LINEARITY OF STABILITY CONDITIONS

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Abstract. We study different concepts of stability for modules over a finite dimensional algebra: linear stability, given by a "central charge", and nonlinear stability given by the wall-crossing sequence of a "green path". Two other concepts, finite Harder-Narasimhan stratification of the module category and maximal forward hom-orthogonal sequences of Schurian modules, which are always equivalent to each other, are shown to be equivalent to nonlinear stability and to a maximal green sequence, defined using Fomin-Zelevinsky quiver mutation, in the case the algebra is hereditary.

This is the first of a series of three papers whose purpose is to determine all maximal green sequences of maximal length for quivers of affine type $\tilde{A}$ and determine which are linear. The complete answer will be given in the final paper [1].

Stability conditions and Harder-Narasimhan filtrations are very active areas of research. Some interesting examples are: [30], [14], [13], [23], [31], [26] and, by the author, [6], [17], [18], [19]. Key references are [24], [29], [15], [3], [4], [28], [22], [25], [10]. Stability functions on quiver representations were introduced by King [24]. This was generalized to abelian categories by Rudakov [29] who also proved the Harder-Narasimhan property [15] under a finiteness property which includes categories of vector bundles which he had been studying earlier and representations of finite dimensional algebras which we consider in this paper. Bridgeland [3] extended linear stability conditions and HN-filtrations to triangulated categories. Reineke [28] showed that the quantum Donaldson-Thomas invariant can be computed using a linear stability function. He wrote it as a sum of terms one for each stable modules of the stability condition. So, it became important to know which sets of modules are given by linear stability conditions. In [22] Keller uses nonlinear stability conditions (equivalent to maximal green sequences for quivers with potential) to give a formula for the refined Donaldson-Thomas invariants of Kontsevich-Soibelman [25]. Bridgeland considers nonlinear stability conditions in more general contexts in [4].

Derksen and Weyman showed that stability conditions, in terms of semi-invariants, can be used to obtain canonical representations of quivers [11] and they also used it to give a new proof of the saturation conjecture for Littlewood-Richardson coefficients [10]. The Stability Theorem in [10] gives the precise relation between semi-invariants and stability conditions. This was later extended to the "virtual Stability Theorem" in [17] and, in the modulated case, in [18]. In [6] these semi-invariant stability conditions were used to prove two conjectures about maximal green sequences. And they will be used in (b) below.

Let Λ be a finite dimensional algebra over a field K. We will always use n to denote the number of simple Λ-modules. Then every Λ-module M has *dimension vector*

$$\underline{\dim}\, M := (x_1, \cdots, x_n) \in \mathbb{N}^n \subset \mathbb{Z}^n$$

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where x_i is the number of times the ith simple Λ-module S_i occurs in the composition series of M. We consider five different concepts of stability for Λ-modules and whether these stability conditions are linear and/or green and/or finite.

(a) Stability functions $Z_t : K_0\Lambda = \mathbb{Z}^n \to \mathbb{C}$.
(b) Wall crossing sequences $D(M_i)$ of a "generic path" $\gamma : \mathbb{R} \to \mathbb{R}^n$.
(c) Harder-Narasimhan stratifications of *mod*-Λ (also called HN-systems).
(d) Maximal forward hom-orthogonal sequences of Schurian modules. (Theorem 0.1(d)) A module M is called *Schurian* if its endomorphism ring is a division algebra.
(e) Sequences of c-vectors for "reddening sequences" given by Fomin-Zelevinsky mutation [12], [22].

These five stability concepts fall into three sets.

(1) (a) and (b) are easily seen to be equivalent in both linear and nonlinear cases. Furthermore, $Z_\bullet$ is "green" if and only if the corresponding wall crossing sequence is green where *green* means the directional velocity of the path γ_Z in the direction $\underline{\dim}\, M_i$ is positive whenever $\gamma_Z(t_i) \in D(M_i)$ (Definition 2.9).
(2) (c) and (d) are show to be equivalent in the finite case (when there are only finitely many strata in the HN-stratification and only one Schurian module in each stratum). Also, (a) implies (c) in the green case.
(3) Fomin-Zelevinsky mutation (e) only makes sense when Λ is a cluster-tilted algebra. This includes hereditary algebras. For these algebras, (e) is equivalent to the finite case of (b). In the green case (e) is equivalent to (c) and (d).

To summarize:

$$(a) \Longleftrightarrow (b) \xrightarrow{green} (c) \overset{finite}{\Longleftrightarrow} (d) \xrightarrow{hereditary} (e) \underset{finite}{\overset{hereditary}{\Longleftrightarrow}} (b)$$

In the finite, green, hereditary case, all five conditions are equivalent. So, we have five equivalent ways to describe the same sequence of Schurian Λ-modules. Furthermore, these modules will be uniquely determined by their dimension vectors (up to isomorphism).

Theorem 0.1. *Let Λ be a finite dimensional hereditary algebra over a field K. Let $\beta_1, \cdots, \beta_m \in \mathbb{N}^n$ be any finite sequence of nonzero, nonnegative integer vectors. Then the following are equivalent.*

(a) *There is a nonlinear stability function $Z_t : K_0\Lambda \to \mathbb{C}$ which is green and has exactly m semistable pairs (M_i, t_i) with $t_1 < t_2 < \cdots < t_m$ so that $\underline{\dim}\, M_i = \beta_i$ for all i.*
(b) *There is a generic green path $\gamma : \mathbb{R} \to \mathbb{R}^n$ which crosses the walls $D(M_i)$, $i = 1, \cdots, m$ in that order, and no other walls, so that $\underline{\dim}\, M_i = \beta_i$ for all i.*
(c) *There exist Λ-modules $M_1, \cdots, M_m$ with $\underline{\dim}\, M_i = \beta_i$ which form a finite Harder-Narasimhan system for Λ.*
(d) *There exist Schurian Λ-modules $M_1, \cdots, M_m$ with $\underline{\dim}\, M_i = \beta_i$ so that*
  (1) $\mathrm{Hom}_\Lambda(M_i, M_j) = 0$ *for $i < j$.*
  (2) *No other modules can be inserted into the sequence preserving (1).*
(e) *There is a maximal green sequence for Λ of length m whose ith mutation is at the c-vector β_i.*

The equivalent $(a) \Leftrightarrow (b)$ is Proposition 3.5, a special case of Theorem 2.13. Theorem 3.8 proves $(c) \Leftrightarrow (d)$. Theorem 4.6 shows that $(a) \Leftrightarrow (e)$. Theorem 4.8 shows $(d) \Leftrightarrow (e)$. Much of this is well-known. The main new idea in this paper is (d) which is a very useful formulation of stability which will be used in the next paper [19] to obtain new results about maximal green sequences of maximal length for cluster-tilted algebras. (In [19] it is shown that (b), (d) and (e) are equivalent for cluster-tilted algebras of finite type over an algebraically closed field.)

This paper is the first of three papers motivated by the question of which stability conditions are "linear". By this we mean that it is given by a linear stability function $Z : K_0\Lambda \to \mathbb{C}$ also called a "central charge". This question originates in a conjecture by Reineke [28] who asks: For any Dynkin quiver Q, is there is a (classical) slope function μ whose corresponding central charge Z makes all KQ-modules stable? See Remark 1.2. Qiu partially solved this problem in [26] where he shows that, for some orientation for each Dynkin, there is a central charge making all indecomposable modules stable.

This series of paper addresses the extension of Reineke's question to a hereditary algebra Λ of affine type $\tilde{A}_{n-1}$ and to cluster-tilted algebras of finite type. We know by [5], [6] that there are only finitely many maximal green sequences. So, there is a longest one. Using all the equivalent formulations, the problem now comes in three parts.

(1) Find the maximum length of all maximal green sequences for Λ. Equivalently, find the maximum size of a finite HN-system for Λ. Call this L.
(2) Describe all L element sets of Λ-modules which can be arranged into a maximal hom-orthogonal sequence.
(3) Which of these sets is the set of stable modules of a linear stability function $Z : K_0\Lambda \to \mathbb{C}$?

