Categories of noncrossing partitions

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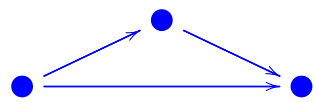
Topology of categories

The classifying space of a small category $\mathcal{C}$ is a union of simplices Δ^k:

$$B\mathcal{C} = \coprod_{X_0 \to X_1 \to \cdots X_k} \Delta^k / \sim$$

$B\mathcal{C}$ has one vertex for every object, one edge for every morphism, one triangle Δ^2 for every commuting triangle



$X = K(G, 1)$ means $\pi_1 X = G$ and universal covering space is contractible: $\widetilde{X} \simeq *$.

Two categories of noncrossing partitions

1. $\mathcal{NP}$: Objects: Noncrossing partitions

Morphisms: (unions of) Binary trees

Constructs $K(\pi, 1)$ for picture group of A_n.

2. $\mathcal{HK}$: Hubery-Krause category of non-crossing partitions.

Q: What are they and how are they related?

A: There is a third category related to both.

$$\mathcal{NP} \to \mathcal{IT} \to \mathcal{HK}$$

3. $\mathcal{IT}$: Cluster morphism category

(This category has clusters as morphisms.)

Co-authors

I-Orr-Todorov-Weyman We introduce the picture group $G(Q)$ for Dynkin quivers Q and we compute its cohomology in type A_n.

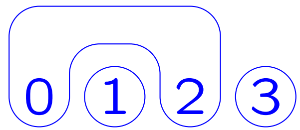
I-Todorov We construct a category $\mathcal{G}(Q)$ for any modulated quiver Q so that, when Q is Dynkin, $B\mathcal{G}(Q) = K(G(Q), 1)$.

I In the special case of $Q = A_n$ with straight orientation, I gave a combinatorial construction of the category $\mathcal{G}(A_n)$.

1st category of noncrossing partitions $\mathcal{NP}$

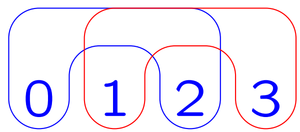
Objects: Noncrossing partitions of the ordered set.

$$[n] = \{0, 1, 2, \cdots, n\}$$

Example of noncrossing partition: 

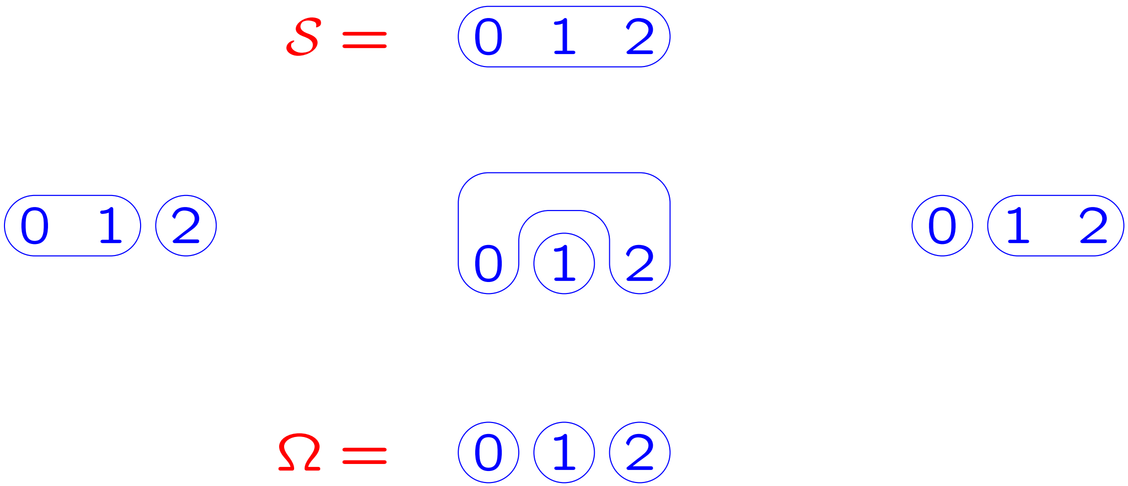
This is a partition of $[3]$ with three parts:

$$\{0, 2\}, \{1\}, \{3\}$$

Here $\{0, 2\}, \{1, 3\}$ is forbidden: 

This is "crossing".

