ by Lemma 3.2.3. We will use this fact to construct certain Schur subroots $\gamma_k \subseteq \beta$.

Claim 1. For each $k = 1, 2, \cdots, n-1$ there is a real Schur subroot γ_k of β so that

(1) $\langle \underline{\dim}\, E_i, \gamma_k \rangle = 0$ if $i \neq k$
(2) $\langle \underline{\dim}\, E_k, \gamma_k \rangle < 0$.

Construction of γ_k: For each k, the category $\mathcal{W}_k = (E_1 \oplus \cdots \oplus \widehat{E_k} \oplus \cdots \oplus E_{n-1})^\perp \subset$ *mod*-Λ is a wide subcategory of rank 2 which contains M_β by definition of the E_is. Let R_k, S_k be the simple objects of $\mathcal{W}_k$. Since M_β is not simple, there exists a nontrivial extension of S_k by R_k (or R_k by S_k):

$$S_k^p \hookrightarrow M_\beta \twoheadrightarrow R_k^q$$

where $p, q \ge 1$. Let $\gamma_k = \underline{\dim}\, S_k$. Then γ_k is a proper real Schur subroot of β.

Properties of γ_k:
(1) $\langle \underline{\dim}\, E_i, \gamma_k \rangle = \langle \underline{\dim}\, E_i, \underline{\dim}\, S_k \rangle = 0$ for all $i \neq k$ since $S_k \in \mathcal{W}_k = (\oplus_{i\neq k} E_i)^\perp$.
(2) $\langle \underline{\dim}\, E_k, \gamma_k \rangle < 0$. We prove this in two steps:

Step 1: Since $\mathrm{Hom}_\Lambda(E_k, M_\beta) = 0$ we must also have $\mathrm{Hom}_\Lambda(E_k, S_j) = 0$. Therefore $\langle \underline{\dim}\, E_k, \gamma_k \rangle = \langle \underline{\dim}\, E_k, \underline{\dim}\, S_k \rangle \le 0$.

Step 2: $\langle \underline{\dim}\, E_k, \underline{\dim}\, S_k \rangle \neq 0$ since all vectors z satisfying $\langle \underline{\dim}\, E_i, z \rangle = 0$ for all i are scalar multiples of β which is not possible since $\gamma_k \subsetneq \beta$. This finishes the proof of Claim 1.

Claim 2: $D^{ss}_{\mathbb{Z}}(\beta) \subseteq \Delta^+(\beta)$.
Proof of claim 2: Since $(M_\beta, E_1, \cdots, E_{n-1})$ is an exceptional sequence, by Proposition 1.3.3, every vector in $\mathbb{Z}^n$ can be expressed uniquely as an integer linear combination of β and the

vectors $\underline{\dim} E_i$. Let $x \in D_{\mathbb{Z}}^{ss}(\beta)$. Then x is an integer linear combination of the roots $\underline{\dim} E_i$, say $x = \sum_{i=1}^{n-1} a_i \underline{\dim} E_i$. The stability conditions which define $D_{\mathbb{Z}}^{ss}(\beta)$ (Definition 3.1.3) and $\langle \underline{\dim} E_i, \gamma_k \rangle = 0$ imply

$$\langle x, \gamma_k \rangle = a_k \langle \underline{\dim} E_k, \gamma_k \rangle \leq 0.$$

Since $\langle \underline{\dim} E_k, \gamma_k \rangle < 0$ by (2), this implies that $a_k \geq 0$ for each k. Since all a_i are integers it follows that $x \in \Delta^+(\beta)$.

This finishes the proof that $D_{\mathbb{Z}}^{ss}(\beta) = D_{\mathbb{Z}}(\beta)$ for β sincere Schur root. □

3.5. **Non-sincere case.** In this subsection we will prove $D_{\mathbb{Z}}^{ss}(\beta) = D_{\mathbb{Z}}(\beta)$ for all real Schur roots β. In order to deal with non-sincere roots, we need the following lemma.

Lemma 3.5.1. *Let Q be a quiver, β a real Schur root which are not sincere. Suppose the j-th coordinate of β is zero, i.e. $\beta_j = 0$. Let $Q_{(j)}$ be the quiver obtained from Q by deleting vertex j and all edges to and from j. Then:*

1. *$D_{\mathbb{Z}}^{ss}(Q, \beta)$, is the set of all integer vectors of the form $x + m \underline{\dim} P_j$ where x lies in $D_{\mathbb{Z}}^{ss}(Q_{(j)}, \beta)$ and $m \in \mathbb{Z}$,*
2. *$D_{\mathbb{Z}}(Q, \beta)$ is the set of all integer vectors of the form $x + m \underline{\dim} P_j$ where x lies in $D_{\mathbb{Z}}(Q_{(j)}, \beta)$ and $m \in \mathbb{Z}$.*

Proof. (1) Since P_j is one dimensional (over F_j) at vertex j, for any integer vector $x \in \mathbb{Z}^n$, $x - x_j \underline{\dim} P_j$ lies in $\mathbb{Z}^{n-1}$. Since $\langle \underline{\dim} P_j, \beta' \rangle = 0$ for all subroots $\beta' \subseteq \beta$, it follows that $x \in D_{\mathbb{Z}}^{ss}(\beta)$ if and only if $x - x_j \underline{\dim} P_j$ lies in $D_{\mathbb{Z}}^{ss}(Q_{(j)}, \beta)$.

(2) Since P_j and $P_j[1]$ are in the perpendicular category ${}^{\perp}M_\beta$, the same is true for $D_{\mathbb{Z}}(Q, \beta)$: $x \in D_{\mathbb{Z}}(Q, \beta)$ iff there is a virtual representation $f : P_1 \to P_0$ of dimension x which lies in ${}^{\perp}M_\beta$. Then $f \oplus |x_j| P_j$ and $f \oplus |x_j| P_j[1]$ are virtual representations in ${}^{\perp}M_\beta$ and one of them has dimension $x - x_j \underline{\dim} P_j$ which lies in $D_{\mathbb{Z}}(Q_{(j)}, \beta)$. Conversely, $D_{\mathbb{Z}}(Q_{(j)}, \beta) + \mathbb{Z} \underline{\dim} P_j$ is contained in $D_{\mathbb{Z}}(\beta)$. So, they are equal. □

Proof. (Virtual Stability Theorem 3.1.4) when the field K is infinite. The proof is by induction on the number of vertices of the quiver Q. Let β be a real Schur root. If β is sincere then $D_{\mathbb{Z}}^{ss}(\beta) = D_{\mathbb{Z}}(\beta)$ by Lemma 3.4.1.

If β is not sincere then by Lemma 3.5.1(1): $D_{\mathbb{Z}}^{ss}(Q, \beta)$, is the set of all integer vectors of the form $x + m \underline{\dim} P_j$ where x lies in $D_{\mathbb{Z}}^{ss}(Q_{(j)}, \beta)$ and $m \in \mathbb{Z}$. Since $Q_{(j)}$ has $n-1$ vertices, by induction $D_{\mathbb{Z}}^{ss}(Q_{(j)}, \beta) = D_{\mathbb{Z}}(Q_{(j)}, \beta)$. Then by Lemma 3.5.1(2) it follows that $D_{\mathbb{Z}}^{ss}(\beta) = D_{\mathbb{Z}}(\beta)$. This finishes the proof of Theorem 3.1.4. □

The extended version of the stability theorem includes also equality $D_{\mathbb{Z}}^{ss}(\beta) = \Delta(\beta)$ which was proved for β sincere in Lemma 3.4.1, which we now extend to the general real Schur root β.

Proposition 3.5.2. *Let β be a real Schur root. Then*

$$D_{\mathbb{Z}}^{ss}(\beta) = D_{\mathbb{Z}}(\beta) = \Delta(\beta).$$

where the terms are defined in Definition 3.1.3 and Equation (3.3).

Proof. When β is sincere, this is Lemma 3.4.1 proved above. So, suppose β is not sincere. Let $j \in J_\beta$ and let $\Lambda_{(j)} = \Lambda / \Lambda e_j \Lambda$ with quiver $Q_{(j)}$ be as in Lemma 3.5.1. Then, by induction on n we have

$$D_{\mathbb{Z}}^{ss}(Q_{(j)}, \beta) = \Delta(Q_{(j)}, \beta) = \left\{ \sum_{i \notin J_\beta} a_i \underline{\dim} E_i' + \sum_{k \in J_\beta, k \neq j} b_k \underline{\dim} P_k \;:\; a_i \in \mathbb{N}, b_k \in \mathbb{Z} \right\},$$

where E_i' are the simple objects of ${}^\perp M_\beta$ in mod-$\Lambda_{(j)}$. By Lemma 3.5.1 we conclude that

$$D_{\mathbb{Z}}^{ss}(\beta) = \Delta(Q_{(j)},\beta) + \mathbb{Z}\,\underline{\dim}\,P_j = \left\{ \sum_{i\notin J_\beta} a_i\,\underline{\dim}\,E_i' + \sum_{k\in J_\beta} b_k\,\underline{\dim}\,P_k \,:\, a_i\in\mathbb{N}, b_k\in\mathbb{Z} \right\}.$$

Since each E_i' and each P_k is a module in ${}^\perp M_\beta$, their dimension vectors are nonnegative integer linear combinations of the dimension vectors of the simple objects E_i of ${}^\perp M_\beta$. Therefore,

$$D_{\mathbb{Z}}^{ss}(\beta) \subseteq \left\{ \sum_{1\le i\le n-1} a_i\,\underline{\dim}\,E_i + \sum_{k\in J_\beta} b_k\,\underline{\dim}\,P_k \,:\, a_i\in\mathbb{N}, b_k\in\mathbb{Z} \right\} = \Delta(\beta).$$

We have already observed that the opposite inclusion $\Delta(\beta)\subseteq D_{\mathbb{Z}}(\beta)\subset D_{\mathbb{Z}}^{ss}(\beta)$ holds. This finishes the proof of the proposition. □

3.6. **Extension to arbitrary fields K.** Suppose that the ground field K is finite. Then, we still have the trivial implication $(1)_\Lambda \Rightarrow (3)_\Lambda$. Since $-\otimes_K K(t)$ is an exact functor, $(3)_\Lambda \Rightarrow (3)_{\Lambda(t)}$ which we have shown to be equivalent to $(1)_{\Lambda(t)}$ and (2) which does not refer to K. It remains to show that these imply $(1)_\Lambda$.

Recall that, for every real Schur root β, the simple objects E_i and projective objects P_j of ${}^\perp M_\beta \cap mod$-$\Lambda(t)$ are exceptional $\Lambda(t)$-modules. By Theorem 1.4.1, these are isomorphic to $E_i'(t), P_j'(t), M_\beta'(t)$ for unique exceptional Λ-modules E_i', P_j', M_β'. Then, for any $x\in D_{\mathbb{Z}}(\beta) = \Delta(\beta)$, we have

$$x = \sum k_i\,\underline{\dim}\,E_i' + \sum \ell_j\,\underline{\dim}\,P_j'$$

where $k_i\in\mathbb{N}$ and $\ell_j\in\mathbb{Z}$. So, there is virtual representation $X = \coprod k_i E_i' \coprod\coprod \ell_j P_j'$ so that, for any presentation $f: P\to Q$ of X, $\mathrm{Hom}_\Lambda(f, M_\beta') : \mathrm{Hom}_\Lambda(Q, M_\beta') \cong \mathrm{Hom}_\Lambda(P, M_\beta')$. This shows that $(2)\Rightarrow(1)_\Lambda$. So, Theorem 3.1.1 holds for finite K.

This completes the proof of Theorem 3.1.1 for all finite dimensional hereditary algebras over any field.

Remark 3.6.1. When K is any perfect field, $\Lambda\otimes_K \overline{K}$ is a hereditary algebra over the algebraically closed field $\overline{K}$ and it should be possible to extend the virtual stability theorem from [IOTW09] to Λ. However, if K is not perfect and $F_i = K(a^{1/p})$ is the division algebra at vertex i then the socle of $S_i\otimes_K \overline{K}$ has infinite projective dimension. So, $\Lambda\otimes_K\overline{K}$ is not hereditary in that case. That is the reason we did not take this approach.