In [1] we completely solve this problem in the case $\tilde{A}_{a,b}$, the tame hereditary algebra given by a cyclic quiver with a arrows pointed one way and $b = n - a$ arrows pointed the other way. For example, $L = \binom{n}{2} + ab$. When $(a, b) = (n - 1, 1)$ this is already known [21]. The case $(a, b) = (n, 0)$ is also solved in [1]. This is cluster-tilted of type D_n. In this case $L = \binom{n}{2} + n - 1$, there are n sets and all are given by linear stability functions.

The purpose of the present paper and the next [19] is to lay the foundations for these result. This paper addresses different notions of stability for hereditary algebras and [19] addresses different notions of stability for cluster-tilted algebras of finite type.

Contents of the paper:

Section 1. We discuss the definition of a linear stability condition for representations of a finite dimensional algebra, or, more generally, for nilpotent representations of any modulated quiver. It is immediate (Theorem 1.4) that a linear stability function Z corresponds to a linear path through semistability sets $D(M)$, also called *walls*, for Z-semistable modules M. We also go through one example, the cyclic quiver with three vertices module rad^k for various $k \geq 2$.

Section 2. We define nonlinear stability functions $Z_\bullet$. The corresponding nonlinear paths cross the walls $D(M)$ for $Z_\bullet$-semistable modules M. Conversely, we show that any "reddening path" (Definition 2.12) comes form a nonlinear stability function. We explain how a "green path" gives a Harder-Narasimhan stratification of the module category.

Section 3. We define a "finite HN-system" (Definition 3.3) for *mod*-Λ to be a special case of a finite HN-stratification and we show that it is equivalent to a "maximal forward hom-orthogonal sequence" (Definition 3.7) of Schurian Λ-modules $M_1, \cdots, M_m$. We observe that a finite, green, nonlinear stability function $Z_\bullet$ gives such a finite HN-system. This is the implication $finite\,green\,(a) \Rightarrow finite\,(c) \Leftrightarrow (d)$ mentioned above.

Section 4. We show that, for any hereditary algebra Λ, the finite nonlinear stability functions give "reddening sequences" and all reddening sequences are given in this way.

As a special case, finite green stability functions give "maximal green sequences" and all maximal green sequences are given in that way. We use the wall-crossing description of maximal green sequences from [18] to make this correspondence. We also show that, in the hereditary case, maximal green sequences are equivalent to maximal forward hom-orthogonal sequences of Schurian modules. This shows that all five notions of stability in Theorem 0.1 above are equivalent in the finite, green hereditary case.

Section 5 contains proofs of some of the key theorems and lemmas. Section 5.1 gives a short proof of the well-known fact that $\mathcal{W}(S)$, the full subcategory of mod-Λ of all M so that $S \subseteq D(M)$, is a wide subcategory of mod-Λ for any subset $S \subset \mathbb{R}^n$. Section 5.2 proves that any green path gives a Harder-Narasimhan filtration of mod-Λ for any finite dimensional algebra Λ. Section 5.3 proves the crucial Lemma A which says that, for any hereditary Λ, the semistability set $D(M)$ for a nonexceptional module M does not meet any codimension 0 or 1 simplex in the cluster complex. This is needed to show that the definition of $D(M)$ given in [18] essentially agrees with the one given here.

Finally, we should mention that many of the results of section 5 have been extended to arbitrary finite dimensional algebras using τ-tilting by Brüstle, Smith and Treffinger [7].

1. Linear stability functions

Linear stability conditions are given in two equivalent ways: by a "central charge" $Z : K_0\Lambda \to \mathbb{C}$ or by a linear "green path" $\gamma : \mathbb{R} \to \mathbb{R}^n$. A Λ-module M is Z-semistable if and only if the corresponding path γ_Z crosses the semistability set $D(M)$. Following a suggestion by Yang-Hui He, we treat Λ as a modulated quiver with unspecified relations. This is equivalent to considering nilpotent representations of the quiver.

1.1. **Nilpotent representations.** Let K be any field and let $\mathcal{M} = (\{F_i\}, \{M_{ij}\})$ be a modulated quiver over K, possibly with oriented cycles. This is define by a finite set $\{F_1, \cdots, F_n\}$ of finite dimensional division algebras over K together with finite dimensional F_i-F_j bimodules M_{ij}. A finite dimensional *representation* X of $\mathcal{M}$ is defined to be a collection of finite dimensional right F_i-vector spaces X_i together with F_j linear maps

$$X_i \otimes_{F_i} M_{ij} \to X_j$$

for all i, j. Representations of $\mathcal{M}$ are the same as right modules over the *tensor algebra* $T\mathcal{M}$ of $\mathcal{M}$ which is defined to be the direct sum $T\mathcal{M} = \coprod_{k\geq 0} T_k\mathcal{M}$ where $T_0\mathcal{M} = \prod F_i$ and, for $k \geq 1$, $T_k\mathcal{M}$ is the direct sum of all tensor paths of length k:

$$T_k\mathcal{M} = \coprod M_{j_0 j_1} \otimes_{F_{j_1}} M_{j_1 j_2} \otimes_{F_{j_2}} \cdots \otimes_{F_{j_{k-1}}} M_{j_{k-1} j_k}$$

Let $R\mathcal{M} = \coprod_{k\geq 1} T_k\mathcal{M}$ and let $nmod\mathcal{M}$ be the category of those finitely generated right $T\mathcal{M}$ modules on which $R\mathcal{M}$ act nilpotent. We call such modules *nilpotent* $\mathcal{M}$-modules and we also refer to the corresponding representations of $\mathcal{M}$ as *nilpotent*.

Each nilpotent module X is a module over $T\mathcal{M}/R\mathcal{M}^m$ for some m and thus a module over a finite dimensional algebra over K. For example, if $\mathcal{M}$ is given by a loop a at a single vertex then $T\mathcal{M} = K[a]$ and $nmod\mathcal{M}$ is the category of all finite dimensional vector spaces X together with a nilpotent endomorphism a. Then $nmodK[a]$ has only one simple module corresponding to the maximal ideal $R = (a)$. In general, the category $nmod\mathcal{M}$ has only n simple modules $S_1, \cdots, S_n$ which are one-dimensional over $F_1, \cdots, F_n$ respectively.

A nilpotent module X gives a nilpotent representation of the modulated quiver $\mathcal{M}$ in a standard way by letting X_i be the F_i-vector space $X_i = Xe_i$ where e_i is unity in F_i and, for every i, j taking $X_i \otimes M_{ij} \to X_j$ to be the F_j-linear map given by the action of $M_{ij} \subseteq T\mathcal{M}$ on X.

The category *nmod-*$\mathcal{M}$ is based on a suggestion by Yang-Hui He at a conference at the Chinese University of Hong Kong. This category is the union or colimit:

$$nmod\text{-}\mathcal{M} = \bigcup mod\text{-}T\mathcal{M}/J = \operatorname{colim} mod\text{-}T\mathcal{M}/J$$

of the module categories *mod-*$T\mathcal{M}/J$ of all finite dimensional algebras of the form $\Lambda = T\mathcal{M}/J$ where J is an admissible ideal $J \subseteq R\mathcal{M}^2$ where *admissible* means that J is a two sided ideal in $T\mathcal{M}$ so that $R\mathcal{M}^k \subseteq J \subseteq R\mathcal{M}^2$ for some $k \geq 2$. Every nilpotent representation of $\mathcal{M}$ is a $T\mathcal{M}/J$-module for some J. Conversely, every module over $\Lambda = T\mathcal{M}/J$ is an object of *nmod-*$\mathcal{M}$ and every subquotient module of such a nilpotent module is also a Λ-module.

1.2. **Linear stability function** Z**.** Let $K_0\mathcal{M}$ be the Grothendieck group of the category *nmod-*$\mathcal{M}$. Then $K_0\mathcal{M} \cong \mathbb{Z}^n$ where we identify $[M] \in K_0\mathcal{M}$ with $\underline{\dim}\, M \in \mathbb{Z}^n$ which is uniquely determined by the dot product equation

$$\dim_K M = (\dim_K S_1, \cdots, \dim_K S_n) \cdot \underline{\dim}\, M. \tag{1.1}$$

Note that $K_0\mathcal{M} = K_0\Lambda$ for any $\Lambda = T\mathcal{M}/J$.