All five objects of $\mathcal{NP}[2]$:



Morphisms in $\mathcal{NP}[2]$

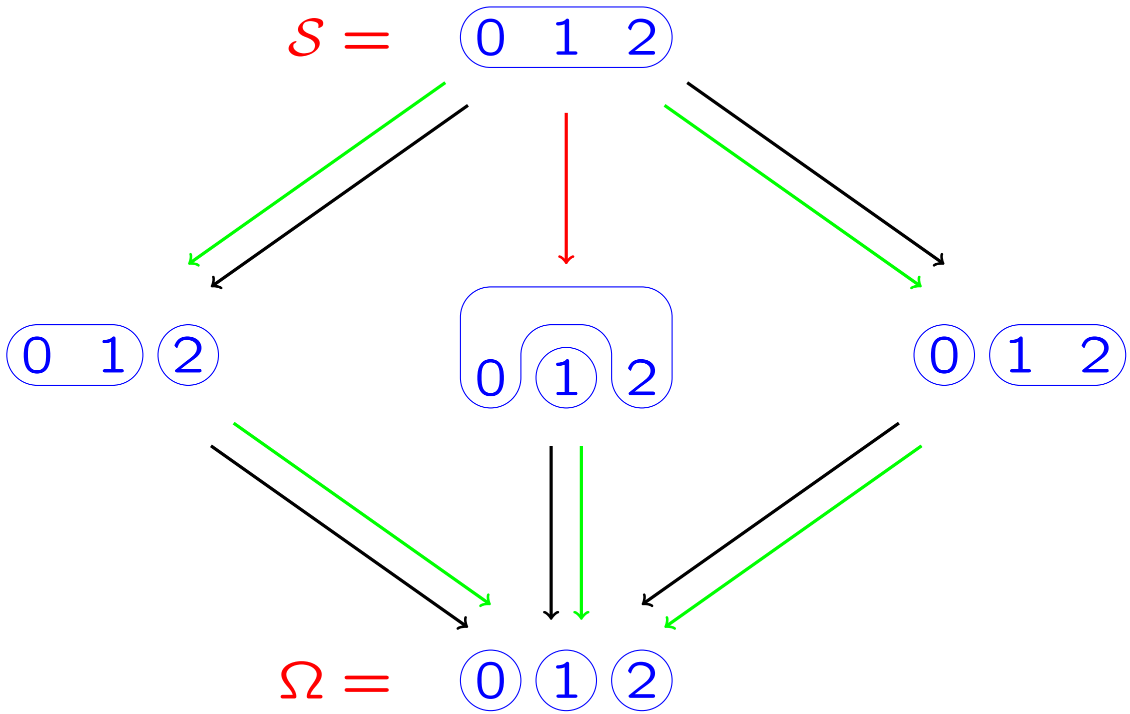
Morphisms are given by "vertical refinement" : When you pull $\{0,1\}$, $\{2\}$ apart, you have to say which is on top. So, there are two morphisms:

$$\{0,1,2\} \rightrightarrows \{0,1\},\{2\}$$

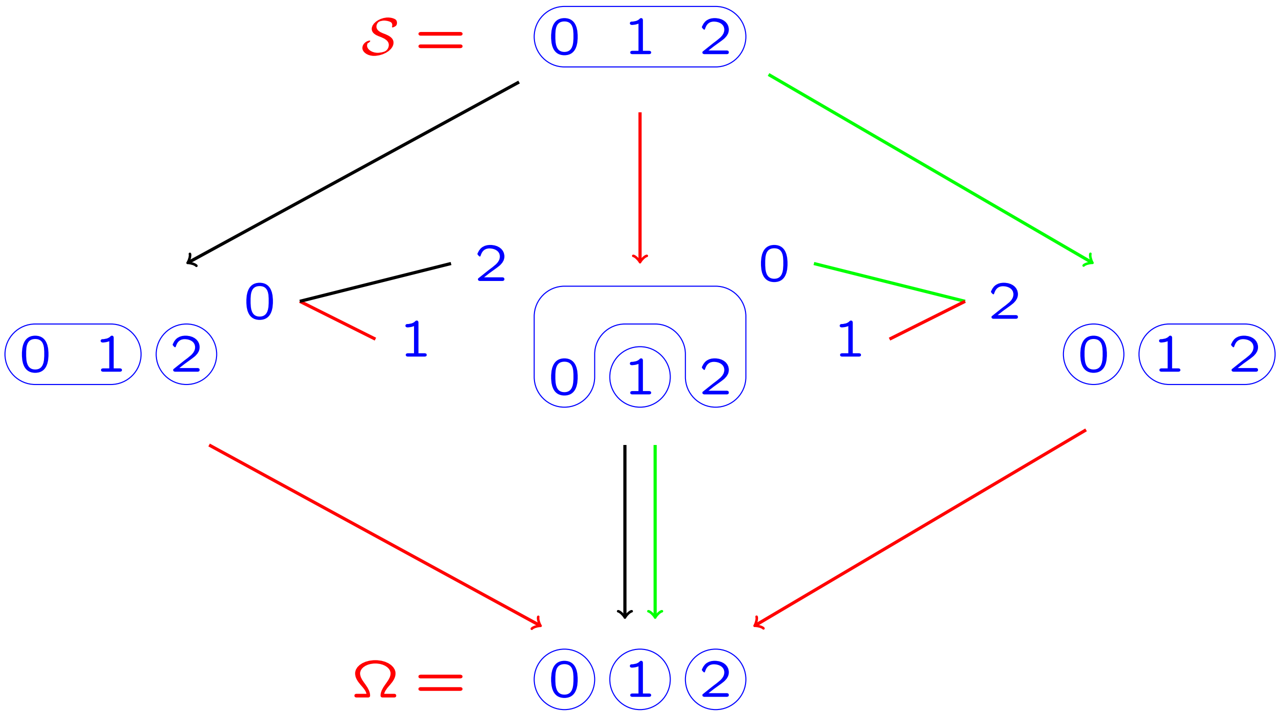
But, when $\{1\}$ is pulled out, it must be underneath. So, there is only one morphism:

$$\{0,1,2\} \longrightarrow \{0,2\},\{1\}$$

Irreducible morphisms in $\mathcal{NP}[2]$:



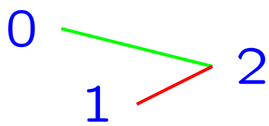
Two of the morphisms $\mathcal{S} \to \Omega$:



Green is green

Green edge i — j is

c-vector $e_{i+1} + e_{i+2} + \cdots + e_j$



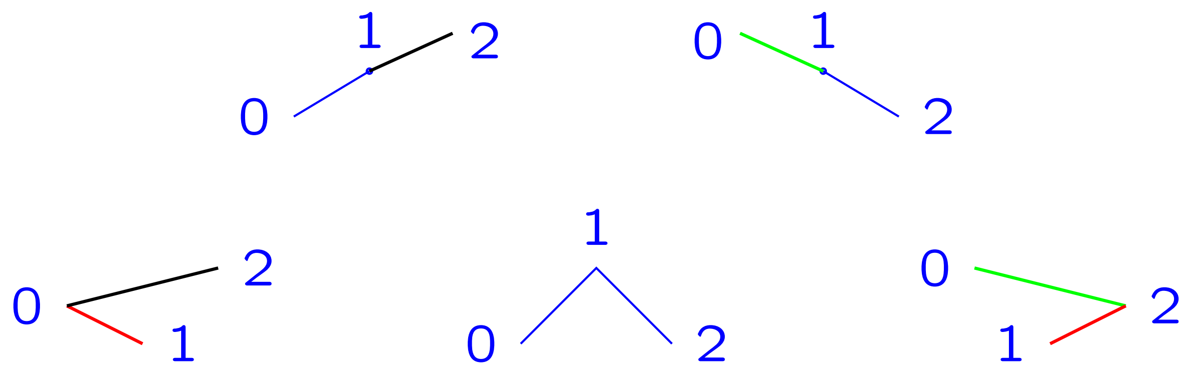
$$C = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = C^{-1}$$

Corresponding cluster are negative rows of

$$C^{-1}E^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

So, cluster is $P_1[1], S_2$

All five morphisms $\mathcal{S} \to \Omega$



These are the five binary tree structures on the ordered set $\{0, 1, 2\}$.