4. c-vectors and semi-invariants

In this section we use the Virtual Stability Theorem for semi-invariants (3.1.1) to prove two fundamental theorems relating determinantal weights of semi-invariants, cluster tilting objects and c-vectors corresponding to a cluster tilting object. These theorems (4.1.4 and 4.1.6) are based on work of Speyer and Thomas [ST]. Variations of these theorems are explained in [IOs], [ITW].

Remark 4.0.2. Since c-vectors come from cluster theory, we use the well-known language of cluster categories [BMRRT]. We recall that there is a bijection between isomorphism classes of objects/exceptional object of the cluster category $\mathcal{C}_\Lambda$ and isomorphism classes of presentations/rigid indecomposable presentations given as follows. The presentation of any Λ-module goes to that module, and the shifted projective $P[1]$ goes to the same shifted projective. This bijection has the property that $\mathrm{Ext}^1_{\mathcal{C}_\Lambda}(X',Y') = 0 = \mathrm{Ext}^1_{\mathcal{C}_\Lambda}(Y',X')$ if and only if $\mathrm{Ext}^1_{Vrep(\Lambda)}(X,Y) = 0 = \mathrm{Ext}^1_{Vrep(\Lambda)}(Y,X)$ where X',Y' are objects in $\mathcal{C}_\Lambda$ corresponding to $X,Y\in Vrep(\Lambda)$. This means we can use properties of cluster tilting objects in a cluster category to work with partial cluster tilting objects in the virtual representation category.

Theorem 4.1.4 gives the relation between semi-invariants and cluster tilting objects. Theorem 4.1.6, relates the semi-invariants of a cluster tilting object to the corresponding c-vectors. The definition of a c-vector is given in section 4.2 below.

4.1. **Structure of semi-invariant domains.** Recall that, if $v_1, \cdots, v_k$ are vectors in $\mathbb{R}^n$, the *complete simplicial fan* spanned by the v_i is the set of all nonnegative linear combinations of the v_i. We also call it a *spherical simplex* since it is determined by its intersection with a unit sphere centered at the origin. A union of complete simplicial fans which meet along common faces is a simplicial fan. When we forget the simplicial subdivision of a simplicial fan, we call it a *polyhedral fan.*

The domains $D(\beta)$ of determinantal semi-invariants with various det-weights β are codimension-one "polyhedral fans" in $\mathbb{R}^n$ by the Virtual Stability Theorem 3.1.1. The relationship with c-vectors arises from the way in which these codimension-one sets meet along codimension-two simplicial fans. Thus, we need to look at partial cluster tilting objects

$$T_0 = T_1 \oplus T_2 \oplus \cdots \oplus T_{n-2}$$

in the cluster category of *mod*-Λ [BMRRT]. The vectors $\underline{\dim}\, T_1, \cdots, \underline{\dim}\, T_{n-2}$ are linearly independent and generate a simplicial fan which has codimension 2 in $\mathbb{R}^n$. The vector $\underline{\dim}\, T_0 = \sum \underline{\dim}\, T_i$ lies in the interior of this simplicial fan.

Given any exceptional object T' so that $T = T_0 \oplus T'$ is a partial cluster tilting object, the right perpendicular category $T^\perp$ contains a unique indecomposable object up to isomorphism and this object is M_β for a uniquely determined real Schur root β. Indeed, $\beta \in \mathbb{N}^n$ is the minimum integer vector satisfying the $n-1$ linear equations $\langle \underline{\dim}\, T_i, \beta \rangle = 0$ and $\langle \underline{\dim}\, T', \beta \rangle = 0$.

We need to determine when the simplex spanned by $\underline{\dim}\, T_i$ lies in the interior of $D(\beta)$. This is equivalent to the question of whether the single vector $\underline{\dim}\, T_0 = \sum \underline{\dim}\, T_i$, which lies in the interior of the simplex, lies in the interior of $D(\beta)$. By the Virtual Stability Theorem 3.1.1, $v \in D(\beta)$ lies in the interior of $D(\beta)$ if and only if $\langle v, \beta' \rangle < 0$ for all proper real Schur subroots $\beta' \subsetneq \beta$.

Lemma 4.1.1. *Let $T_0 = T_1 \oplus \cdots \oplus T_k$ be any partial cluster tilting object. Let M_β be an exceptional module in $T_0^\perp$ (equivalently, $\underline{\dim}\, T_0 \in D(\beta)$). Then $\underline{\dim}\, T_0$ lies in the interior of $D(\beta)$ if and only if M_β is a simple object of $T_0^\perp$.*

Remark 4.1.2. Since $T_0^\perp$ has $n-k$ simple objects, say $M_{\alpha_1}, \cdots, M_{\alpha_{n-k}}$, $\underline{\dim}\, T_0$ lies in the interior of exactly $n-k$ semi-invariant domains $D(\alpha_1), \cdots, D(\alpha_{n-k})$.

Proof. Suppose M_β is a simple object of $T_0^\perp$. To show that $\underline{\dim}\, T_0$ lies in the interior of $D(\beta)$, suppose not. Then $\underline{\dim}\, T_0 \in \partial D(\beta)$ then $\langle \underline{\dim}\, T_0, \beta' \rangle = 0$ for some proper real Schur subroot $\beta' \subsetneq \beta$. Then, it follows from the Virtual Stability Theorem 3.1.1 that T_0 lies in $D(\beta')$ and therefore $M_{\beta'}$ is an object of $T_0^\perp$. But $M_{\beta'}$ is a subobject of M_β contradicting the assumption that M_β is simple in $T_0^\perp$. So, $\underline{\dim}\, T_0$ is in the interior of $D(\beta)$.

Suppose M_β is not simple in $T_0^\perp$. Then M_β contains a simple subobject $M_\alpha \in T_0^\perp$. Any such α is a real Schur root. So, $\langle \underline{\dim}\, T_0, \alpha \rangle = 0$ which implies that $\underline{\dim}\, T_0 \in D(\beta) \cap D(\alpha) \subset \partial D(\beta)$. So, $\underline{\dim}\, T_0$ lies on the boundary of $D(\beta)$ when M_β is not simple in $T_0^\perp$. □

Proposition 4.1.3. *Let $T_0 = T_1 \oplus \cdots \oplus T_{n-2}$ be a partial cluster tilting object with $n-2$ summands. Let M_β be an exceptional object of $T_0^\perp$. Then*

(1) *If M_β is one of the two simple objects in $T_0^\perp$ there are two nonisomorphic objects T', T'' in the cluster category of mod-Λ so that $T_0 \oplus T'$ and $T_0 \oplus T''$ are partial cluster tilting objects and so that T', T'' lie in ${}^{\perp_V} M_\beta$.*

(2) *If M_β is a nonsimple exceptional object of $T_0^\perp$ there is, up to isomorphism, only one object $T(\beta)$ so that $T_0 \oplus T(\beta)$ is a partial cluster tilting object and $\underline{\dim}\, T(\beta) \in D(\beta)$.*

In Figure 3 below we denote T', T'' in (1) by $T(\alpha_i)', T(\alpha_i)''$ for β equal to the two simple roots α_1, α_2 of $T_0^\perp$.

Proof. In Case (2), by the lemma, the partial cluster tilting object T_0 lies on the boundary of the polyhedral region $D(\beta)$. Therefore, there is at most one way to complete it to a cluster tilting object in $D(\beta)$. Thus, it suffices to show the existence of a nonzero object $T(\beta) \in {}^\perp M_\beta$ so that $T_0 \oplus T(\beta)$ is a partial cluster tilting object and, in Case (1), we need to show that there are two objects T', T'' where either $T', T'' \in {}^\perp M_\beta$ or $T' \in {}^\perp M_\beta$ and $T'' = P[1]$ where $P \in {}^\perp M_\beta$ is a projective Λ-module.

The existence of $T(\beta)$ is straightforward using basic properties of cluster tilting objects [BMRRT]. Since T_0 is an almost complete cluster tilting object in the cluster category of ${}^\perp M_\beta$, there are two objects T', T'' in this cluster category which complete the cluster tilting object. At least one of them, say T', is a module in ${}^\perp M_\beta$. Letting $T(\beta) = T'$, this proves Case (2).

Case (1) M_β is simple in $T_0^\perp$. If both T', T'' are modules we are done. So, suppose that $T'' = P[1]$ for some projective object $P \in {}^\perp M_\beta$. Then we claim that P is projective in *mod*-Λ making $T'' = P[1]$ an object of the cluster category of *mod*-Λ [BMRRT].

Suppose that P is the projective cover of the simple object $E_k \in {}^\perp M_\beta$. Then the dimension vectors of the objects $T_1, \cdots, T_{n-2}$ lie on the face of the positive simplex $\Delta^+(\beta)$ opposite the vertex E_k. By Lemma 4.1.1, $D(\beta)$ contains a small neighborhood of the point $\underline{\dim}\, T_0$. After rescaling, any such neighborhood contains an integer point having negative E_k-coordinate. By the Virtual Stability Theorem 3.1.1 such a point has the form $\sum k_i\, \underline{\dim}\, E_i + \sum \ell_j\, \underline{\dim}\, P_j$ where E_i are the simple objects of ${}^\perp M_\beta$ and P_j are the projective objects of ${}^\perp M_\beta$ which are also projective in *mod*-Λ. By construction, at least one of these P_j must have E_k in its composition series. But then the projective cover P of E_k in ${}^\perp M_\beta$ is a submodule of P_j which is projective in both ${}^\perp M_\beta$ and *mod*-Λ. So, $T'' = P[1]$ lies in the cluster category of *mod*-Λ as claimed. □

Theorem 4.1.4. *Let $T = T_1 \oplus \cdots \oplus T_n$ be a cluster tilting object for Λ. Then:*

(a) *The dimension vectors $\underline{\dim}\, T_i$ span a complete simplicial fan in $\mathbb{R}^n$ whose walls are $D(\beta_i)$ for uniquely determined real Schur roots β_i.*

(b) $\mathrm{End}_\Lambda(M_{\beta_i}) \cong \mathrm{End}(T_i)$ *for each i.*

(c) *The interior of this spherical simplex does not meet any $D(\beta)$.*

(d) *Furthermore,*

$$\langle \underline{\dim}\, T_i, \beta_j \rangle = \delta_{ij} \varepsilon_j f_j \tag{4.1}$$

where $f_j = \dim \mathrm{End}_\Lambda(M_{\beta_i}) = \dim \mathrm{End}(T_i)$ *and* $\varepsilon_j = \pm 1$.

(e) *The objects T_i can be numbered in such a way that* $\mathrm{End}(T_i) = \mathrm{End}_\Lambda(S_i) = F_i$.

Proof. (a) Each face of the spherical simplex is spanned by $\underline{\dim}\, T_i$ with one T_j deleted. Then $(T/T_j)^\perp$ has a unique simple object, say, M_{β_j} and $\underline{\dim}\, T_i, i \neq j$ lie in $D(\beta_j)$.

(b) By Schofield (2.1.2), the T_i can be renumbered to form an exceptional sequence $(T_1, \cdots, T_n)$. For each j we have another exceptional sequence $(M_{\beta_j}, T_1, \cdots, \widehat{T_j}, \cdots, T_n)$. By Proposition 1.3.3 (5), this implies (b).

(c) The interior of the spherical simplex σ cannot lie in any $D(\alpha)$. If it did, then $D_{\mathbb{Z}}(\alpha)$ would contain a integer point, say v, in the interior of σ. But then the general virtual representation $P \to Q$ with dimension vector v would lie in $D_{\mathbb{Z}}(\alpha)$. However, by the virtual canonical decomposition theorem this representation is a direct sum of the representations T_i and each T_i occurs. So, $\underline{\dim}\, T_i \in D_{\mathbb{Z}}(\alpha)$ for all i. This would make $D_{\mathbb{Z}}(\alpha)$ n dimensional contradicting the fact that it has codimension one.

(4.1) follows from (a): Since $T_i \in D(\beta_j)$ for $i \neq j$, we have $\langle \underline{\dim} T_i, \beta_j \rangle = 0$ for $i \neq j$. And $\langle \underline{\dim} T_j, \beta_j \rangle \neq 0$ since otherwise $\underline{\dim} T_i$ would be n linearly independent vectors in the same hyperplane $H(\beta_j)$. And $\langle \underline{\dim} T_j, \beta_j \rangle \neq 0$ is a multiple of f_j by Proposition 1.2.3. It remains to show that this multiple is ± 1.