Definition 1.1. A *linear stability function* for $\mathcal{M}$ is an additive map:

$$Z : K_0\mathcal{M} \to \mathbb{C}$$

which we write as:

$$Z(x) = a \cdot x + ib \cdot x = r(x)e^{i\theta(x)}$$

where $a \in \mathbb{R}^n$, $b \in (0, \infty)^n$ are fixed and $0 < \theta(x) < \pi$. For any $M \in nmod\text{-}\mathcal{M}$, let

$$\mu(M) = \mu_Z(M) := \frac{a \cdot \underline{\dim}\, M}{b \cdot \underline{\dim}\, M} = \cot\theta(M)$$

where $\theta(M) = \theta(\underline{\dim}\, M)$. This is called the *slope* of M. Note that $\mu(M)$ is a monotonically decreasing function of $\theta(M)$. A nonzero nilpotent module M is called Z*-semistable*, resp. Z*-stable*, if $\mu(M') \geq \mu(M)$, resp. $\mu(M') > \mu(M)$, for all nonzero submodules $M' \subsetneq M$.

Remark 1.2. An important special case is the *classical choice* $b = (\dim_K S_1, \cdots, \dim_K S_n)$. For any $a \in \mathbb{R}^n$ the resulting function $\mu(M) = a \cdot \underline{\dim}\, M/\dim_K M$ is a *classical slope function.* Reineke's original conjecture [28] is that, for every Dynkin quiver, there is a classical slope function making all indecomposable modules stable.

We observe that simple modules are always stable. Often the restriction on θ is taken to be $0 \leq \theta < \pi$ and $Z : K_0\Lambda \to \mathbb{C}$ is called a *central charge.* We take $\theta > 0$ so that the slope function μ is defined.

1.3. **Linear green path** γ**.** For any linear stability function $Z : K_0(\mathcal{M}) \to \mathbb{C}$ we will show that there is a corresponding linear path $\gamma_Z : \mathbb{R} \to \mathbb{R}^n$ which crosses the wall $D(M)$, defined below, whenever M is Z-semistable.

Definition 1.3. For any nilpotent module $M \in nmod\text{-}\mathcal{M}$, we define $H(M)$ to be the hyperplane in $\mathbb{R}^n$ perpendicular to $\underline{\dim}\, M$:

$$H(M) := \{x \in \mathbb{R}^n \mid x \cdot \underline{\dim}\, M = 0\}$$

The *semistability set* of M is defined to the subset $D(M) \subseteq H(M)$ given by

$$D(M) = \{x \in H(M) \mid x \cdot \underline{\dim}\, M' \leq 0 \text{ for all } M' \subseteq M\}$$

The *interior* $int\, D(M)$ of $D(M)$ is defined to be the subset of all $x \in D(M)$ so that $x \cdot \underline{\dim}\, M' < 0$ for all nonzero proper submodules $M' \subsetneq M$. The *boundary* of $D(M)$ is the complement: $\partial D(M) := D(M) - int\, D(M)$.

Given a linear stability function $Z(x) = a \cdot x + ib \cdot x$, the *corresponding path* $\gamma_Z : \mathbb{R} \to \mathbb{R}^n$ is the linear path given by

$$\gamma_Z(t) = tb - a.$$

Theorem 1.4. *Let M be a nilpotent module. Then M is Z-semistable, resp. Z-stable, if and only if $\gamma_Z(t) \in D(M)$, resp. $int\, D(M)$, for some $t \in R$. Furthermore, in that case, $t = \mu_Z(M)$.*

Proof. We prove the Z-semistable statement. $\gamma_Z(t_0)$ lies in $H(M)$ if and only if $t_0 = \mu_Z(M)$. For any $M' \subset M$ let $t_1 = \mu_Z(M')$. Then $\gamma_Z(t_1) \cdot \underline{\dim}\, M' = 0$. So,

$$\gamma_Z(t_0) \cdot \underline{\dim}\, M' = (t_0 - t_1) b \cdot \underline{\dim}\, M'$$

Since $b \in \mathbb{R}^n$ has positive coordinates, this quantity is ≤ 0 for all $M' \subset M$, making $\gamma_Z(t_0) \in D(M)$, iff $t_1 = \mu(M') > t_0 = \mu(M)$ for all $M' \subset M$, i.e., M is Z-semistable. □

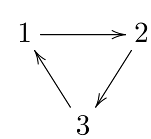
Corollary 1.5. *If M is Z-stable then M is Schurian.*

Proof. If M has a nontrivial endomorphism with image $M' \subsetneq M$ then $D(M)$ is contained in $D(M')$. So, $int\, D(M)$ is empty. By the Theorem above, M cannot be Z-stable. □

1.4. **Example of cyclic A_3 with three possible algebras.** When $\Lambda = T\mathcal{M}/J$ is a modulated quiver modulo a fixed admissible ideal, Theorem 1.4 still holds, the only difference being that not all nilpotent modules M will be Λ-modules since they might not be zero on the ideal J. But, if M is a Λ-module then so are all of its subquotient modules. Therefore, the set $D(M)$ will be the same. It will be the set of all $x \in \mathbb{R}^n$ so that $x \cdot \underline{\dim}\, M = 0$ and $x \cdot \underline{\dim}\, M' \leq 0$ for all Λ-submodules M' of M. We give an example where $\mathcal{M}$ is the simply laced quiver Q given by a single oriented cycle of length 3.

Recall that the modulated quiver given by a directed graph Q has division algebras all equal to the ground field K and bimodules M_{ij} equal to $K^{m_{ij}}$ where m_{ij} is the number of arrows from i to j. The tensor algebra is called the *path algebra* of Q and denoted KQ. As a vector space it has a basis given by the paths including those of length zero which are the vertices of Q.

$Q:$



We consider the algebra $\Lambda_k = KQ/R^{k+1}$ where $R = RQ$ is the ideal generated by all paths of length ≥ 1. These are string algebras and all indecomposable modules are string modules [9]. Thus the algebra Λ_k has $3(k+1)$ indecomposable modules (up to isomorphism) given by paths of length $\leq k$ starting at any vertex.

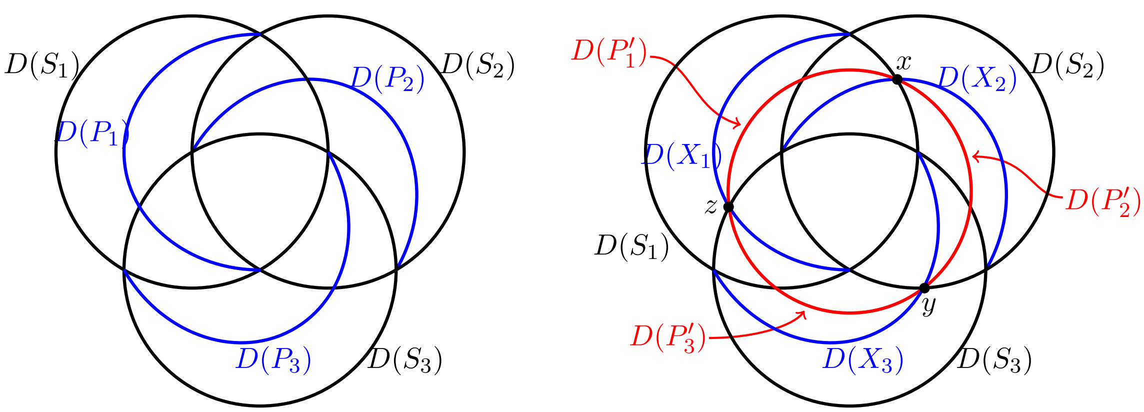
In Figure 1, the left side shows the semistability sets for all six modules over $\Lambda_1 = KQ/R^2$ and the right side shows all nine modules over $\Lambda_2 = KQ/R^3$ and all 12 modules over $\Lambda_3 = KQ/R^4$.