Geometric realization of $\mathcal{NP}[n]$

THEOREM The geometric realization of the category $\mathcal{NP}[n]$ is

$$B\mathcal{NP}[n] = K(G(A_n), 1)$$

where $G(A_n)$ is the picture group of type A_n (straight orientation). This group has generators: x_{ij}, $0 \le i < j \le n$ and relations:
$[x_{ij}, x_{jk}] = x_{ik}$ and
$[x_{ij}, x_{k\ell}] = 1$ if $\{i, j\}, \{k, \ell\}$ are noncrossing ($i < j < k < \ell$, $k < i < j < \ell$ or ...)

$$[x, y] := y^{-1}xyx^{-1}$$

noncrossing partitions = exceptional sequences.

DEFINITION Suppose that $\mathcal{A} \cong mod$-Λ for some fin dim hereditary algebra Λ. An object E in $\mathcal{A}$ is exceptional if it is indecomposable without self-extensions. A sequence of exceptional objects:

$$E_1, E_2, \cdots, E_k$$

is called exceptional if

$Hom(E_j, E_i) = 0 = Ext(E_j, E_i)$ when $j > i$.

The sequence is complete if k is maximal.

(i.e, $k = n$, the number of vertices in quiver.)

Linearization of exceptional sequences

Exceptional objects E are uniquely determined by their dimension vectors $\underline{\dim}\, E$. A sequence of such vectors $\beta_1, \cdots, \beta_n$ is an exceptional sequence iff

$$\langle \beta_j, \beta_i \rangle = 0$$

for all $i < j$. Hubery and Krause define a category abstractly and show that their category can be described as follows.

Hubery-Krause category: $\mathcal{HK}$

For hereditary algebra Λ, let $K_0(\Lambda) = \mathbb{Z}^n$ with the Euler pairing: $\langle -, - \rangle$ so that

$$\langle \underline{\dim} M, \underline{\dim} N \rangle = \dim Hom(M, N) - \dim Ext(M, N)$$

The objects of $\mathcal{HK}$ are pairs (Γ, E) where $\Gamma = K_0(\Lambda)$ and $E = (\beta_1, \cdots, \beta_n)$ so that $\beta_i \in \Gamma$ are the dimension vectors of a complete exceptional sequence in mod-Λ. For example:

$$(\mathbb{Z}^2, (2e_1, -4e_2))$$

Morphisms in $\mathcal{HK}$

A morphism $(\Gamma_1, E_1) \to (\Gamma_2, E_2)$ is an isometric embedding $\Gamma_1 \to \Gamma_2$ so that the image of E_1 in Γ_2 is, up to sign, an exceptional sequence. Two examples:
Multiplication by -1 is an automorphism of any (Γ, E),

$$Aut(K_0(KA_2)) = \mathbb{Z}/6 = \{id, \phi, \phi^2, \phi^3, \phi^4, \phi^5\}$$

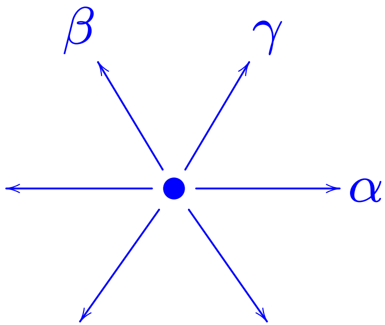
Linearization functor

$\mathcal{E}(K)$ is the category whose
objects are hereditary abelian categories $\mathcal{A}$ over K and whose
morphisms are exact embeddings $\mathcal{A} \hookrightarrow \mathcal{B}$ whose images are extension closed.
Take functor:

$$K_0 : \mathcal{E}(K) \to \mathcal{HK}$$

sending a category $\mathcal{A}$ to its root space and the exceptional sequence given by the dimension vectors of its simple objects in admissible order.