This follows from the properties of exceptional sequences. Namely, by Proposition 1.3.3 (3) and a little linear algebra we can conclude that $\underline{\dim} T_k = \pm \underline{\dim} M_{\beta_k}$ plus a linear combination of $\underline{\dim} T_j$ for $j \neq k$. Since $\langle \underline{\dim} T_j, \beta_k \rangle = 0$ for all $j \neq k$, this implies that

$$\langle \underline{\dim} T_k, \beta_k \rangle = \pm \langle \beta_k, \beta_k \rangle = \pm f_j$$

We denote the sign by ε_j. This completes the proof of (d).

Finally (e) follows from Corollary 1.3.5. □

By the Virtual Stability Theorem 3.1.1 we have the following immediate corollary.

Corollary 4.1.5. *Let Γ_T be the $n \times n$ matrix with columns $\gamma_i = \varepsilon_i \beta_i$ where ε_i are the signs given in Theorem 4.1.4 (d) above. Let V be the $n \times n$ matrix with columns $\underline{\dim} T_i$. Then*

$$V^t E \Gamma_T = D$$

where E is the Euler matrix and D is the diagonal matrix with diagonal entries f_i.

We can now state the main theorem of this section. The definition of c-vectors is given in section 4.2 below.

Theorem 4.1.6 (c-vector theorem)**.** *Given that the initial cluster tilting object in the cluster category $\mathcal{C}_\Lambda$ is the cluster tilting object of shifted projective modules $\Lambda[1] = \coprod P_i[1]$, the c-vectors associated to the cluster tilting object $T = \coprod T_i$ are $-\gamma_i$ as given in Corollary 4.1.5 above. In other words,*

$$c_i = -\varepsilon_i \beta_i \quad \textit{for} \quad i = 1, \cdots, n.$$

4.2. **Definition of c-vectors.** An $n \times n$ integer matrix B is called *skew symmetrizable* if there is a diagonal matrix D with positive integer diagonal entries so that DB is skew-symmetric. D is called the *symmetrizer* of B. An *extended exchange matrix* is defined to be a $2n \times n$ matrix $\tilde{B} = \begin{bmatrix} B \\ C \end{bmatrix}$ whose top half B is skew-symmetrizable.

Definition 4.2.1. [FZ07] For any extended exchange matrix $\tilde{B} = (b_{ij})$ and any $1 \leq k \leq n$, the *mutation $\mu_k \tilde{B}$ of $\tilde{B}$ in the k-direction* is defined to be the matrix $\tilde{B}' = (b'_{ij})$ defined by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + b_{ik}|b_{kj}| & \text{if } b_{ik}b_{kj} > 0 \\ b_{ij} & \text{otherwise} \end{cases} \tag{4.2}$$

For any finite sequence of positive integers $k_1, k_2, \cdots, k_r \leq n$ we have the *iterated mutation* $\mu_{k_r} \cdots \mu_{k_1} \tilde{B}$ of $\tilde{B}$.

Definition 4.2.2. [FZ07] Let B_0 be a fixed skew-symmetrizable matrix which we call the *initial exchange matrix.* Then $\tilde{B}_0 = \begin{bmatrix} B_0 \\ I_n \end{bmatrix}$ is called the *initial extended exchange matrix.* Consider the set of all $n \times n$ matrices C which appear at the bottom of matrices $\tilde{B} = \begin{bmatrix} B \\ C \end{bmatrix}$ given by iterated mutation of the initial extended exchange matrix. The columns of all such matrices C are called the *c-vectors* of B_0. The matices C will be called the *c-matrices* of B_0.

We recall that a vector v is called *sign coherent* if its nonzero coordinates have the same sign. We write $v > 0$ if this sign is positive and $v \neq 0$. We will use a theorem of Nakanishi and Zelevinsky which can be phrased as follows.

Theorem 4.2.3. [NZ] *Let B_0 be a skew-symmetrizable matrix with symmetrizer D and let $\mathcal{X}$ be a set of $n \times n$ integer matrices C with the following properties.*

(1) $I_n \in \mathcal{X}$
(2) *For any $C \in \mathcal{X}$, the columns of C are sign coherent and nonzero.*
(3) *For any $C \in \mathcal{X}$, the matrix*

$$B_C := D^{-1}C^t D B_0 C$$

has integer entries b_{ij}.

(4) *Let $C \in \mathcal{X}$ and $1 \le k \le n$, then $\mathcal{X}$ contains the matrix $C' = \mu_k C$ with columns c'_j given as follows.*

$$c'_j = \begin{cases} -c_k & \text{if } j = k \\ c_j + |b_{kj}|c_k & \text{if } b_{kj}c_k > 0 \\ c_j & \text{otherwise} \end{cases} \tag{4.3}$$

(5) *$\mathcal{X}$ is minimal with the above properties.*

Then $\mathcal{X}$ is the set of c-matrices of B_0 and the columns of $C \in \mathcal{X}$ are the c-vectors of B_0.

To specify the c-vectors of a cluster tilting object, we need to choose an initial cluster tilting object. Let $B_0 = L^t - R$ (see section 1.1.). Then B_0 is an $n \times n$ skew-symmetrizable matrix since $DB_0 = E^t - E$ is skew-symmetric. We will use B_0 as the initial exchange matrix and $\widetilde{B}_0 = \begin{bmatrix} B_0 \\ I_n \end{bmatrix}$ as the initial extended exchange matrix. The following easy observation will be useful.

Lemma 4.2.4. *For any two vectors $x, y \in \mathbb{R}^n$ we have*

$$\langle y, x\rangle - \langle x, y\rangle = x^t D B_0 y.$$

4.3. **c-vector theorem.** The plan for the proof of the c-vector theorem (Theorem 4.1.6) goes as follows. We let $\mathcal{X}$ denote the set of all matrices of the form $C = -\Gamma_T$ where Γ_T is given in Corollary 4.1.5. We will use Theorem 4.2.3 to show that $\mathcal{X}$ is the set of all c-matrices of the chosen initial exchange matrix $B_0 = L^t - R$.

Lemma 4.3.1. *Let $T = T_1 \oplus \cdots \oplus T_n$ be a cluster tilting object and Γ_T the associated matrix with j-th column γ_j given in Corollary 4.1.5. Then the matrix $B_\Gamma = B_{-\Gamma} = D^{-1}\Gamma^t D B_0 \Gamma$ has integer entries verifying Condition (3) in Theorem 4.2.3.*

Proof. The columns γ_i of Γ_T are, up to sign, dimension vectors of indecomposable modules M_{β_i} which satisfy the following (with suitable ordering of the objects T_i)

$$\mathrm{End}_\Lambda(M_{\beta_i}) \cong \mathrm{End}(T_i) \cong F_i = \mathrm{End}_\Lambda(S_i)\,. \tag{4.4}$$

By Lemma 4.2.4, the entries of B_Γ are

$$b_{ij} = f_i^{-1}(\langle\gamma_j, \gamma_i\rangle - \langle\gamma_i, \gamma_j\rangle) \tag{4.5}$$

which are integers by (4.4) and Proposition 1.2.3. □

Using Theorem 4.2.3, it remains to prove the following.

Proposition 4.3.2. *Under the mutation μ_k of T, the matrix Γ_T changes as follows.*

(1) $\gamma'_k = -\gamma_k$
(2) *For $j \neq k$ the vector γ_j changes to*

$$\gamma'_j = \begin{cases} \gamma_j + |b_{kj}|\gamma_k & \text{if } b_{kj}\gamma_k < 0 \\ \gamma_j & \text{if not} \end{cases}$$

where b_{ij} are the entries of B_Γ.

The inequality $b_{kj}\gamma_k < 0$ is reversed from (4.3) since γ_j, γ_k are (claimed to be) negative c-vectors.

We will prove Proposition 4.3.2 first in the special case when $n = 2$. The general case will follow easily from the special case.

4.4. **Rank 2 case.** The results of this section are well-known. We include them for clarity. Let $\mathcal{H}$ be a wide subcategory of *mod*-Λ of rank 2. Then, we have a complete list of all cluster tilting objects in the category $\mathcal{H}$. Inspection of this list will give a proof of Proposition 4.3.2 for $\mathcal{H}$.

By the assumption of rank 2, $\mathcal{H}$ has two simple objects, call them X_1, X_2, with dimension vectors α_1, α_2. Keeping in mind the general case, we will refrain from using the standard notation $\alpha_1 = (1,0), \alpha_2 = (0,1)$. So, the dimension vectors of all objects of $\mathcal{H}$ have the form $a_1\alpha_1 + a_2\alpha_2$ where $a_1, a_2 \in \mathbb{N}$. The cluster category of $\mathcal{H}$, denoted $\mathcal{C}_\mathcal{H}$, has two additional exceptional objects $P_1^\mathcal{H}[1], P_2^\mathcal{H}[1]$ whose dimension vectors are the negatives of the dimension vectors of $P_1^\mathcal{H}, P_2^\mathcal{H}$ which are the projective covers of X_1, X_2 in $\mathcal{H}$. (See [BMRRT].)

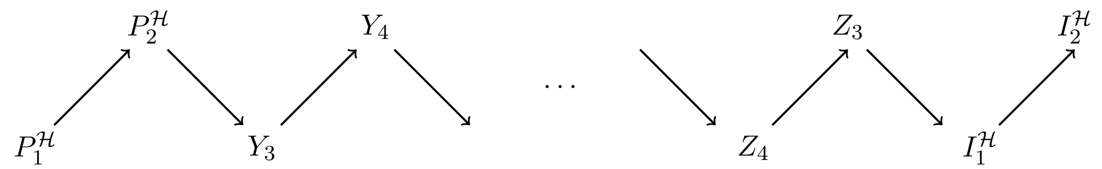
By renumbering X_1, X_2 if necessary, we may assume that $\mathrm{Ext}^1_\Lambda(X_1, X_2) = 0$ (so that $X_1 = P_1^\mathcal{H}$ is projective and $X_2 = I_2^\mathcal{H}$ is injective). Let $F_i = \mathrm{End}_\Lambda(X_i)$ and $M = \mathrm{Ext}^1_\Lambda(X_2, X_1)$. Then M is an $F_1 - F_2$ bimodule with $\dim_K M = m = f_1d_1 = f_2d_2$ where $d_i = \dim_{F_i} M$ and $\mathcal{H}$ is equivalent to the category of representations of the modulated quiver

$$F_1 \overset{M}{\longleftarrow} F_2.$$

All exceptional objects in $\mathcal{H}$ have dimension vectors of the form $a_1\alpha_1 + a_2\alpha_2$ and this corresponds to the modulated quiver representation $(F_1^{a_1}, F_2^{a_2}, F_1^{a_1} \leftarrow M \otimes F_2^{a_2})$.

It is well-known that $\mathcal{H}$ has *finite type*, i.e., has only finitely many exceptional objects up to isomorphism, if and only if $d_1d_2 \le 3$. Equivalently, $\min(d_1, d_2) = 1$ and $\max(d_1, d_2) = 1, 2$ or 3. These are the quivers A_2, B_2, G_2.