Any Λ_k-module is a module over Λ_j for all $j > k$. Therefore, the set $\bigcup D(M)$ for Λ_1 is a subset of $\bigcup D(M)$ for Λ_2. It will turn out that $\bigcup D(M)$ for Λ_3 is equal to that of Λ_2 which is the reason that the same figure (the right side of Figure 1) illustrates both. The figure shows the stereographic projection of the intersections of $D(M)$'s with the unit sphere S^2.

For $\Lambda_1 = KQ/R^2$, $D(S_i) = H(S_i)$ are the hyperplanes perpendicular to the unit vectors $\underline{\dim}\, S_i = e_i$. These intersect S^2 in great circles which stereographically project to three circles in $\mathbb{R}^2$. Each set $D(P_i) = D(X_i)$ lies inside the $D(S_i)$ circle and outside the $D(S_{i+1})$ circle (with index modulo 3) since S_i is a quotient of P_i and S_{i+1} is a submodule.

For $\Lambda_2 = KQ/R^3$, we have projective modules P_i' of length 3. The sets $D(P_i')$ all lie in the hyperplane $(1,1,1)^\perp$. But $D(P_1')$ lies inside $D(S_1)$ and outside $D(S_3)$ since $S_3 \subset P_1'$ and $S_1 = P_1'/X_2$. So, $D(P_1')$ is the part of the red circle between points x and z. Similarly $D(P_2')$ is the part of the red circle from x to y and $D(P_3')$ is the part from y to z.

For $\Lambda_3 = KQ/R^4$, the projective modules P_i'' have length 4 and they have the same simple S_i on the top and bottom. This forces $D(P_i'')$ to lie in the line $H(P_i'') \cap H(S_i) = (1,1,1)^\perp \cap e_i^\perp$. Also, X_i is a quotient of P_i'', so $D(P_i'')$ lies on the positive side of the hyperplane $D(X_i)$. In the figure we get the single points x for $D(P_1'')$, y for $D(P_2'')$ and z for $D(P_3'')$.



6 $D(M)$'s for $\Lambda_1 = KQ/R^2$ 9 $D(M)$'s for $\Lambda_2 = KQ/R^3$

Figure 1. Stereographic projections of the intersections of $D(M)$'s with the unit sphere $S^2 \subset \mathbb{R}^3$. The modules X_i over Λ_2 are equal to the projective Λ_1-modules P_i. The projective Λ_2-modules P_i' have identical dimension vectors $\underline{\dim}\, P_i' = (1,1,1)$. And, the union of $D(P_i')$, $i = 1,2,3$, form the hyperplane $(1,1,1)^\perp$ which is the red circle in the figure.

2. Nonlinear stability functions

In this section we define nonlinear stability functions $Z_t : K_0(\mathcal{M}) \to \mathbb{C}$, $t \in \mathbb{R}$, and show that the corresponding nonlinear paths $\gamma_Z : \mathbb{R} \to \mathbb{R}^n$ cross the semistability sets $D(M)$ of the $Z_\bullet$-semistable modules. When $Z_\bullet$ is "green" we obtain a Harder-Narasimhan stratification of the category *mod*-Λ or *nmod*-$\mathcal{M}$, depending on point of view. Details of the proof are postponed to Section 5.

2.1. **Definition of nonlinear $Z_\bullet$.**

Definition 2.1. We define a *nonlinear stability function* on the modulated quiver $\mathcal{M}$ to be a smooth (C^1) family of linear functions

$$Z_t : K_0(\mathcal{M}) \cong \mathbb{Z}^n \to \mathbb{C}, \quad t \in \mathbb{R}$$

given by

$$Z_t(x) = a_t \cdot x + b_t \cdot x\sqrt{-1} = r_t(x)e^{i\theta_t(x)}$$

where:

(1) $a_t \in \mathbb{R}^n$, $b_t \in (0,\infty)^n$, $0 < \theta_t(x) < \pi$. Thus, $b_t^k > 0$ for all k, t.
(2) a_t and b_t have velocity 0 for $|t|$ large, giving four constants: $a_\infty, b_\infty, a_{-\infty}$ and $b_{-\infty}$.

For any nilpotent module $M \in$ *nmod*-$\mathcal{M}$ let

$$\mu_t(M) = \cot\theta_t(M) = \frac{a_t \cdot \underline{\dim} M}{b_t \cdot \underline{\dim} M}$$

where $\theta_t(M)$ is short for $\theta_t(\underline{\dim} M)$.

Definition 2.2. Let $Z_\bullet$ be a nonlinear stability function. We say that $M \in$ *nmod*-$\mathcal{M}$ is $Z_\bullet$*-stable/semistable* if, for some $t_0 \in \mathbb{R}$ we have the following.

(1) M is stable/semistable with respect to Z_{t_0}.
(2) $\mu_{t_0}(M) = t_0$.

The pair (M, t_0) will be called a $Z_\bullet$*-stable/semistable pair.* Such a pair (M, t_0) is *green* or *red* if

$$\frac{d}{dt}\mu_t(M)\bigg|_{t=t_0} < 1 \tag{2.1}$$

or > 1, respectively. We say the pair (M, t_0) is *generic* if it is green or red, i.e., the expression (2.1) is not equal to 1. We say that $Z_\bullet$ is *green*, resp. *generic*, if all $Z_\bullet$-semistable pairs are green, resp. generic. We say that $Z_\bullet$ is *green/generic for the algebra* $\Lambda = T\mathcal{M}/J$ if those $Z_\bullet$-semistable pairs (M, t_0) for which M is zero on J (making M a Λ-module) are green/generic.

All linear stability functions are green since $0 < 1$. For nonlinear stability functions, the condition is needed to obtain the Harder-Harasimhan filtration for any module M.

2.2. **Harder-Harasimhan filtration.**

Recall that a *wide subcategory* of an abelian category $\mathcal{A}$ is a full subcategory $\mathcal{W}$ which is closed under direct summands, extensions, kernels and cokernels. We observe that, if $\mathcal{B}$ is an exactly embedded abelian full subcategory of $\mathcal{A}$, then $\mathcal{W} \cap \mathcal{B}$ is a wide subcategory of $\mathcal{B}$.

Lemma 2.3. *Given a nonlinear stability function $Z = Z_\bullet$ and $t_0 \in \mathbb{R}$ let $\mathcal{S}_Z(t_0)$ be the full subcategory of nmod-$\mathcal{M}$ consisting of all nilpotent $\mathcal{M}$-modules M so that (M, t_0) is a Z-semistable pair. Then $\mathcal{S}_Z(t_0)$ is a wide subcategory of nmod-$\mathcal{M}$.*

Proof. This is an easy statement which we verify in the last section 5.1. □

Theorem 2.4. *Let $Z_\bullet$ be green. Then, for any $M \in$ nmod-$\mathcal{M}$, there exists a unique finite sequence $t_1 < t_2 < \cdots < t_m$ and a unique filtration*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = M$$

having the property that $M_k/M_{k-1} \in \mathcal{S}_Z(t_k)$ for all k.

This is the well-known *Harder-Narasimhan (HN) filtration* of M.

Proof. This is a rewording of Theorem 5.13 proved below. □

Definition 2.5. A *Harder-Narasimhan stratification* of *nmod*-$\mathcal{M}$ is defined to be any family of wide subcategories $\{\mathcal{S}_t\}, t \in \mathbb{R}$, in *nmod*-$\mathcal{M}$ satisfying Theorem 2.4 above.

Remark 2.6. We observe that, for the proof of Theorem 5.13, it suffices for $Z_\bullet$ to be green on an algebra $\Lambda = T\mathcal{M}/J$ where J is contained in the annihilator of M (so that M is a Λ-module).