Example: A_2



$\alpha = dim\, S_1$, $\beta = dim\, S_2$, $\gamma = dim\, P_2 \in K_0(KA_2)$.
(S_2, S_1) is an exceptional sequence. So

$$K_0(KA_2) = (\mathbb{Z}^2, (\beta, \alpha)) \in \mathcal{HK}$$

Cluster morphism category $\mathcal{IT}$

Given an exact extension closed embedding

$$j : \mathcal{A} \hookrightarrow \mathcal{B} \quad \text{in } \mathcal{E}(K)$$

There exists a partial cluster tilting object T in the cluster category $\mathcal{C}_{\mathcal{B}}$ of $\mathcal{B}$ so that $\mathcal{A} = T^{\perp}$. But T is not unique.

Adding a choice of T gives a larger category $\widetilde{\mathcal{E}}(K)$ with the same objects as $\mathcal{E}(K)$ but with morphisms

$$(j, T) : \mathcal{A} \hookrightarrow \mathcal{B}$$

DEFINITION $\qquad \mathcal{IT}_K = \widetilde{\mathcal{E}}(K)^{op}$

Composition of $\mathcal{C} \xrightarrow{(i,T)^*} \mathcal{B} \xrightarrow{(j,T')^*} \mathcal{A}$

$(\mathcal{A} \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{C}, \quad i\mathcal{B} = T^{\perp})$

$$(j,T')(i,T) = (i \circ j, \tau_+^{-1}(\tau_+(T'), \tau_+(T))$$

τ_+ is the bijection from ordered clusters to signed exceptional sequences given by

$$\tau_+(v_1, \cdots, v_k) = (w_1, \cdots, w_k)$$

(1) $w_j - v_j$ is a linear combination of v_i for $i > j$.

(2) $\left\langle v_i, w_j \right\rangle = 0$ for $i > j$.

Example: A_2

Let $\mathcal{A} = mod\text{-}KA_2$ where $A_2 : \bullet \leftarrow \bullet$. There is a unique morphism

$$j : 0 \hookrightarrow \mathcal{A}$$

in $\mathcal{E}(K)$. But there are five morphisms

$$(j, T) : 0 \hookrightarrow \mathcal{A}$$

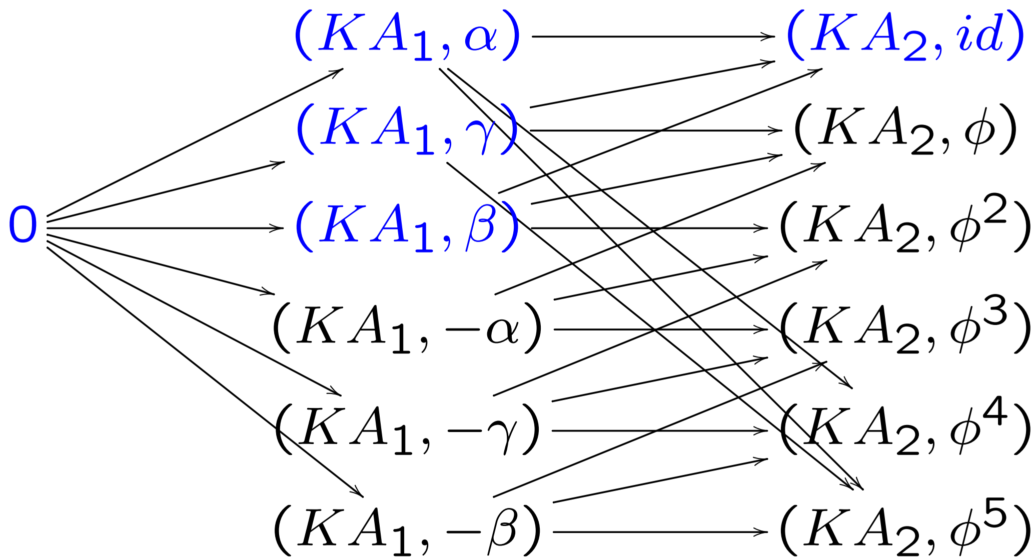
in $\widetilde{\mathcal{E}}(K)$. Let $F : \widetilde{\mathcal{E}}(K) \to \mathcal{E}(K)$ be the forgetful functor.