Auslander-Reiten quiver (preprojective and preinjective terms) of the category $\mathcal{H}$:



The arrows are the irreducible maps $Y_i \to Y_{i+1}$ where $Y_1 = X_1 = P_1^\mathcal{H}, Y_2 = P_2^\mathcal{H}$ are the projective objects of $\mathcal{H}$ and $Z_j \to Z_{j-1}$ where $Z_1 = I_1^\mathcal{H}, Z_2 = X_2 = I_2^\mathcal{H}$ are the injective objects of $\mathcal{H}$. The Auslander-Reiten translation $\tau^\mathcal{H}$ acts by "shifting two spaces to the left" and we have $\mathcal{H}$-almost split sequences (subscripts of d should be taken modulo 2).

$$Y_i \rightarrowtail Y_{i+1}^{d_{i+1}} \twoheadrightarrow Y_{i+2}, \quad Z_{j+2} \rightarrowtail Z_{j+1}^{d_j} \twoheadrightarrow Z_j \quad (i \ge 1).$$

We also have short exact sequences:

$$Y_1^{d_1} \rightarrowtail Y_2 \twoheadrightarrow X_2, \quad X_1 \rightarrowtail Z_2 \twoheadrightarrow Z_1^{d_2}$$

These imply that the dimension vectors of Y_i are given recursively by $\underline{\dim}\, Y_1 = \alpha_1, \underline{\dim}\, Y_2 = d_1\alpha_1 + \alpha_2$ and, for $i \ge 2$ (for $2 \le i \le s$ if $\mathcal{H}$ has finite type and $Y_s = Z_1$),

$$\underline{\dim}\, Y_i = d_{i-1}\, \underline{\dim}\, Y_{i-1} - \underline{\dim}\, Y_{i-2}\,. \tag{4.6}$$

The vectors $\underline{\dim}\, Z_j$ are given by $\underline{\dim}\, Z_1 = \alpha_2, \underline{\dim}\, Z_2 = \alpha_1 + d_2\alpha_2$ and, for $j \ge 2$ (for $2 \le j \le s$ if $\mathcal{H}$ has finite type),

$$\underline{\dim}\, Z_j = d_j\, \underline{\dim}\, Z_{j-1} - \underline{\dim}\, Z_{j-2}\,. \tag{4.7}$$

Another important observation is the following.

Lemma 4.4.1. *Let X, Y be distinct objects in the Auslander-Reiten quiver of $\mathcal{H}$ and suppose that X is to the left of Y. Then $\operatorname{Hom}_{\mathcal{H}}(Y,X) = 0$ and $\operatorname{Hom}_{\mathcal{H}}(X,Y)$ is nonzero except in the case $X = X_1 = Y_1$, $Y = X_2 = Z_1$ in which case $\operatorname{Hom}_{\mathcal{H}}(Y_1, Z_1) = 0$.*

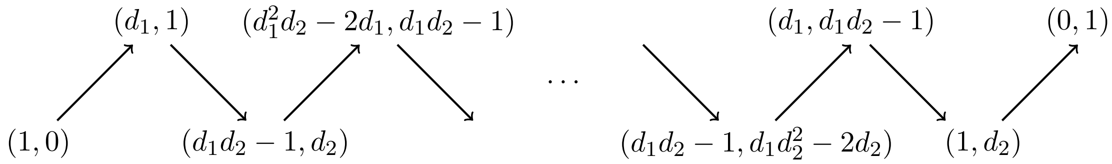
Proof. The fact that morphisms only go to the right is well-known. For the second statement, note that $\operatorname{Hom}_{\mathcal{H}}(P_2^{\mathcal{H}}, Y) \neq 0$ except in the case Y is the simple object $X_1 = Y_1$. By properties of almost split sequences, every morphism $P_2^{\mathcal{H}} \to Y$ factors through every object which lies between $P_2^{\mathcal{H}}$ and Y. Therefore, $\operatorname{Hom}_{\mathcal{H}}(X,Y) \neq 0$ for all $X \neq X_1$ and all Y to the right of X. Similarly $\operatorname{Hom}_{\mathcal{H}}(X,Y) \neq 0$ for all $Y \neq X_2 = I_2^{\mathcal{H}}$ and all X to the left of Y. □

This lemma implies the well-known properties of the rank 2 case.

Proposition 4.4.2. (1) *The tilting objects of $\mathcal{H}$ are $Y_i \oplus Y_{i+1}$ and $Z_i \oplus Z_{i+1}$ for $i \geq 1$.*
(2) *The complete exceptional sequences for $\mathcal{H}$ are $(Y_i, Y_{i+1}), (Z_{i+1}, Z_i)$ for $i \geq 1$ and (Z_1, Y_1).*

Proof. Suppose that X, Y are exceptional objects of $\mathcal{H}$ and X is to the left of Y in the AR-quiver of $\mathcal{H}$. If X, Y are not consecutive then $\tau^{\mathcal{H}}Y$ lies to the right of X. So, $\operatorname{Ext}^1_{\mathcal{H}}(Y,X) = D\operatorname{Hom}_{\mathcal{H}}(X, \tau^{\mathcal{H}}Y) \neq 0$ by the lemma. So, components of a tilting object must be consecutive. For exceptional sequences we have the sequence (Z_1, Y_1) and any other sequence (X,Y) must have X to the left of Y by the lemma and $\operatorname{Hom}_{\mathcal{H}}(X,Y) \neq 0$. The condition $\operatorname{Ext}^1_{\mathcal{H}}(Y,X) = 0$ forces X, Y to be consecutive objects as claimed. □

By (4.6) and (4.7), the dimension vectors of the objects Y_i, Z_i have the form $a_1\alpha_1 + a_2\alpha_2$ where $(a_1, a_2) =$



Furthermore, the following facts are well-known.

Proposition 4.4.3. (1) *In the Auslander-Reiten quiver of $\mathcal{H}$, the modules are arranged in order of the ratio a_2/a_1 starting with $P_1^{\mathcal{H}} = X_1$ with $a_2/a_1 = 0$ and ending with $I_2^{\mathcal{H}} = X_2$ with $a_2/a_1 = \infty$.*
(2) *Given any two consecutive terms X, Y in the AR-quiver of $\mathcal{H}$ we have:*

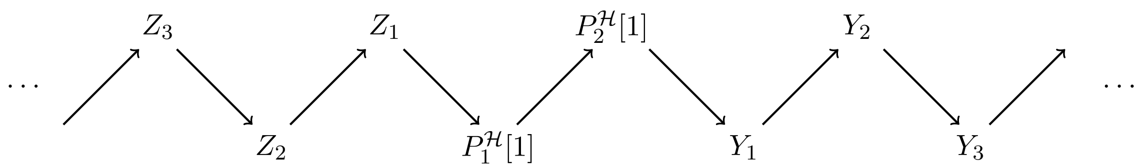
$$\dim_K \operatorname{Hom}_{\mathcal{H}}(X,Y) = m = f_1 d_1 = f_2 d_2 \,.$$

Let $\Psi = \{\underline{\dim}\, Y_i\} \cup \{\underline{\dim}\, Z_j\}$ be the set of (positive) real Schur roots of $\mathcal{H}$. If $\psi \in \Psi$ is not simple then $\alpha_1 \subsetneq \psi$. Therefore,

$$D(\psi) = \left\{ x \in \mathbb{R}^2 \mid \langle x, \psi \rangle = 0, \langle x, \alpha_1 \rangle \leq 0 \right\} \,.$$

Using the change of coordinates $\pi : \mathbb{R}^2 \cong \mathbb{R}^2$ given by $\pi(x) = (\langle x, \alpha_1 \rangle, \langle x, \alpha_2 \rangle)$ and property (1) from the above proposition, we see that the domains $D(\psi)$ are arranged counterclockwise in Quadrant II as shown in Figure 1 below.

The Auslander-Reiten quiver of the cluster category $\mathcal{C}_{\mathcal{H}}$ of $\mathcal{H}$ [BMRRT] has two more exceptional objects, $P_1^{\mathcal{H}}[1]$ and $P_2^{\mathcal{H}}[1]$, which come before the P_i and after the I_i:



This union is called the *transjective component* of the Auslander-Reiten quiver of $\mathcal{C}_\mathcal{H}$. The cluster tilting objects of $\mathcal{C}_\mathcal{H}$ are sums of pairs of consecutive objects in this quiver. These are $Y_i \oplus Y_{i+1}$, $Z_i \oplus Z_{i+1}$ plus three more cluster tilting objects:

$$Z_1 \oplus P_1^\mathcal{H}[1], \quad P_1^\mathcal{H}[1] \oplus P_2^\mathcal{H}[1], \quad P_2^\mathcal{H}[1] \oplus Y_1 .$$

By Proposition 4.4.3 the semi-invariant matrix Γ and exchange matrix B_Γ corresponding to these cluster tilting objects is given as follows. We use the notation $\eta_i = \underline{\dim}\, Y_i, \zeta_j = \underline{\dim}\, Z_j$ and arrange the objects of the cluster tilting object in the order they appear in the transjective component of the AR-quiver as given above.

cluster tilting object	matrix V	matrix Γ	B_Γ
$Z_{j+1} \oplus Z_j$	(ζ_{j+1}, ζ_j)	$(\zeta_{j+1}, -\zeta_{j+2})$	$\begin{bmatrix} 0 & -f_j^{-1}m \\ f_{j+1}^{-1}m & 0 \end{bmatrix} = B_j$
$Z_1 \oplus P_1^\mathcal{H}[1]$	$(\zeta_1, -\eta_1)$	$(\zeta_1, -\zeta_2)$	$\begin{bmatrix} 0 & -d_2 \\ d_1 & 0 \end{bmatrix} = B_1$
$P_1^\mathcal{H}[1] \oplus P_2^\mathcal{H}[1]$	$(-\eta_1, -\eta_2)$	$(-\eta_1, -\zeta_1)$	$\begin{bmatrix} 0 & -d_1 \\ d_2 & 0 \end{bmatrix} = B_0$
$P_2^\mathcal{H}[1] \oplus Y_1$	$(-\eta_2, \eta_1)$	$(-\zeta_1, \eta_1)$	$\begin{bmatrix} 0 & -d_2 \\ d_1 & 0 \end{bmatrix} = B_1$
$Y_1 \oplus Y_2$	(η_1, η_2)	(η_1, ζ_1)	$\begin{bmatrix} 0 & -d_1 \\ d_2 & 0 \end{bmatrix} = B_0$
$Y_i \oplus Y_{i+1}$, $(i \geq 2)$	(η_i, η_{i+1})	$(\eta_i, -\eta_{i-1})$	$\begin{bmatrix} 0 & -f_i^{-1}m \\ f_{i-1}^{-1}m & 0 \end{bmatrix} = B_{i-1}$

The subscript of B should be taken modulo 2 where B_0 and B_1 are as given in the table. Note that $B_1 = -B_0^t$. We will use this table to prove Proposition 4.3.2 for $\mathcal{H}$.

Proof of Proposition 4.3.2 in rank 2. The initial cluster tilting object is $P_1^\mathcal{H}[1] \oplus P_2^\mathcal{H}[1]$ which is given on the third line of the table above. Since $n = 2$, there are only two indices $k = 1, 2$. By symmetry we examine the case $k = 1$. Thus we need to verify the formula for the mutation μ_1 which is mutation of the first object.

In the table, mutation of the first object is given by descending lines of the table (and by descending values of j on the first line and ascending values of i on the last line). Since the order of the objects is switched at each step, we need to rephrase the proposition.

Restatement of Proposition 4.3.2: Under the mutation μ_1 of $T = T_1 \oplus T_2$ to $T^* = T_2 \oplus T_1^*$, the matrix $\Gamma_T = (\gamma_1, \gamma_2)$ changes to (γ_1', γ_2') as follows.

(1) $\gamma_2' = -\gamma_1$.
(2)

$$\gamma_1' = \begin{cases} \gamma_2 + |b_{12}|\gamma_1 & \text{if } |b_{12}|\gamma_1 < 0 \\ \gamma_2 & \text{otherwise} \end{cases}$$

where $(b_{ij}) = B_\Gamma$.