Let $A \in \mathcal{S}_Z(s)$, $B \in \mathcal{S}_Z(t)$ where $s < t$. By considering the HN-filtration of $A \oplus B$ we see that $\mathrm{Hom}(A, B) = 0$. More generally we have the following well-known corollary where, for any connected subset $S \subseteq \mathbb{R}$, $\mathcal{S}_Z(S)$ is the full subcategory of *nmod*-$\mathcal{M}$ so that the numbers $t_1, \cdots, t_m$ all lie in S. We recall that a *torsion pair* is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{A}$ so that

1. An object $A \in \mathcal{A}$ lies in $\mathcal{T}$ if and only if $\mathrm{Hom}_{\mathcal{A}}(A, B) = 0$ for all $B \in \mathcal{F}$.
2. An object $B \in \mathcal{A}$ lies in $\mathcal{F}$ if and only if $\mathrm{Hom}_{\mathcal{A}}(A, B) = 0$ for all $A \in \mathcal{T}$.

Corollary 2.7. *Let $Z_\bullet$ be green. Then, for any $t_0 \in \mathbb{R}$, $\mathcal{S}_Z(-\infty, t_0]$ and $\mathcal{S}_Z(t_0, \infty)$ form a torsion pair as do $\mathcal{S}_Z(-\infty, t_0)$ and $\mathcal{S}_Z[t_0, \infty)$.*

Proof. It is clear that $\mathrm{Hom}(A, B) = 0$ for any $A \in \mathcal{S}_Z(-\infty, t_0]$ and $B \in \mathcal{S}_Z(t_0, \infty)$. Conversely, $M \in \mathcal{S}_Z(t_0, \infty)$ if and only if it does not have a submodule in $\mathcal{S}_Z(-\infty, t_0]$ and $M \in \mathcal{S}_Z(-\infty, t_0]$ iff it does not have a quotient module in $\mathcal{S}_Z(t_0, \infty)$. So, $\mathcal{S}_Z(-\infty, t_0]$ and $\mathcal{S}_Z(t_0, \infty)$ form a torsion pair. The other case is similar. □

2.3. **Comparison with corresponding path.** Given any nonlinear stability function

$$Z_t(x) = a_t \cdot x + b_t \cdot x\sqrt{-1},$$

let $\gamma_Z : \mathbb{R} \to \mathbb{R}^n$ be the path

$$\gamma_Z(t) = tb_t - a_t$$

Then, $\gamma_Z(t) \in H(M)$ iff

$$t = \frac{a_t \cdot \underline{\dim}\, M}{b_t \cdot \underline{\dim}\, M} = \mu_t(M).$$

Lemma 2.8. (1) *$\gamma_Z(t_0) \in D(M)$ if and only if M is Z_{t_0}-semistable.*
(2) *$\gamma_Z(t_0) \in \mathrm{int}\, D(M)$ if and only if M is Z_{t_0}-stable. And, in that case, M is Schurian.*

Proof. This follows from Theorem 1.4 applied to the linear stability function Z_{t_0}. □

Definition 2.9. Let $v(t) = d\gamma_Z(t)/dt$ be the velocity vector of the smooth path γ_Z at time t. If the path $\gamma_Z(t)$ crosses $D(M) \subset H(M)$ at $t = t_0$, we say that the crossing is *green* if $v(t_0) \cdot \underline{\dim}\, M > 0$. We say it is *red* if $v(t_0) \cdot \underline{\dim}\, M < 0$. Thus any transverse intersection is either green or red.

Lemma 2.10. *Suppose that $\gamma(t_0) \in D(M)$. Then the derivative of $\mu_t(M)$ at $t = t_0$ is not equal to 1 if and only if γ crosses the hyperplane $H(M)$ transversely at $t = t_0$. Furthermore the crossing is green/red if and only if the sign of $d\mu_t(M)/dt|_{t=t_0} - 1$ is negative/positive, respectively.*

Proof. The statement is that the following have the same sign for $\beta = \underline{\dim} M$.

$$1 - \frac{d}{dt}\,\mu_t(\beta)|_{t=t_0} = 1 - \frac{(a'_{t_0}\cdot\beta)(b_{t_0}\cdot\beta) - (a_{t_0}\cdot\beta)(b'_{t_0}\cdot\beta)}{(b_t\cdot\beta)^2} \tag{2.2}$$

$$v(t_0)\cdot\beta = \frac{d}{dt}\,\gamma_Z(t)\cdot\beta|_{t=t_0} = b_{t_0}\cdot\beta + t_0\beta'_{t_0}\cdot\beta - a'_{t_0}\cdot\beta \tag{2.3}$$

Substituting $t_0 = \mu_{t_0}(M) = \frac{a_{t_0}\cdot\beta}{b_{t_0}\cdot\beta}$ in (2.3), we see that this second expression is equal to (2.2) times $b_{t_0}\cdot\underline{\dim} M > 0$. So, the two expressions have the same sign. □

Lemma 2.11. *The coordinates of $\gamma_Z(t)$ are all negative for $t << 0$ and they are all positive for $t >> 0$.*

Proof. When $|t|$ is very large, $\gamma_Z(t) = tb_t - a_t$ is dominated by the term tb_t whose coordinates are nonzero with the same sign as t. □

The important properties of γ_Z are summarized by the following definition and theorem which we state for $\Lambda = T\mathcal{M}/J$.

Definition 2.12. A *reddening path* for Λ is defined to be a C^1 path $\gamma : \mathbb{R} \to \mathbb{R}^n$ with the following two properties.

(1) All coordinates of $\gamma(t)$ are negative for $t << 0$ and positive for $t >> 0$.
(2) Whenever γ crosses a semistability set $D(M)$ of a Λ-module M, the crossing is transverse, i.e., the curve γ is not tangent to the hyperplane $H(M)$.

A reddening path will be called a *green path* for Λ if all crossings are green.

Theorem 2.13. *Let $Z_\bullet$ be a nonlinear stability function for Λ. Then $Z_\bullet$ is generic, resp. green, if and only if the corresponding path γ_Z is a reddening path, resp. a green path. Conversely, for any reddening path γ there is a nonlinear $Z_\bullet$ so that $\gamma(t) \in D(M)$ if and only if M is Z_t-semistable, equivalently, if $\gamma_Z(t) \in D(M)$.*

Proof. The relation between $Z_\bullet$ and γ_Z is proved by the lemmas.

To prove the second statement, let γ be any reddening path. Since each $H(M)$ and thus each $D(M)$ contains no points whose coordinates are all positive or all negative, the two tails of an arbitrary reddening path γ can be modified to be stationary so that γ becomes equal to some γ_Z without changing when, where or with what velocity it meets any semistability set $D(M)$ for $M \in$ *mod*-Λ. The second statement follows. □

3. Finite HN-systems and forward hom-orthogonality

In this section we consider stability functions and paths which are finite with respect to a fixed finite dimensional algebra Λ. The resulting HN-stratification of *mod*-Λ is of course finite. We call it a "finite HN-system" for Λ and we show that it is equivalent to a "forward hom-orthogonal sequence".

3.1. **Finite HN-systems.** For any Λ-module M let $\mathcal{E}(M)$ denote the full subcategory of *mod*-Λ of all modules X having a filtration with all subquotients isomorphic to M, i.e., X is an "iterated self-extension" of M. The following is an easy exercise.

Lemma 3.1. *If M is Schurian then $\mathcal{E}(M)$ is an abelian category.* □

Lemma 3.2. *Let M be the unique indecomposable object of a wide subcategory $\mathcal{W}$ of mod-Λ. Then M is Schurian and $\mathcal{W} = \mathcal{E}(M)$.*

Proof. Since $\mathcal{W}$ is closed under extensions and contains M, we have $\mathcal{E}(M) \subseteq \mathcal{W}$. Conversely, let X be the minimal object of $\mathcal{W}$ which is not in $\mathcal{E}(M)$. Let $X_0 \subset X$ be the smallest submodule of X which lies in $\mathcal{W}$. Then X_0 must be Schurian since the image of any nonzero endomorphism of X_0 must also lie in $\mathcal{W}$. By assumption, $X_0 = M$. By minimality of X we have $X/M \in \mathcal{E}(M)$. So, $X \in \mathcal{E}(M)$. We conclude that $\mathcal{W} = \mathcal{E}(M)$. □

Definition 3.3. By a *finite Harder-Narasimhan (HN) system* for *mod*-Λ we mean a finite sequence of abelian subcategories $\mathcal{E}(M_1), \cdots, \mathcal{E}(M_m) \subset$ *mod*-Λ with M_i Schurian which give an HN-stratification of *mod*-Λ. I.e., for any Λ-module X there is a unique filtration

$$0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_m = X$$

so that each X_k/X_{k-1} is in $\mathcal{E}(M_k)$.