Comma category

Given a functor $F : \mathcal{C} \to \mathcal{D}$ and an object X in $\mathcal{D}$, the comma category $F \downarrow X$ is the category with objects pairs (Y, f) where $Y \in \mathcal{C}$ and $f : FY \to X$ is a morphism in $\mathcal{D}$. A morphism $(Y, f) \to (Z, g)$ is a morphism $h : Y \to Z$ in $\mathcal{C}$ so that $f = g \circ Fh : FY \to FZ \to X$.

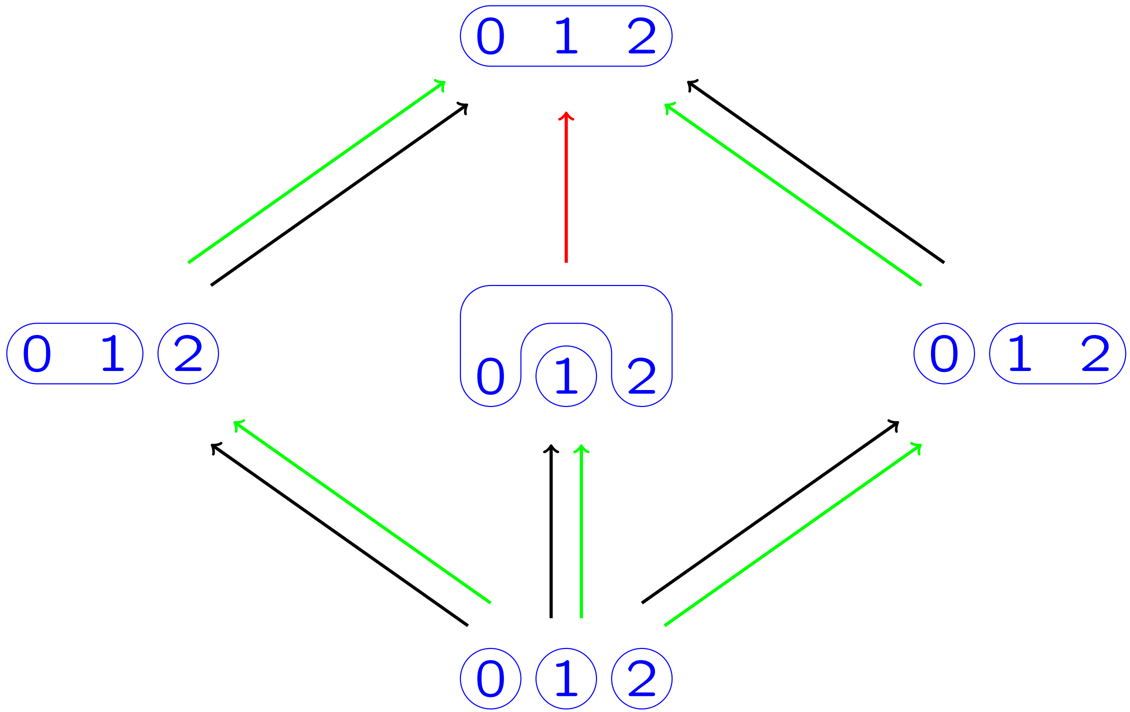
THEOREM The opposite category $\mathcal{NP}[n]^{op}$ is isomorphic to a full subcategory of the comma category $K_0 \circ F \downarrow K_0(KA_n)$.