Proof: Condition (1) holds by inspection of the table: The second column of Γ in each line is the negative of the first column of Γ in the line above. To verify Condition (2) note that $b_{12} < 0$ in all cases. So, the first column of Γ has the same sign in lines 3 and 4. As predicted, the first columns of Γ in lines 4,5 are equal to the second columns of Γ in the lines above. In all other cases, the sign of the first column of Γ is positive (and $b_{12} < 0$). So, the predicted value of γ_1' in those cases is

$$\gamma_1' = \gamma_2 + |b_{12}|\gamma_1$$

This is verified by the following four calculations which follow from (4.6) and (4.7) with the observation that $\alpha_1 = \eta_1$.

matrix Γ	b_{12}	$\gamma_2 + \lvert b_{12}\rvert\gamma_1$	Γ'
$(\zeta_{j+1}, -\zeta_{j+2})$	$-d_j$	$-\zeta_{j+2} + d_j\zeta_{j+1} = \zeta_j$	$(\zeta_j, -\zeta_{j+1})$
$(\zeta_1, -\zeta_2)$	$-d_2$	$-\zeta_2 + d_2\zeta_1 = -\alpha_1$	$(-\eta_1, -\zeta_1)$
(η_1, ζ_1)	$-d_1$	$\zeta_1 + d_1\eta_1 = \eta_2$	$(\eta_2, -\eta_1)$
$(\eta_i, -\eta_{i-1})$	$-d_{i-1}$	$-\eta_{i-1} + d_{i-1}\eta_i = \eta_{i+1}$	$(\eta_{i+1}, -\eta_i)$

Since $\gamma_2 + |b_{12}|\gamma_1$ is equal to the first column of Γ' in all cases, this completes the proof of Proposition 4.3.2 for $n = 2$. $\square$

The images under the change of coordinates $\pi(x) = (\langle x, \alpha_1\rangle, \langle x, \alpha_2\rangle)$ of the dimension vectors $\zeta_j = \underline{\dim}\, Z_j$, $-\eta_1 = \underline{\dim}\, P_1^{\mathcal{H}}[1]$, $-\eta_2 = \underline{\dim}\, P_2^{\mathcal{H}}[1]$, $\eta_i = \underline{\dim}\, Y_i$ are arranged counter-clockwise starting and ending in Quadrant II as shown in Figure 3 below.

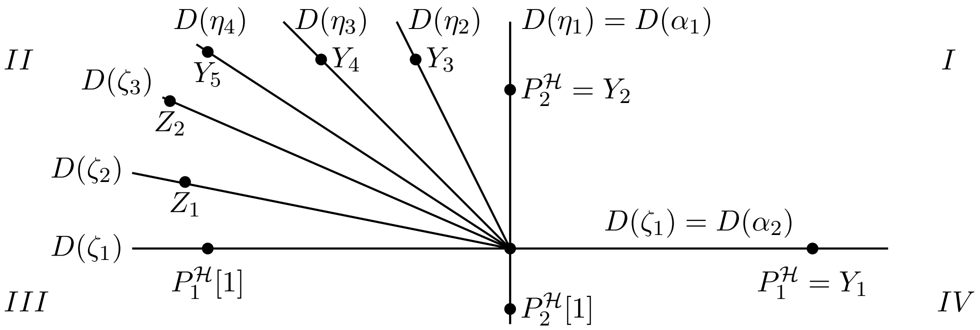


FIGURE 1. Since (Y_i, Y_{i+1}) is exceptional, $\underline{\dim}\, Y_{i+1} = \eta_{i+1} \in D(\eta_i)$ for all $i \geq 1$. Similarly, $\underline{\dim}\, Z_j = \zeta_j \in D(\zeta_{j+1})$ for all $j \geq 1$.

4.5. **Proof in general case.**

Proof of Proposition 4.3.2. We use Proposition 4.1.3 to reduce the proof to the rank 2 case. Throughout the proof we work with a fix k and $j \neq k$. We consider a fixed cluster tilting object $T = \coprod T_i$ in the cluster category $\mathcal{C}_\Lambda$ [BMRRT].

The mutation μ_k replaces T_k with T_k' in the cluster tilting object $T = \coprod T_i$. Let $\mathcal{R}$ be the simplicial fan in $\mathbb{R}^n$ spanned by $\underline{\dim}\, T_1, \cdots, \underline{\dim}\, T_n$. Let $\mathcal{R}'$ be the simplicial fan spanned by $\underline{\dim}\, T_i$, $i \neq k$, and $\underline{\dim}\, T_k'$. By Theorem 4.1.4(c), there are no semi-invariant domains $D(\beta)$ which meet the interiors of $\mathcal{R}, \mathcal{R}'$.

(1) $\gamma_k' = -\gamma_k$. The regions $\mathcal{R}, \mathcal{R}'$ are n-dimensional and meet along the wall $D(\beta_k)$ which is part of the hyperplane $H(\beta_k)$. The vectors $\underline{\dim}\, T_k, \underline{\dim}\, T_k'$ are on opposite sides of this hyperplane. The vector γ_k is equal to $\pm\beta_k$ with the sign chosen so that $\langle \underline{\dim}\, T_k, \gamma_k\rangle > 0$. This implies $\langle \underline{\dim}\, T_k^*, \gamma_k\rangle < 0$. Since $T_i, i \neq k$ are unchanged by mutation, $\beta_k' = \beta_k$. But $\gamma_k' = \pm\beta_k$ must have sign opposite to that of γ_k in order to have $\langle \underline{\dim}\, T_k^*, \gamma_k'\rangle > 0$. Therefore, $\gamma_k' = -\gamma_k$. This proves (1).

(2) Formula for γ_j', $j \neq k$. This breaks into several cases depending on the sign of γ_k and γ_j. We will explain the general case and prove the exceptional cases by direct computation.

Consider the objects T_i, $i \neq j, k$. This is a partial cluster tilting object missing two summands. The vectors $\underline{\dim}\, T_i$ lie in $D(\beta_j)$ and $D(\beta_k)$. This is equivalent to the statement that the corresponding exceptional objects M_{β_j}, M_{β_k} lie in the right perpendicular category $T_0^\perp$ where $T_0 = \coprod_{i\neq j,k} T_i$. Since T_0 is a partial cluster tilting object which is missing only

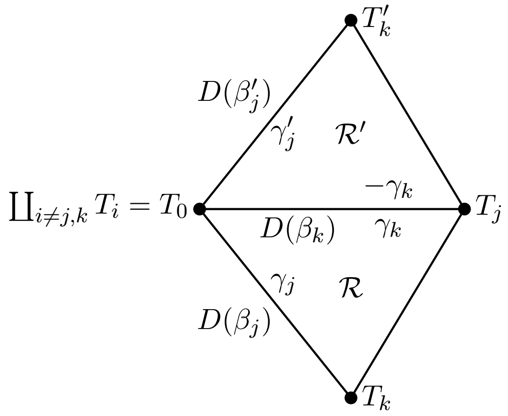


FIGURE 2. $\mu_k(T) = T_0 \oplus T_j \oplus T'_k$ where $T_0 = \coprod_{i\neq j,k} T_i$. The vectors $\underline{\dim}\, T_i$ span the spherical simplex $\mathcal{R}$. The components of $\mu_k T$ spans $\mathcal{R}'$. $\underline{\dim}\, T_0$ lies in $D(\beta_j)\cap D(\beta_k)\cap D(\beta'_j)$ and $\mathcal{R},\mathcal{R}'$ lie on opposite sides of $D(\beta_k)$. $\gamma_j = \varepsilon_j\beta_j$ and $\gamma_k = \varepsilon_k\beta_k$ where $\varepsilon_j, \varepsilon_k = \pm 1$ are such that $\langle \underline{\dim}\, T_j, \gamma_j\rangle, \langle \underline{\dim}\, T_k, \gamma_k\rangle > 0$.

two summands, the perpendicular category $T_0^\perp$ is a wide subcategory of *mod*-Λ of rank 2. We refer to the analysis of the cluster tilting objects and exceptional sequences in $\mathcal{H} = T_0^\perp$ as detailed in the last subsection.

We recall all of the notation from the last subsection. $\mathcal{H} = T_0^\perp$ has two simple objects X_1, X_2, with dimension vectors α_1, α_2. $\mathrm{Ext}^1_\Lambda(X_1, X_2) = 0$, $F_i = \mathrm{End}_\Lambda(X_i)$ and $M = \mathrm{Ext}^1_\Lambda(X_2, X_1)$. Then M is an $F_1 - F_2$ bimodule with $\dim_K M = m = f_1 d_1 = f_2 d_2$ where $d_i = \dim_{F_i} M$. The exceptional objects in $T_0^\perp$ are Y_i, Z_i, $i \geq 1$ whose dimension vectors form the set Ψ. By Proposition 4.4.2, the exceptional sequences in $T_0^\perp$ are (Y_p, Y_{p+1}), (Z_{q+1}, Z_q) for $p, q \geq 1$ and $(Z_1, Y_1) = (X_2, X_1)$. With a suitable ordering of the objects $T_i, i \neq j, k$, these extend to complete exceptional sequences

$$(Y_p, Y_{p+1}, |T_1|, \cdots, \widehat{|T_j|}, \widehat{|T_k|}, \cdots),\ (Z_{q+1}, Z_q, \cdots, \widehat{|T_j|}, \widehat{|T_k|}, \cdots),\ (Z_1, Y_1, \cdots, \widehat{|T_j|}, \widehat{|T_k|}, \cdots)$$

where $|T_i|$ indicates $|P[1]| = P$ or $|M| = M$.

Let $\eta_p = \underline{\dim}\, Y_p$, $\zeta_q = \underline{\dim}\, Z_q$ (we use p, q instead of i, j to avoid confusion). Then the semi-invariant domains $D(\eta_p)$, $D(\zeta_q)$ contain all $\underline{\dim}\, T_i$, $i \neq j, k$.

As before we take the projection $\pi : \mathbb{R}^n \to \mathbb{R}^2$ given by $\pi(x) = (\langle x, \alpha_1\rangle, \langle x, \alpha_2\rangle)$ with kernel the span of $\underline{\dim}\, T_i$ for $i \neq j, k$. Then, for all nonsimple $\beta \in \Psi$, $\pi(D(\beta)) \subset \mathbb{R}^2$ lies in the second quadrant as shown in Figure 3. The semi-invariant domains of the simple roots $\alpha_1 = \eta_1$ and $\alpha_2 = \zeta_1$ map to the y- and x-axes respectively.

The next step deviates from the details given in the last section. Instead of using Proposition 4.4.2, we need to use Proposition 4.1.3 which describes all possible ways to complete the partial cluster tilting object T_0. By Proposition 4.1.3 there are unique exceptional objects $T(\beta) \in D(\beta)$ for nonsimple $\beta \in \Psi$ and $T(\alpha_i)', T(\alpha_i)'' \in D(\alpha_i)$ for $i = 1, 2$ so that all ways of completing T_0 to a cluster tilting object are given by adding two consecutive objects in the following list.

$$\cdots, T(\zeta_3), T(\zeta_2), T(\alpha_2)', T(\alpha_1)'', T(\alpha_2)'', T(\alpha_1)', T(\eta_2), T(\eta_3), T(\eta_4), \cdots$$

These appear counterclockwise in Figure 3. In the figure two regions are labeled $\mathcal{R}, \mathcal{R}'$ to depict the example when $T_k = T(\eta_2)$, $T_j = T(\eta_3)$ and $T'_k = T(\eta_4)$.

We now claim that the remainder of the proof has already been carried out in detail in the rank 2 case. More explicitly, the claim is:

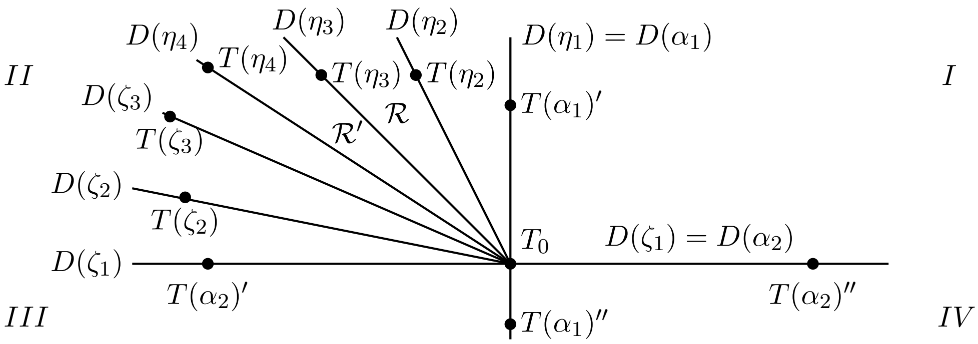


FIGURE 3. Images of $D(\beta)$, $\beta \in \Psi$, under $\pi : \mathbb{R}^n \to \mathbb{R}^2$ given by $\pi(x) = (\langle x, \alpha_1 \rangle, \langle x, \alpha_2 \rangle)$. All $D(\beta)$ with $\beta \neq \alpha_1, \alpha_2$ lie in Quadrant II. All $D(\beta)$ have positive side facing Quadrant I.

Claim The cluster tilting objects $T = T_0 \oplus T_k \oplus T_j$ and $T' = T_0 \oplus T_j \oplus T_k'$ are two consecutive items in the following list.