A finite HN-system is exactly the kind of HN-stratification that we get from a "finite" nonlinear stability function $Z_\bullet$.

Definition 3.4. A nonlinear stability function $Z_\bullet$ for Λ will be called *finite* if it is generic and there are only finitely many semistable pairs (M_i, t_i), up to isomorphism, with indecomposable M_i and the t_i are all distinct.

A reddening path $\gamma : \mathbb{R} \to \mathbb{R}^n$ will be called *finite* if there are only finitely many real numbers t_i so that $\gamma(t_i)$ lie in some $D(M_i)$ for M_i indecomposable and if, for each t_i, M_i is uniquely determined up to isomorphism.

The statement that, for the semistable pairs (M_i, t_i) of a finite $Z_\bullet$, the t_i are all distinct means that M_i is uniquely determined by t_i. This implies the following.

Proposition 3.5. *Let $Z_\bullet$ be a nonlinear stability function for Λ with corresponding path γ_Z. Then $Z_\bullet$ is finite if and only if γ_Z is finite.* □

Theorem 3.6. *Let $Z_\bullet$ be a finite, green stability function with semistable pairs (M_i, t_i). Then each pair is $Z_\bullet$-stable, each M_i is Schurian and $\mathcal{E}(M_1), \cdots, \mathcal{E}(M_m)$ form a finite HN-system.*

Proof. The pair (M_i, t_i) is stable iff $\gamma_Z(t_i) \in int\, D(M_i)$. So, suppose not. Then $\gamma_Z(t_i) \in \partial D(M_i)$ which implies that $\gamma_Z(t_i) \in D(M')$ for some proper submodule $M' \subsetneq M_i$. The minimal such M' must be Schurian which contradicts the uniqueness of M_i.

By Theorem 2.4, we obtain an HN-stratification $\mathcal{S}_Z(t_1), \cdots, \mathcal{S}_Z(t_m)$ of *mod*-Λ. By Lemma 3.2, $\mathcal{S}_Z(t_i) = \mathcal{E}(M_i)$ proving that we have a finite HN-system. □

3.2. **Forward hom-orthogonality.** A powerful reformulation of a finite HN-system.

Definition 3.7. We call a sequence of Λ-modules $M_1, \cdots, M_m$ *weakly forward hom-orthogonal* if $\mathrm{Hom}_\Lambda(M_i, M_j) = 0$ for all $i < j$. We call it *maximal* if

(1) It cannot be embedded in a longer weakly forward hom-orthogonal sequence.
(2) Each M_i is Schurian.

An example of a weakly forward hom-orthogonal sequence of modules, with length $m = 1$ is $M_1 = \bigoplus S_i$, the sum of all simple modules. An example of a maximal forward hom-orthogonal sequence is given by taking Λ of finite representation type so that its Auslander-Reiten quiver has no oriented cycles. Then the indecomposable Λ-modules are all Schurian and form a maximal forward hom-orthogonal sequence when they are ordered from right to left in the Auslander-Reiten quiver. In [19] we use this observation to construct maximal green sequences of maximal length for many cluster-tilted algebras of finite type.

Theorem 3.8. *Given a finite sequence of Schurian Λ-modules $M_1, \cdots, M_m$, the following are equivalent.*

(1) *The sequence is maximal forward hom-orthogonal.*
(2) *$\mathcal{E}(M_1), \cdots, \mathcal{E}(M_m)$ is a finite HN-system for mod-Λ.*

Proof. The implication (2) $\Rightarrow$ (1) is trivial and well-known. The converse (1) $\Rightarrow$ (2) is also easy but we will go through it to make sure it is stated correctly.

Let (M_i) be maximal forward hom-orthogonal and let X be any module. Then, there is at least one k so that $\mathrm{Hom}(M_k, X) \neq 0$. Let k be minimal and let $f : M_k \to X$ be nonzero. Then f must be a monomorphism. Otherwise, by minimality of k, the image of f would fit between M_k and M_{k-1} contradicting the maximality of (M_i). So, we have a short exact sequence $M \hookrightarrow X \twoheadrightarrow Y$.

By induction on the length of X, we have an HN-filtration $Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_m = Y$ of $Y = \operatorname{coker} f$ where $Y_i/Y_{i-1} \in \mathcal{E}(M_i)$. Then it suffices to show:

Claim: $Y_i = 0$ for all $i < k$.

Then we obtain an HN-filtration: $0 \subset X_k \subset X_{k+1} \subset \cdots \subset X_m = X$ of X where each X_j is the inverse image of Y_j in X.

Pf: Let j be minimal so that $Y_j \neq 0$ and suppose $j \leq k$. Then we will show that $j = k$. Let $E \subseteq X$ be the inverse image of Y_j in X. Then we have an exact sequence:

$$0 \to M_k \to E \to Y_j \to 0$$

(1) $\mathrm{Hom}_\Lambda(M_i, E) = 0$ for $i < k$ since $E \subseteq X$ and $\mathrm{Hom}(M_i, X) = 0$.
(2) $\mathrm{Hom}_\Lambda(E, M_p) = 0$ for $p > k$ since $\mathrm{Hom}_\Lambda(M_k, M_p) = 0$ making any map $E \to M_p$ factor through $Y_j \in \mathcal{E}(M_j)$. But $j \leq k < p$. So, the map is zero.

By maximality of (M_i) there must be a nonzero map $g : E \to M_k$. If the restriction of g to M_k is nonzero then $E = M_k \oplus Y_j$ and $\mathrm{Hom}_\Lambda(M_j, E) \neq 0$ which implies $\mathrm{Hom}_\Lambda(M_j, X) \neq 0$ since $E \subseteq X$. So, $j = k$. If $g|M_k = 0$ then $\mathrm{Hom}_\Lambda(Y_j, M_k) \neq 0$ showing again that $j = k$. So, $j = k$ in all cases. The proves the Claim. The Theorem follows. □

4. Finite Z and maximal green sequences

In this section we restrict to the case when Λ is the finite dimensional hereditary algebra. The main result is that a finite green nonlinear stability function gives a maximal green sequence (MGS) for Λ and all MGS's are given in this way.

We first review the characterization of MGS's and more general reddening sequences by exceptional wall crossing sequences from [18]. In particular, we already know that reddening sequences are given by paths which meet finitely many exceptional semistability sets $D(M_i)$ at distinct times t_i where exceptional means that M_i are exceptional modules. The only thing left to prove is that such a path cannot meet any $D(M)$ for nonexceptional M. The proof will be reduced to Lemma A (4.4) which we prove in the last section.

We also show that, in the hereditary case, a MGS is equivalent to a maximal forward hom-orthogonal sequence. This implies that all five notions of stability outlined in the introduction are equivalent in the finite, green hereditary case.

4.1. **Review of the cluster complex.** The cluster complex is well-known and there are several different constructions. (See, e.g., [27], [16].) Here we follow [18].

Let Λ be a hereditary algebra over K. Recall that an *exceptional module* is a finite dimensional module M which is *rigid*, i.e., $\mathrm{Ext}^1_\Lambda(M, M) = 0$ and Schurian. For example, the simple modules S_i and their projective covers P_i are exceptional. Exceptional modules are indecomposable and uniquely determined by their dimension vectors which are called (positive) *real Schur roots* and, for each positive real Schur root β we used the notation M_β for the unique exceptional module with dimension vector β and we use $D(\beta)$ to denote $D(M_\beta)$. We call these *exceptional walls*.