Example: $K_0 \circ F \downarrow K_0(KA_2)$

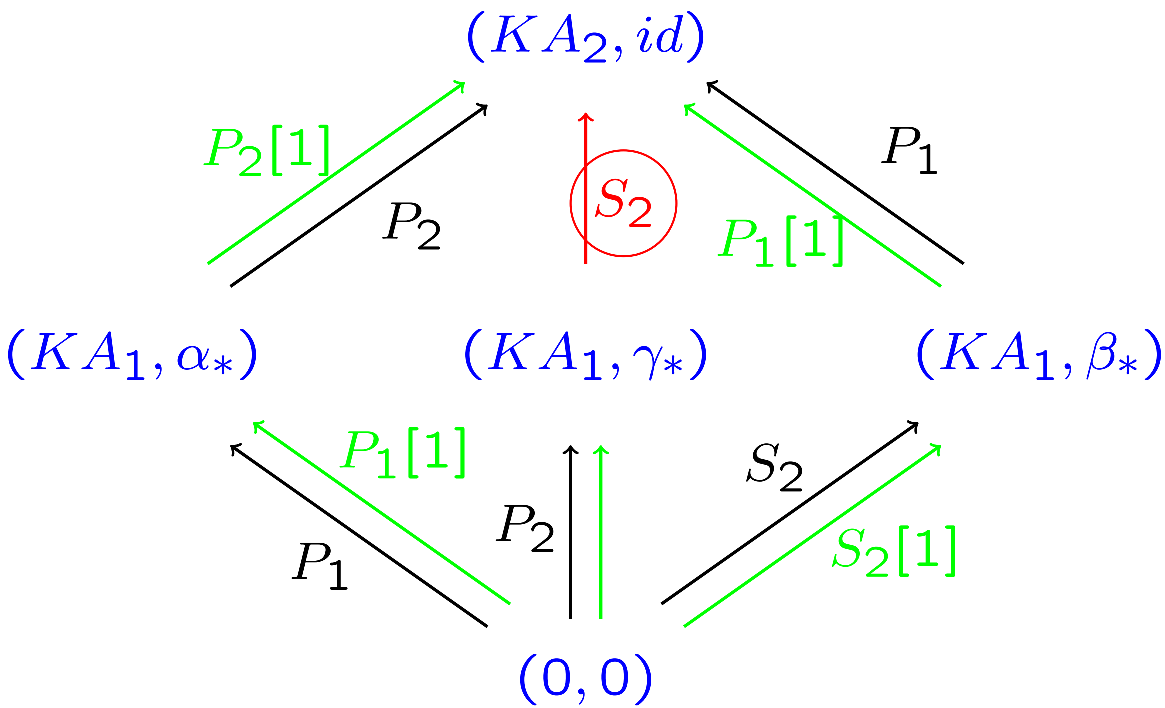


$\mathcal{NP}[2]^{op}$ is the full subcategory in blue.

Irreducible morphisms in $\mathcal{NP}[2]^{op}$:



As full subcategory of $K_0 \downarrow K_0(KA_2)$:



Signed exceptional sequences

There are $n!C(n+1) = 2^1 5 = 10$ signed exceptional sequence:

$$(\pm P_1, \pm P_2), (\pm P_2, S_2), (\pm S_2, \pm P_1)$$

Each corresponds to a total ordering of the five clusters under the twist map τ_+. E.g, for the ordered cluster $(P_1[1], S_2)$:

$$\tau_+(-\alpha, \beta) = (-\alpha - \beta, \beta) = (-\gamma, \beta)$$

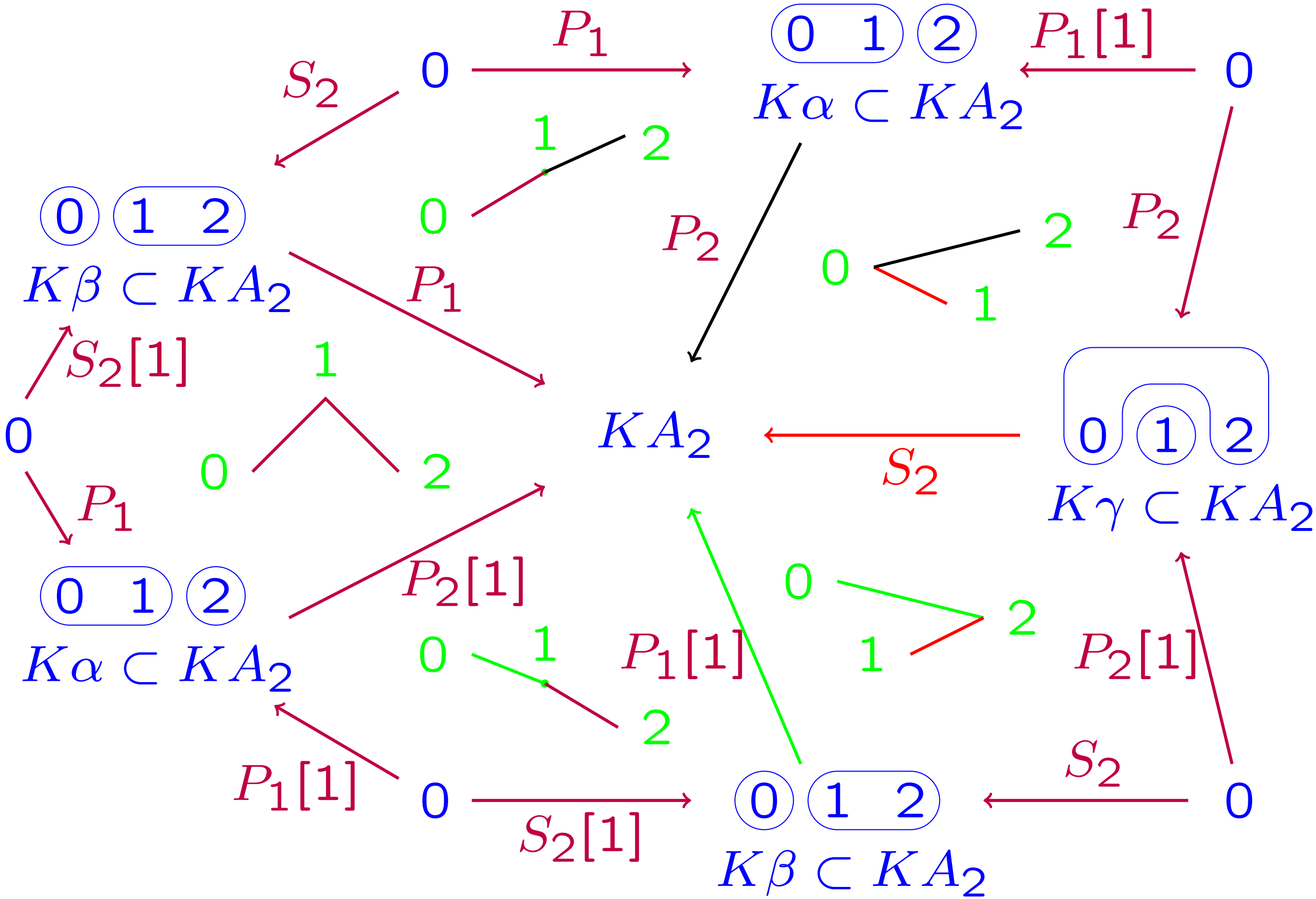
gives signed exceptional sequence $(-P_2, S_2)$.

Repeating the formula:

(1) $w_j - v_j$ is a linear comb. of v_i for $i > j$.

(2) $\left\langle v_i, w_j \right\rangle = 0$ for $i > j$.

Summary of KA_2 example on last slide:


S_2
0
P_1
0 1 2
Kα ⊂ KA_2
P_1[1]
0
1
2
0 1 2
Kβ ⊂ KA_2
0
P_2
P_2
2
0
1
P_1
S_2[1]
1
0
KA_2
S_2
0 1 2
Kγ ⊂ KA_2
0
2
P_1
0 1 2
Kα ⊂ KA_2
P_2[1]
0
2
1
P_1[1]
P_2[1]
0
1
2
P_1[1]
0
S_2[1]
0 1 2
Kβ ⊂ KA_2
S_2
0