$n = 2$ case	cluster tilting object	(γ_k, γ_j)	b_{kj}
$Z_{q+1} \oplus Z_q$	$T_0 \oplus T(\zeta_{q+2}) \oplus T(\zeta_{q+1})$	$(\zeta_{q+1}, -\zeta_{q+2})$	$-d_{q+1}$
$Z_1 \oplus P_1^{\mathcal{H}}[1]$	$T_0 \oplus T(\zeta_2) \oplus T(\alpha_2)'$	$(\zeta_1, -\zeta_2)$	$-d_2$
$P_1^{\mathcal{H}}[1] \oplus P_2^{\mathcal{H}}[1]$	$T_0 \oplus T(\alpha_2)' \oplus T(\alpha_1)''$	$(-\eta_1, -\zeta_1)$	$-d_1$
$P_2^{\mathcal{H}}[1] \oplus Y_1$	$T_0 \oplus T(\alpha_1)'' \oplus T(\alpha_2)''$	$(-\zeta_1, \eta_1)$	$-d_2$
$Y_1 \oplus Y_2$	$T_0 \oplus T(\alpha_2)'' \oplus T(\alpha_1)'$	(η_1, ζ_1)	$-d_1$
$Y_2 \oplus Y_3$	$T_0 \oplus T(\alpha_1)' \oplus T(\eta_2)$	$(\eta_2, -\eta_1)$	$-d_2$
$Y_p \oplus Y_{p+1}$, $(p \geq 3)$	$T_0 \oplus T(\eta_{p-1}) \oplus T(\eta_p)$	$(\eta_p, -\eta_{p-1})$	$-d_p$

Take for example the case $T = T_0 \oplus T(\eta_2) \oplus T(\eta_3)$. If $T_k = T(\eta_2)$ then $\mu_k T = T_0 \oplus T(\eta_3) \oplus T(\eta_4)$. The domain $D(\beta_k)$ separating these two cluster tilting objects is the unique one containing T_0 and T_j. This is $D(\eta_3)$. Furthermore, T is on the positive side since all objects in $D(\eta_2)$ are on the positive side of $D(\eta_3)$. So, $\gamma_k = \eta_3$. Also, $\gamma_j = -\eta_2$ since

$$\langle \underline{\dim}\, T_j, \eta_2 \rangle = \langle \eta_3, \eta_2 \rangle = -f_2 < 0.$$

This is one example and there are many other items in this list. However, it is not necessary to verify more than one example! The reason is: cluster mutation always passes from one compartment to an adjacent compartment. We know the semi-invariants of the walls from the rank 2 case. These are given in the third column of the table above. If we fix T_0 and mutate the other two objects we will move up and down this list once we know we are on the list. So, it suffices to check only one example, as we have done.

The numbers b_{kj} depend only on γ_k, γ_j:

$$b_{kj} = f_k^{-1}(\langle \gamma_j, \gamma_k \rangle - \langle \gamma_k, \gamma_j \rangle).$$

Therefore, Proposition 4.3.2 is a statement about the items in the third column of the table. But we already verified in the rank 2 case that exactly these vectors satisfy Proposition 4.3.2. So, we are done. □

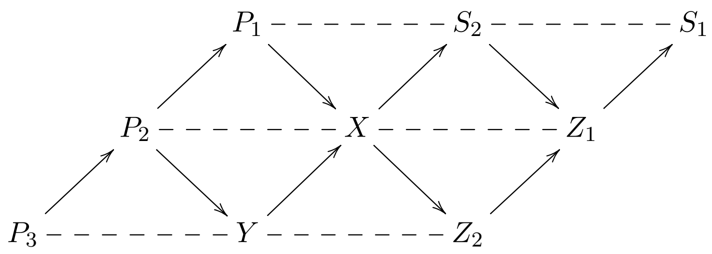
We can now prove the c-vector theorem.

Proof of Theorem 4.1.6. The statement is that the c-vectors of a cluster tilting object T are $-\gamma_i$. This holds for the initial cluster tilting object $\Lambda[1]$ by definition. It is well-known that cluster mutation acts transitively on the set of cluster tilting objects. (See [Hu].) Therefore, it suffices to show that the equation $c_i = -\gamma_i$ remains true under mutation. But this is what is shown in Proposition 4.3.2 with the aid of Theorem 4.2.3. □

4.6. **Example.** Figure 4 illustrates several concepts discussed in the paper. Take the modulated quiver

$$\mathbb{R} \xrightarrow{\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{C}} \mathbb{C}$$

In other words, $F_1 = F_2 = \mathbb{R}$ is the ground field, $F_3 = \mathbb{C}$ and the nonzero bimodules are $M_{12} = \mathbb{R}$, $M_{23} = \mathbb{C}$. The tensor algebra of this quiver is of finite type with 9 indecomposable objects:



Consider L, the intersection with the unit sphere $S^2 \subseteq \mathbb{R}^3$ with the union $\bigcup D(\beta)$ of all semi-invariant domains (all nine). Figure 4 shows the stereographic projection of L onto the plane.

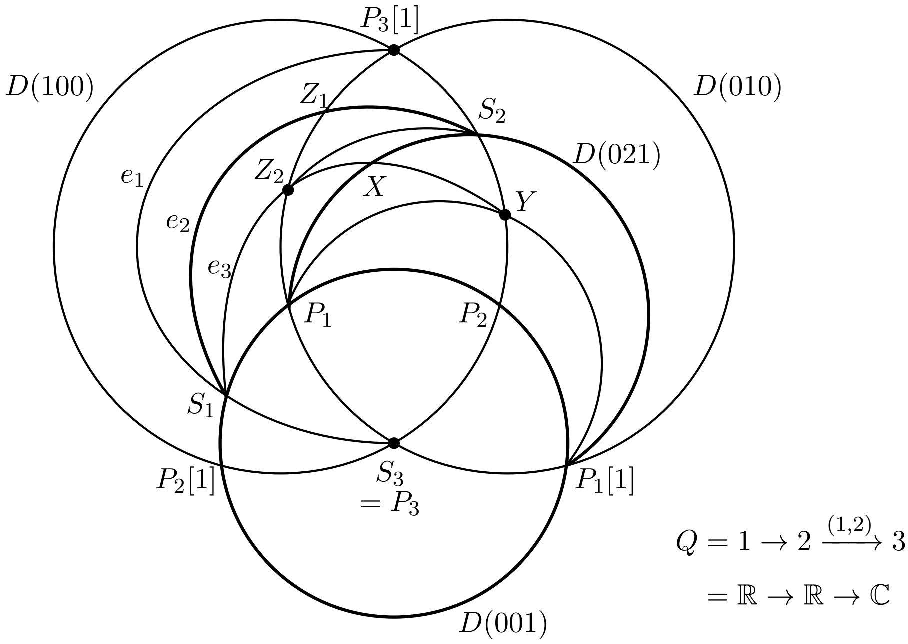
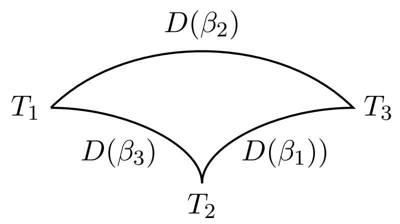


FIGURE 4. The three circles are domains of semi-invariants with simple det-weight vectors. Other det-weights are given by extensions of the corresponding modules. For example, edges e_1, e_2, e_3 are domains of $(1,1,0), (2,2,1), (1,1,1)$. The four dark vertices $S_3, Z_2, Y, P_3[1]$ indicate the objects with endomorphism ring $\mathbb{C}$. The semi-invariant domains $D(0,0,1), D(0,2,1)$ and $e_2 = D(2,2,1)$ which correspond to S_3, Y, Z_2 by Proposition 4.6.1 are also darkened.

In reading Figure 4 the following easy observation is helpful.

Proposition 4.6.1. *Let $T = \coprod T_i$ be a cluster tilting object with associated matrix $\Gamma_T = (\gamma_i)$. Suppose that $\gamma_k = \beta_k$ is positive and all other columns of Γ_T are negative. Then $\underline{\dim}\, T_k = \beta_k$. In other words, when the following triangle appears in a picture, $T_2 = M_{\beta_2}$.*



Proof. As we say in the proof of the c-vector theorem, rank 2 case, for any $j \neq k$, the modules M_{γ_j} and M_{γ_k} are consecutive objects in the Auslander-Reiten quiver of the rank 2 perpendicular category $T_0^\perp$ where $T_0 = \coprod_{i \neq j,k} T_i$. Therefore, $\langle \gamma_k, \gamma_j \rangle = 0$ for all $j \neq k$. We also have $\langle \underline{\dim}\, T_k, \gamma_j \rangle = 0$ for all $j \neq k$. Since Γ_T is an invertible matrix (by Corollary 4.1.5), this implies that $\underline{\dim}\, T_k$ is a scalar multiple of γ_k. So, $\underline{\dim}\, T_k = \beta_k$. □

Example 4.6.2. Examples of Proposition 4.6.1 in Figure 4.

(1) $\underline{\dim}\, Z_1 = (1, 1, 0)$ and $e_1 = D(1, 1, 0)$
(2) $\underline{\dim}\, Z_2 = (2, 2, 1)$ and $e_2 = D(2, 2, 1)$ which extends from S_1 to S_2
(3) $\underline{\dim}\, P_1 = (1, 1, 1)$ and $e_3 = D(1, 1, 1)$ which extends from S_1 to Y and passes through Z_2, X.
(4) $\underline{\dim}\, X = (1, 2, 1)$ and $D(1, 2, 1)$ is the edge connecting Z_2 and S_2.

Figure 4 also illustrates the following concepts used in the paper. For $\beta = (1, 1, 1)$, the simple objects of the category ${}^\perp M_\beta$ are S_1 and Y with dimension vectors $\alpha_1 = (1, 0, 0)$ and $\alpha_2 = (0, 2, 1)$. These form the corners (endpoints in this dimension) of the convex region $D(\beta)$. The other roots in this region are positive integer linear combinations: $\underline{\dim}\, Z_2 = 2\alpha_1 + \alpha_2$ and $\underline{\dim}\, X = \alpha_1 + \alpha_2$.

Also, a specific example of Figure 3 appears. Take $T_0 = S_1$. Then $D(\zeta_i) = e_i$ for $i = 1, 2, 3$, $Z_1 = T(\zeta_2)$, $Z_2 = T(\zeta_3)$. The labels in Figure 4 do not quite match the ones in Figure 1 since $n = 3$ in Figure 4.

4.7. **Applications.** In concurrently written papers we use the results of this paper to:

(1) Develop the theory of signed exceptional sequences and show they are in bijection with ordered cluster tilting objects [IT16a].
(2) Develop the theory of semi-invariant picture groups and compute their cohomology in type A_n [IOTW4].
(3) Show that, for acyclic modulated quivers of finite type, the maximal green sequences are in bijection with the positive expressions for the Coxeter element in the picture group [IT16b].
(4) For any acyclic modulated quiver with a bimodule $M_{ij} : i \to j$ of infinite type, any maximal green sequence mutates at j before i [BHIT].

Finally, we point out that Theorem 4.1.6 implies the sign coherence of c-vectors (that in each c-vector the coordinates have the same sign) a theorem which has been proven many times and in fact the present version of this paper grew out of a desire to understand the proof given by Speyer-Thomas [ST]. Proposition 2.2.5 gives the conceptual proof of this fact. Namely, semi-invariants defined on presentation spaces are necessarily sign coherent.

In future work, we plan to extend the results of this paper to modulated quivers with oriented cycles.

5. Appendix A: Associated modulated quiver

In this appendix we discuss the problem of when a finite dimensional hereditary algebra over a field K is Morita equivalent to the tensor algebra of its associated modulated quiver.

Theorem 5.0.1. *Λ is Morita equivalent to $T(Q,\mathcal{M})$ if and only if, for each arrow $i \to j$, the F_i-F_j-bimodule epimorphism*

$$\mathrm{Hom}_\Lambda(P_j, rP_i) \twoheadrightarrow M_{ij} = \mathrm{Hom}_\Lambda(P_j, rP_i/r^2P_i) \tag{5.1}$$

has a section. (Recall that $F_i = \mathrm{End}_\Lambda(P_i)$.)