Let $\mathcal{C}_\Lambda$ be the cluster category [8] of Λ which is the orbit category of the bounded derived category of *mod*-Λ by the equivalence $F=\tau^{-1}[1]$. We represent indecomposable objects of $\mathcal{C}_\Lambda$ by representatives in the fundamental domain of F which consists of Λ-modules and shifted projective modules $P_i[1]$. Then the exceptional objects of $\mathcal{C}_\Lambda$ are the exceptional modules M_β and these shifted indecomposable projective modules $P_i[1]$. To each exceptional object $T=M_\beta$ or $P_i[1]$ we associate the *g-vector* $g(T)\in\mathbb{Z}^n$ which is characterized by the following properties where $f_i=\dim_K S_i$.

(1) If $T=P_i[1]$ then $g(P_i[1])=-f_ie_i$ where e_i is the i-th unit vector.
(2) If $T=M_\beta$ and $\coprod n_iP_i\to\coprod m_iP_i\to M_\beta$ is a minimal projective presentation then the i-th coordinate of $g(M_\beta)$ is $f_i(m_i-n_i)$.

Thus, dot product with $g(T)$ gives the *Euler-Ringel pairing*:

$$g(T)\cdot\underline{\dim}\,M=\langle\underline{\dim}\,T,\underline{\dim}\,M\rangle=\dim_K\operatorname{Hom}_{\mathcal{D}^b}(T,M)-\dim_K\operatorname{Ext}^1_{\mathcal{D}^b}(T,M).$$

Here we need to take Hom and Ext in the bounded derived category $\mathcal{D}^b$ of *mod*-Λ where, e.g., $\operatorname{Ext}^1_{\mathcal{D}^b}(P[1],M)=\operatorname{Hom}_\Lambda(P,M)$. In this notation, we have the following.

Theorem 4.1 (Virtual Stability Theorem)**.** [18] $g(X)$ *lies in the exceptional wall*

$$D(M)=\{x\in\mathbb{R}^n\mid x\cdot\underline{\dim}\,M=0,\ x\cdot\underline{\dim}\,M'\le 0\ \forall M'\subset M\}$$

if and only if $\operatorname{Hom}_\Lambda(X,M)=0=\operatorname{Ext}^1_\Lambda(X,M)$.

We repeat that, in [18], the sets $D(M)$ are defined only for exceptional M.

Example 4.2. Take the modulated quiver of type B_2 given by

$$\mathbb{R}\overset{\mathbb{C}}{\leftarrow}\mathbb{C}$$

with $F_1=\mathbb{R}$, $F_2=\mathbb{C}$ with $f_1=1,f_2=2$ and M_{21} is the $\mathbb{C}$-$\mathbb{R}$-bimodule $M_{21}=\mathbb{C}$. Two cluster tilting objects for this quiver are $T=S_2\oplus P_1[1]$ and $T'=S_2\oplus I_1$ with

(1) $g(P_1[1])=-f_1e_1=(-1,0)$
(2) $g(S_2)=(-2,2)$ since $P_1^2\to P_2\to S_2$ is a minimal presentation of S_2.
(3) $g(I_1)=(-1,2)$ since $P_1\to P_2\to I_1$ is a minimal presentation of I_1.

The cluster complex is given in Figure 2.

A *cluster tilting object* for Λ is an object $T=T_1\oplus\cdots\oplus T_n$ in the cluster category of Λ with n nonisomorphic components each of which is an exceptional object so that $\operatorname{Ext}^1_{\mathcal{C}_\Lambda}(T,T)=0$. This condition of having no self-extensions in the cluster category $\mathcal{C}_\Lambda$ is equivalent to the condition that it has no self-extensions in the derived category.

For each cluster tilting object $T=\coprod T_i$, there is a conical simplex $R(T)\subset\mathbb{R}^n$ given as follows. The vertices of $R(T)$ are the n rays generated by the g-vectors of the components of T. The faces of $R(T)$ are the subsets of the semistability sets $D(M_i)$ in the convex hull of these n rays where $M_i=M_{\beta_i}$ are the unique exceptional modules having the property that, in the derived category, $\operatorname{Hom}_{\mathcal{D}^b}(T_i,M_j)=0=\operatorname{Ext}^1_{\mathcal{D}^b}(T_i,M_j)$ or, equivalently, $g(T_i)\in D(M_j)$ for all $i\neq j$.

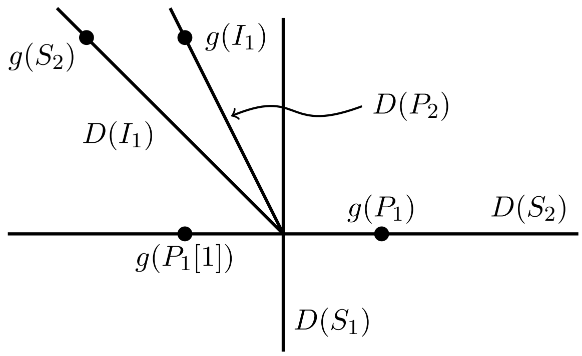


FIGURE 2. The cluster complex for B_2 where $B_2 : \mathbb{R} \leftarrow \mathbb{C}$. Since $S_1 \subset I_1, P_2$, $D(I_1)$ and $D(P_2)$ occur only on the negative side of $D(S_1)$. Also, $g(I_1) \in D(P_2)$ since $\mathrm{Hom}(I_1, P_2) = 0 = \mathrm{Ext}^1(I_1, P_2)$. And

$$\langle I_1, P_2 \rangle = g(I_1) \cdot \underline{\dim}\, P_2 = (-1, 2) \cdot (2, 1) = 0.$$

Similarly, $g(S_2) \in D(I_1)$ and $g(P_1[1]), g(P_1) \in D(S_2)$.

The *cluster complex* is defined to be the union of the conical simplices $R(T)$. This union is a *simplicial fan* which means that the intersection of any two $R(T) \cap R(T')$ is either empty or a common face of codimension ≥ 1. In particular, any codimension one simplex of the cluster complex is the common face of exactly two simplices $R(T), R(T')$.

Theorem 4.3. [18] *For every cluster tilting object $T = \bigoplus T_i$ of the cluster category of Λ the codimension 1 wall of $R(T)$ are $D(\beta_i)$ for $\varepsilon_1\beta_1, \cdots, \varepsilon_n\beta_n$ the c-vectors of the cluster tilting object T where $\varepsilon_i = \pm 1$ is the negative of the sign of $g(T_i) \cdot \beta_i$.*

Recall from [18] that a *maximal green sequence* consists of a sequence of cluster tilting objects $T_0, \cdots, T_m$ where $T_0 = \Lambda[1]$ and $T_m = \Lambda$ so that each $R(T_k)$ shares a wall, say $D(\beta_k)$ with $R(T_{k+1})$ and so that the mutation $T_k \to T_{k+1}$ is *green* in the sense that $R(T_{k+1})$ is on the positive side of $D(\beta_k)$. This gives a path γ in $\mathbb{R}^n$ going through the regions $R(T_k)$ in order and passing through the walls $D(\beta_k)$.

However, in this paper we have additional walls $D(M)$ for modules M which may not be exceptional. The following lemma proved in Corollary 5.18 allows us to ignore these other walls.

Lemma 4.4 (Lemma A)**.** *If M is not an exceptional module then $D(M)$ does not meet the interior of any codimension 0 or 1 face of the cluster complex.*

This implies that a path which meets $D(M)$ for non-exceptional M must either pass through infinitely many walls $D(\beta)$ of the cluster complex or it must pass through a simplex of codimension ≥ 2.