Proof. This condition is necessary since it holds on the category of representations of $T(Q,\mathcal{M})$. Conversely, suppose the condition holds on *mod*-Λ. Choose a section $\sigma_{ij} : M_{ij} \to \mathrm{Hom}_\Lambda(P_j, rP_i)$ of (5.1) for every $i \to j$ in Q_1. For every Λ-module X, let $X_i = \mathrm{Hom}_\Lambda(P_i, X)$. This is a right F_i-module. For each arrow $i \to j$ in Q_1, define the morphism $X_i \otimes_{F_i} M_{ij} \to X_j$ to be the composition:

$$X_i \otimes_{F_i} M_{ij} \xrightarrow{r\otimes\sigma_{ij}} \mathrm{Hom}_\Lambda(rP_i, rX) \otimes_{F_i} \mathrm{Hom}_\Lambda(P_j, rP_i) \xrightarrow{c} \mathrm{Hom}_\Lambda(P_j, rX) \hookrightarrow \mathrm{Hom}_\Lambda(P_j, X) = X_j$$

where $r : \mathrm{Hom}_\Lambda(P_i, X) \to \mathrm{Hom}_\Lambda(rP_i, rX)$ is the restriction map and c is composition. Since each morphism in this sequence is natural in X, this defines a functor

$$\varphi : \textit{mod-}\Lambda \to \textit{Rep}(Q,\mathcal{M})$$

which is clearly exact and faithful since it takes nonzero objects to nonzero objects.

We claim that φP_i is the projective cover P_i^T of S_i in $\textit{Rep}(Q,\mathcal{M})$. This follows by induction on the length of P_i and the fact that the structure maps $c(r \otimes \sigma_{ij}) : M_{ij} \to \mathrm{Hom}_\Lambda(P_j, rP_i)$ of φP_i are, together, adjoint to the isomorphism $\coprod_j M_{ij} \otimes_{F_j} P_j \cong rP_i$.

Thus, $\mathrm{Hom}_\Lambda(P_i, X) = X_i = \mathrm{Hom}_{T(Q,\mathcal{M})}(P_i^T, \varphi X)$ and it follows that φ is an equivalence between the full subcategories of projective objects of *mod*-Λ and $\textit{Rep}(Q,\mathcal{M})$. Being exact, φ extends to an equivalence of the module categories. $\square$

Example 5.0.2. Let $L = \mathbb{F}_2(t)$ with subfields $K = \mathbb{F}_2(t^4) \subset F = \mathbb{F}_2(t^2) \subset L$. We have a short exact sequence of L-bimodules:

$$0 \to L \otimes_F L \xrightarrow{j} L \otimes_K L \xrightarrow{p} L \otimes_F L \to 0 \tag{5.2}$$

where j sends $1 \otimes 1$ to $t^2 \otimes 1 + 1 \otimes t^2$ and p takes $1 \otimes 1$ to $1 \otimes 1$. This sequence does not split since $L \otimes_K L$ is indecomposable as an L-bimodule. This follows from the L-algebra isomorphism $\varphi : L[X]/(X^4) \to L \otimes_K L$ given by $\varphi(X) = t \otimes 1 + 1 \otimes t$ where we consider $L \otimes_K L$ as an L-algebra using $L \otimes 1$.

Let Λ be the tensor algebra of the modulated quiver

$$\begin{array}{ccccccccc} & & F_2 & & & & & F & \\ & \overset{M_{12}}{\nearrow} & & \overset{M_{23}}{\searrow} & & & \overset{L}{\nearrow} & & \overset{L}{\searrow} \\ F_1 & & \xrightarrow{\widetilde{M}_{13}} & & F_3 & = & L & \xrightarrow{L\otimes_K L} & L \end{array}$$

modulo the relation that the composition $L \otimes_F L$ of the top two arrows is identified with the image of j in $L \otimes_K L$. Then Λ is hereditary since the radical of each projective module is projective, e.g., $rP_1 \cong P_2 \oplus P_3^2$. However, the bimodule morphism $\widetilde{M}_{13} = \mathrm{Hom}_\Lambda(P_3, rP_1) \twoheadrightarrow M_{13}$ is not split because it is equal to the map p in (5.2). By Theorem 5.0.1, Λ is not Morita equivalent to the tensor algebra of its associated modulated quiver.

6. Appendix B: Reduced norm

This appendix reviews the definition and properties of the reduced norm [J] and uses them to compare the determinantal weight with the "true weight" of a semi-invariant on presentation spaces as claimed in Remark 2.4.3. We assume that K is infinite.

For any finite dimensional algebra A over any field K, the *general element* of A is

$$a(\xi) = \sum \xi_i u_i \in A \otimes_K K(\xi)$$

where $u_1, \cdots, u_n$ is a vector space basis for A over K and $\xi_1, \cdots, \xi_n$ are a transcendence basis for $K(\xi) = K(\xi_1, \cdots, \xi_n)$. Let

$$m_{a(\xi)}(\lambda) = \lambda^m + c_1(\xi)\lambda^{m-1} + \cdots + c_m(\xi) \quad \in K(\xi)[\lambda]$$

be the minimal polynomial of $a(\xi)$ over $K(\xi)$. The degree m of $m_{a(\xi)}(\lambda)$ is called the *degree* of A over K. We call it the *reduced degree* in cases where the word "degree" is already defined as in the case of field extensions.

It is easy to see that the reduced degree of a finite separable extension of K is equal to its vector space dimension over K (the usual notion of degree). However, this is not true in general for inseparable extensions and division algebras.

If D is a finite dimensional division algebra over its center C then $\dim_C D = d^2$ where d is the degree of D over C. Furthermore, there is an open dense subset of D consisting of all elements $b \in D$ so that $C(b)$ is a separable field extension of C of degree d. Each of these is called a *maximal separable subfield* of D.

Example 6.0.3. Let $A = \mathbb{H}$ and $K = \mathbb{R}$. The minimal polynomial of the general element $a = t + xi + yj + zk \in \mathbb{H}$ is $m_a(\lambda) = \lambda^2 - 2t\lambda + t^2 + x^2 + y^2 + z^2$. So, $\mathbb{H}$ has degree 2 over $\mathbb{R}$. For any $b \in \mathbb{H}$ which is not in $\mathbb{R}$, $\mathbb{R}(b) \cong \mathbb{C}$ is a maximal (separable) subfield of $\mathbb{H}$.

Lemma 6.0.4. [J] *$m_{a(\xi)}(\lambda)$ is a polynomial in $\xi_1, \cdots, \xi_n, \lambda$ and $c_j(\xi) \in K[\xi]$ is a homogeneous polynomial of degree j in the variables ξ_i.*

The *reduced characteristic polynomial* of $b \in A$ is the specialization of $m_{a(\xi)}(\lambda)$ given by

$$m_b(\lambda) = \sum_{i=0}^{m} c_i(b_1, \cdots, b_n)\lambda^{m-i} \in K[\lambda]$$

where $b = \sum b_i u_i$, $b_i \in K$ and $c_0 = 1$. We will use the notation $c_i(b) = c_i(b_1, \cdots, b_n)$.

Proposition 6.0.5. [J]

(0) *$m_b(\lambda)$ depends only on $b \in A$. (The coefficients $c_i(b)$ are independent of the choice of basis $u_1, \cdots, u_n$.)*
(1) *$m_b(b) = 0$. Equivalently, the minimal polynomial $\mu_b(\lambda)$ of b is a factor of $m_b(\lambda)$.*
(2) *Every root of $m_b(\lambda)$ is a root of $\mu_b(\lambda)$.*
(3) *The set of all $b \in A$ for which $m_b(\lambda)$ is the minimal polynomial of b is an open dense subset of A.*
(4) *$m_b(\lambda)$ is invariant under extension of scalars, i.e., $m_b(\lambda) = m_{b\otimes 1}(\lambda)$ if $b \otimes 1 \in A \otimes_K L$ is the image of b for any extension field L of K.*

The following observation follows easily from Property (3).

Lemma 6.0.6. *The reduced degree of a finite purely inseparable extension F of K is the smallest power $q = p^\mu$ of $p = char\, K$ so that $F^q \subseteq K$. Furthermore the reduced characteristic polynomial is $m_b(\lambda) = \lambda^q - b^q$ for every $b \in F$.*

Example 6.0.7. Let $A = \mathbb{F}_p(s,t)$ and $K = \mathbb{F}_p(s^p, t^p)$. Then $a^p \in K$ for any $a \in A$ and the minimimal polynomial of the general element $a \in A$ is $m_a(\lambda) = \lambda^p - a^p$. So, the reduced degree of A over K is p although A is a field extension of K of degree p^2.

Definition 6.0.8. The *reduced norm* $\overline{n} : A \to K$ is defined to be the homogeneous polynomial function of degree m, the degree of A over K, given on any $b \in A$ by

$$\overline{n}(b) = (-1)^m c_m(b).$$

The main properties of the reduced norm are the following.

$$\overline{n}(ab) = \overline{n}(a)\overline{n}(b), \quad \overline{n}(1) = 1.$$

Any polynomial function $\chi : A \to K$ satisfying these two properties will be called a *character* on A.

Another easy consequence of Propery (3) is the following. If A, B are finite dimensional algebras over K and $(a, b) \in A \times B$, then

$$m_{(a,b)}(\lambda) = m_a(\lambda)m_b(\lambda).$$

This implies in particular that the degree of $A \times B$ over K is the sum of the degrees of A, B over K. Also the reduced norm over $A \times B$ is the product:

$$\overline{n}_{A\times B}(a, b) = \overline{n}_A(a)\overline{n}_B(b).$$

The main theorem relevant to our paper is the following.

Theorem 6.0.9. *Let D be a finite dimensional division algebra over K which has degree d over its center C and suppose that C has reduced degree c over K. Then, for any $k \geq 1$, $M_k(D)$ has degree dkc. Furthermore, any character $M_k(D) \to K$ is a nonnegative power of the reduced norm.*

Remark 6.0.10. This implies that every character $M_k(D) \to K$ is a nonnegative fractional power of the K-determinant det_K: $det_K = \overline{n}^{f/dc}$ where $f = \dim_K D$. Thus, the "true weight" of a semi-invariant with determinantal weight β is the vector whose i-th coordinate is $\beta_i f_i/d_i c_i$ where $d_i c_i$ is the (reduced) degree of F_i over K. In particular, if m is the least common multiple of the integers $f_i/d_i c_i$ then the m-th power σ^m of any semi-invariant on a presentation space $\text{Hom}_\Lambda(P(\gamma_1), P(\gamma_0))$ has determinantal weight.

Proof. We first compute the degree of $M_k(D)$ over K. Let L be the maximal separable subfield of D. Then L is separable over C of degree d. Let E be the separable closure of K in L. Then $F = E \cap C$ is the separable closure of K in C. Let $s = [F : K]$. Then $c = qs$ where $q = p^\mu$ is the reduced degree of C over F. By Lemma 6.0.6, the reduced degree of L over E is also q and q is minimal so that $L^q \subseteq E$. Let S be the splitting field of E over K. Then $C \otimes_F S \cong CS$ is a field which is separable over C. So,

$$M_k(D) \otimes_F S = M_k(D) \otimes_C C \otimes_F S = M_k(D) \otimes_C CS \cong M_{dk}(CS).$$

<u>Claim</u> The degree of $M_{dk}(CS)$ over S is qdk and the reduced norm $M_{dk}(CS) \to S$ is the q-th power of the determinant over CS.

Proof: Any $a \in M_{dk}(CS)$ satisfies its characteristic polynomial $f(\lambda) = \det(\lambda - a) \in CS[\lambda]$ with degree dk. Then $f(\lambda)^q$ is a polynomial in $S[\lambda]$ of degree qdk satisfied by a. So, the degree of $M_{dk}(CS)$ over S is $\leq qdk$. Now consider the inclusion of the diagonal matrices:

$$CS^{dk} = CS \times \cdots \times CS \hookrightarrow M_{dk}(CS)$$

Since the general element of CS has degree q over S, the general element of CS^{dk} has degree qdk over S. So, the degree of $M_{dk}(CS)$ over S is $\geq qdk$. So, it is equal to qdk. Furthermore, the reduced characteristic polynomial is $\det(\lambda - a)^q$ and the reduced norm is $\det(a)^q$.

We return to the proof of the theorem. Since S is the splitting field of E over K and F is an intermediate field, we have $F \otimes_K S \cong S^s$ where $s = [F : K]$. Since (reduced) degree is invariant under extension of scalars, the degree of $M_k(D)$ over K is equal to the degree of $M_k(D) \otimes_K S$ over S. But

$$M_k(D) \otimes_K S = M_k(D) \otimes_F F \otimes_K S = M_k(D) \otimes_F S^s = M_{dk}(CS)^s$$

which has degree s times the degree of $M_{dk}(CS)$ over S. By the claim above this is s times qdk which is $dkqs = dkc$ proving the first statement of the theorem.

By the argument above, the reduced norm $\overline{n} : M_k(D) \to K$ has degree $qkc = qdsk$. Now consider any character $\chi : M_k(D) \to K$. We note that arbitrary (polynomial) characters must be homogeneous polynomials. By extending scalars we get a character $\chi_S : M_k(D) \otimes_K S \cong M_{dk}(CS)^s \to S$ which must be a product of s characters $(\chi_S)_i : M_{dk}(CS) \to S$. By symmetry given by the action of $Gal(S/K)$, these s characters are equal. By restriction to diagonal matrices we get a character $CS^{dks} \to S$. But a character on CS^{dks} is a product of characters one for each factor: $\chi_S|CS^{dks} = (\chi_0)^{dks}$. By symmetry, these characters must all be equal. But each character $\chi_0 : CS \to S$ is a power of the reduced norm $\overline{n}_{CS} : CS \to S$ since $\chi_0(x) = x^m$ and this lies in S only when m is a multiple of q, say $m = qt$, $\chi_0 = \overline{n}_{CS}^t$. Therefore,

$$\chi_S|CS^{dks} = (\chi_0)^{dks} = \overline{n}_{CS}^{tdks}$$

which has degree equal to $qtdks$. When χ is the reduced norm $\overline{n}$ we get $t = 1$. Therefore, in general we get $(\chi_S)_i = \overline{n}_i^t$ when restricted to the diagonal matrices where $\overline{n}_i$ is the reduced norm of $M_{dk}(CS)$ over S. However, any invertible matrix is equivalent to a diagonal matrix under row and column operations which are given by multiplication by elements of the commutator subgroup of $GL(dk, CS)$. Since S^* is abelian, each group homomorphism $(\chi_S)_i : GL(dk, CS) \to S^*$ is uniquely determined by its restriction to diagonal invertible matrices. So, $\chi_S = \overline{n}^t$ for all elements of $GL(dk, CS)^s$. Since this is an open dense subset of $M_{dk}(CS)^s$, $\chi_S = \overline{n}^t$ as homogeneous polynomials over S. But both polynomials have coefficients in K. So, they give $\chi = \overline{n}^t$ as characters $M_k(D) \to K$. □

Corollary 6.0.11. *Suppose that F_1, F_2 are division algebras over K of dimensions f_1, f_2 and degrees n_1, n_2 over K. Let M be an F_1-F_2-bimodule with* $\dim_K M = m$. *Then*

$$\frac{mn_1}{f_1 n_2}, \quad \frac{mn_2}{f_2 n_1}$$

are integers.

Proof. The reduced norm gives a character

$$\mathrm{End}_{F_1}(M) \cong M_{m/f_1}(F_1) \xrightarrow{\overline{n}_1} K$$

which is polynomial of degree mn_1/f_1. Composing with the inclusion $F_2 \hookrightarrow \mathrm{End}_{F_1}(M)$ we get a character $\chi : F_2 \to K$ of degree mn_1/f_1. By Theorem 6.0.9, χ is an integer power of the reduced norm $\overline{n}_2 : F_2 \to K$ which has degree n_2. Therefore n_2 divides mn_1/f_1 making mn_1/f_1n_2 an integer. The other case is similar. □

Definition 6.0.12. Let Λ be a finite dimensional hereditary algebra over a field K. Let B_Λ be the exchange matrix of Λ. We define the *reduced exchange matrix* of Λ to be

$$\overline{B}_\Lambda = ZB_\Lambda Z^{-1}$$

where Z is the diagonal matrix with entries $z_i = f_i/n_i$ where n_i is the degree of F_i over K and $f_i = \dim_K F_i$. By Corollary 6.0.11, the entries of $\overline{B}_\Lambda$ are integers.

Given a cluster tilting object T with exchange matrix B_T and c-matrix C_T, we define the *reduced exchange matrix* and the matrix of *reduced c-vectors* by $\overline{B}_T = ZB_TZ^{-1}$ and $\overline{C}_T = ZC_TZ^{-1}$.

Since mutation of exchange matrices and extended exchange matrices commutes with conjugation, $\overline{B}_T$ and $\overline{C}_T$ have integer coordinates and are obtained from $\begin{bmatrix}\overline{B}_\Lambda\\ I_n\end{bmatrix}$ by mutation. We claim that the reduced c-vectors are the reduced weights of the reduced norm semi-invariants which we now define.

Definition 6.0.13. We define the *reduced norm semi-invariant* $\overline{\sigma}_\beta$ to be the polynomial function

$$Pres_\Lambda(\gamma_1,\gamma_0)=\mathrm{Hom}_\Lambda(P(\gamma_1),P(\gamma_0))\to K$$

which sends $f:P(\gamma_1)\to P(\gamma_0)$ to the reduced norm of

$$\mathrm{Hom}(f,1):\mathrm{Hom}_\Lambda(P(\gamma_0),M_\beta)\to \mathrm{Hom}_\Lambda(P(\gamma_1),M_\beta)$$

considered as a linear map of F_β-vector spaces. We define the *reduced weight* of a semi-invariant σ on presentation space $Pres_\Lambda(\gamma_1,\gamma_0)$ to be the vector $w\in\mathbb{N}^n$ so that $\sigma(gfh)=\prod \overline{n}_i(g)^{w_i}\sigma(f)\overline{n}_i(h)^{w_i}$ where $\overline{n}_i(g)$ is the reduced norm of the $GL(\gamma_{0i},F_i)$-component of $g\in\mathrm{Aut}_\Lambda(P(\gamma_0))$ and similarly for $\overline{n}_i(h)$.

Lemma 6.0.14. *The reduced weight of* $\overline{\sigma}_\beta$ *is*

$$\overline{\beta}=\frac{1}{z_\beta}(z_1\beta_1,z_2\beta_2,\cdots,z_n\beta_n).$$

Proof. By Theorem 6.0.9, the reduced norm semi-invariant $\overline{\sigma}_\beta$ is related to the determinantal semi-invariant σ_β by: $\sigma_\beta=\overline{\sigma}_\beta^{z_\beta}$. Since the det-weight of σ_β is β we have:

$$\begin{aligned}\overline{\sigma}_\beta(gfh) &= \sigma_\beta(gfh)^{1/z_\beta}\\ &= \prod \chi_i(g)^{\beta_i/z_\beta}\sigma_\beta(f)^{1/z_\beta}\chi_i(h)^{\beta_i/z_\beta}\\ &= \prod \overline{n}_i(g)^{n_i\beta_i/z_\beta}\overline{\sigma}_\beta(f)\overline{n}_i(g)^{n_i\beta_i/z_\beta}\end{aligned}$$

where $\chi_i(g)=\overline{n}_i(g)^{z_i}$ is the det-weight of the $GL(\gamma_{0i},F_i)$-component of $g\in\mathrm{Aut}_\Lambda(P(\gamma_0))$. So, the reduced weight of $\overline{\sigma}_\beta$ is $(n_i\beta_i/z_\beta)=\overline{\beta}$. □

In the notation of Corollary 4.1.5 we have the following.

Lemma 6.0.15. *For any cluster tilting object* T *of* Λ *we have*

$$V^t\overline{E}\,\overline{\Gamma}_T=N$$

where $\overline{E}=EZ^{-1}=LN$, $\overline{\Gamma}_T=Z\Gamma_TZ^{-1}$.

Proof. By Corollary 4.1.5 we have: $V^t\overline{E}\,\overline{\Gamma}_T=V^tE\Gamma_TZ^{-1}=DZ^{-1}=N$. □

We can now restate the c-vector theorem in terms of reduced c-vectors.

Theorem 6.0.16 (Reduced Norm c-vector Theorem)**.** *The reduced c-vectors associated to a cluster tilting object* T *are*

$$\overline{c}_j=-\varepsilon_j\overline{\beta}_j$$

where $\overline{\beta}_j$ *is the reduced weight of the reduced norm semi-invariant* $\overline{\sigma}_{\beta_j}$.

Proof. Since conjugation of exchange matrices and c-matrices commutes with mutation, given that c_j is the j-th c-vector of the object T, the reduced vector $\overline{c}_j$ is the j-th c vector of T using $\overline{B}_\Lambda$ as initial exchange matrix. Since $c_j=-\varepsilon_j\beta_j$ by Theorem 4.1.6, reduction of both sides, using the fact that $z_j=z_{\beta_j}$, gives $\overline{c}_j=-\varepsilon_j\overline{\beta}_j$. □

Example 6.0.17. Consider the modulated quiver $\mathbb{C} \to \mathbb{C} \to \mathbb{H}$. This has 9 indecomposable modules giving the same picture as Figure 4. Using the same label for these modules as in Figure 4 (which originates in Figure 1) we have:

label	β	z_β	$\overline{\beta} = \frac{1}{z_\beta}(\beta_1, \beta_2, 2\beta_3)$
S_3	$(0,0,1)$	2	$(0,0,1)$
P_2	$(0,1,1)$	1	$(0,1,2)$
P_1	$(1,1,1)$	1	$(1,1,2)$
Y	$(0,2,1)$	2	$(0,1,1)$
X	$(1,2,1)$	1	$(1,2,2)$
S_2	$(0,1,0)$	1	$(0,1,0)$
Z_2	$(2,2,1)$	2	$(1,1,1)$
Z_1	$(1,1,0)$	1	$(1,1,0)$
S_1	$(1,0,0)$	1	$(1,0,0)$

As an example, take the injective module Z_1. This has a determinantal semi-invariant of det-weight $\beta = (2,2,1)$ since a presentation for Z_1 is $P_2 \oplus P_3 \to P_1 \oplus P_2 \to Z_1$. We take homomorphisms to Z_2 to get:

$$\mathrm{Hom}_\Lambda(P_1 \oplus P_2, Z_2) = \mathbb{C}^2 \oplus \mathbb{C}^2 \to \mathrm{Hom}_\Lambda(P_2 \oplus P_3, Z_2) = \mathbb{C}^2 \oplus \mathbb{H} \tag{6.1}$$

The determinantal semi-invariant σ_β is given by considering this as an isomorphism of 8-dimensional real vector spaces and taking determinant. This has determinantal weight $(2,2,1)$ since the automorphism of P_1 given by $z = a + bi \in \mathbb{C}^*$ has real determinant $\begin{vmatrix} a & b \\ -b & a \end{vmatrix} = a^2 + b^2$ and multiplies the 8×8 determinant by $(a^2+b^2)^2$ (since it multiplies the first two $\mathbb{C}$ coordinates) which is the second power of the det-weight $|z|^2$ of z. Similarly any $z \in \mathrm{Aut}(P_2)$ also changes the 8×8 determinant by $|z|^4$ making the det-weight of σ_β equal to $(2,2,?)$. The third coordinate of the det-weight is 1 since $h \in \mathrm{Aut}(P_3)$ changes the 8×8 determinant by $|h|^4$ which is the det-weight of h.

The reduced norm semi-invariant $\overline{\sigma}_\beta$ is given by considering (6.1) as an isomorphism of 2-dimensional vector spaces over $\mathbb{H}$ and taking the reduced norm over $\mathbb{H}$ which is the square root of the real determinant. So, any automorphism of P_1 or P_2 given by $z \in \mathbb{C}^*$ will change the reduced norm semi-invariant by $|z|^2$ which is the norm of z. Also, any automorphism of P_3 given by $h \in \mathbb{H}^*$ will change $\overline{\sigma}_\beta$ by $|h|^2 = \overline{n}(h)$. So, the reduced weight is $(1,1,1)$.

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