Lemma 4.5 (Lemma B)**.** *Let γ be any path in $\mathbb{R}^n$ which meets only a finite number of exceptional walls $D(\beta)$, and at least one, and which is transverse to the cluster complex, i.e., is disjoint from the codimension ≥ 2 simplices. Then γ does not meet $D(M)$ for any M which is not exceptional.*

Proof. The assumption that γ meets at least one wall means that part of γ lies in the union of the two $n-1$ simplices (with codimension 0) which contain that wall. The boundary of each $n-1$ simplex is a union of exactly n faces which are codimension 1 simplices. On the other side of each face, there is another $n-1$ simplex. If γ crosses k walls then, by induction on k, it will be in the union of $k+1$ open codimension 0 simplices and k open codimension 1 simplices. By Lemma A, γ cannot meet any $D(M)$ for M not exceptional. □

4.2. **Maximal green sequences.** For Λ hereditary, the finite, green nonlinear stability functions $Z_\bullet$ correspond to maximal green sequences:

Theorem 4.6. *Let $Z_t : K_0\Lambda \to \mathbb{C}$ be a finite, green nonlinear stability function with stable pairs (M_i, t_i). Then each $M_i = M_{\beta_i}$ is exceptional and the sequence of real Schur roots $\beta_1, \beta_2, \cdots, \beta_m$ in increasing order of t_i, form the c-vectors of a unique maximal green sequence for Λ. Furthermore, all maximal green sequences for Λ are given in this way.*

Remark 4.7. If we weaken the assumption that (M_i, t_i) are all green to the assumption that each pair is generic then $\varepsilon_1\beta_1, \cdots, \varepsilon_m\beta_m$, where $\beta_i = \underline{\dim}\, M_i$, are the c-vectors of a reddening sequence of Λ and all reddening sequence for Λ are obtained in this way. Furthermore ε_i is red in the reddening sequence iff (M_i, t_i) is red for $Z_\bullet$.

Proof of Theorem 4.6. We prove the stronger version of the Theorem as stated in Remark 4.7. One direction follows easily from the lemmas. Given $Z_\bullet$ finite, by Lemma 4.5, all $Z_\bullet$-stable modules are exceptional, say $M_k = M_{\beta_k}$. Lemma 2.11 implies that the path γ_Z starts in the "green region" where all coordinates are negative and ends in the "red region" where all coordinates are positive. Lemma 2.8 implies that $\gamma_Z(t)$ is disjoint from any $D(M)$ except at time t_k when $\gamma_Z(t_k) \in D(\beta_k)$. Thus, for each k, there is a cluster tilting object T_k so that $\gamma(t) \in R(T_k)$ for $t_{k-1} < t < t_k$. Lemma 2.10 implies that, at $t = t_k$, $\gamma_Z(t)$ crosses $D(\beta_k)$ in the green or red direction depending on whether the expression (2.1) is < 1 or > 1, respectively. Therefore, $T_0, \cdots, T_m$ is a reddening sequence with c-vectors $\varepsilon_1\beta_1, \cdots, \varepsilon_m\beta_m$.

Conversely, let $T_0, \cdots, T_m$ be a reddening sequence, i.e., a sequence of mutation from $T_0 = \Lambda[1]$ to $T_m = \Lambda$. Suppose that $\gamma : \mathbb{R} \to \mathbb{R}^n$ is a smooth path starting in $R(T_0)$, ending in $R(T_m)$ and passing through the regions $R(T_k)$ in order of k and transverse to the exceptional walls separating these regions. We may assume $\gamma(t) = t(f_1, \cdots, f_n)$ for $|t|$ large where $f_i = \dim_K S_i$ since these points lie in $R(\Lambda[1])$ and $R(\Lambda)$. Equivalently, the path $a_t = t(f_1, \cdots, f_n) - \gamma(t)$ in $\mathbb{R}^n$ has bounded support. So

$$\gamma(t) = -a_t + t(f_1, \cdots, f_n).$$

This corresponds to the nonlinear stability function $Z_t(x) = a_t \cdot x + b_t \cdot x\sqrt{-1}$ where $b_t = (f_1, \cdots, f_n)$ for all $t \in \mathbb{R}$. This clearly satisfies the conditions of Definition 2.1.

By construction, the path γ crosses exactly m exceptional walls $D(M_1), \cdots, D(M_m)$ at times $t_1 < \cdots < t_m$ with $M_i = M_{\beta_i}$ exceptional. By Lemma B (4.5) γ crosses no other walls. Therefore, γ is a finite reddening path corresponding to a finite stability function $Z_\bullet$. Lemma 2.8 implies that M_i are Z_{t_i}-stable and there are no other $Z_\bullet$ semistable modules. Lemma 2.10 implies that (M_{β_i}, t_i) is green or red for $Z_\bullet$ iff γ passes transversely through $D(\beta_i)$ in the green or red direction respectively at time t_i. Thus the sequence of stable roots β_i given by $Z_\bullet$ is the same up to the correct sign as the c-vectors of the arbitrary reddening sequence that we started with. □

4.3. **Hom-orthogonality and MGSs.** In this subsection we show that a maximal forward hom-orthogonal sequence is equivalent to a maximal green sequence. This is a reformulation of the well-known statement that a maximal green sequence is equivalent to a maximal chain in the poset of finitely generated torsion classes in *mod*-Λ. An alternate proof which works for quivers with potential is explained in [19].

Theorem 4.8. *Let $M_1, \cdots, M_m$ be a finite sequence of Schurian modules over a hereditary algebra Λ. Then the following are equivalent.*

(1) *The sequence is maximal forward hom-orthogonal.*
(2) *$\beta_k = \underline{\dim}\, M_k$ are the c-vectors of a maximal green sequence for Λ. In particular, all M_k are exceptional.*

We have already shown that (2) $\Rightarrow$ (1) since, by Theorem 4.6, any MGS is given by a finite green path which, by Theorem 3.6 gives an HN-system which is equivalent to (1) by Theorem 3.8. So, it suffices to show that (1) $\Rightarrow$ (2). We use the representation theoretic definition of a maximal green sequence explained in subsection 4.1. We use Lemma B (4.5) to insure that all walls that we encounter are exceptional walls.

We recall some standard cluster theory in the language of [20], [18] and [8]. In checking that the proofs in [20] work for modulated quivers we noticed one misprint: In the proof of Lemma 2.8 in [20], the reference should be to "proof of Lemma VI.6.1 in [2] using the trace of R in U_0, not $g(R)$."

Recall that a *support tilting module* is a Λ-module T so that $\mathrm{Ext}_\Lambda(T,T) = 0$ and the number of (nonisomorphic) summands of T is equal to the size of its support. In the cluster category [8], T can be completed to a cluster tilting object with n elements, n being the number of vertices of the modulated quiver $\mathcal{M}$ of Λ, by adding the shift $P_i[1]$ of the projective cover P_i of each vertex i not in the support of T. Let $R(T)$ denote the simplicial cone of the cluster tilting object $T \oplus \coprod P_i[1]$.

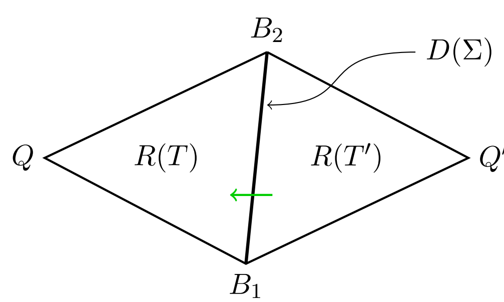


FIGURE 3. $T' \mapsto T$ is a green mutation if and only if the wall separating $R(T)$ and $R(T')$ is $D(\Sigma)$ for some simple object Σ in the wide subcategory corresponding to $Gen(T)$ and Q is the unique split projective in $Gen(T)$ which maps onto Σ.

Recall from [20] that, for each support tilting module T, there is a corresponding torsion class $Gen(T)$ in which T is the sum of the Ext-projective objects. There are two kinds of components of T: *split projectives* and *nonsplit projectives* where, by [20], Proposition 2.16, the split projectives of T are the projective objects in the wide subcategory $\mathcal{W}(T)$ of *mod*-Λ corresponding to $Gen(T)$ and, by [20], Proposition 2.24, the nonsplit projectives are those components of T which do not map nontrivially to any object in $\mathcal{W}(T)$. We will need the following which follows from [20], Theorem 2.35 and Proposition 2.24.

Lemma 4.9. *The "red" walls $D(\Sigma_k)$ of the region $R(T)$ are those opposite (the g-vector of) the split projective summands Q_k of T so that Σ_k is the top of the projective object Q_k in the wide subcategory $\mathcal{W}(T)$ of $Gen(T)$.*

Proof. Let $\Sigma_k \in \mathcal{W}(T)$ be the simple top of the projective object Q_k. Then

$$\mathrm{Hom}_\Lambda(T/Q_k, \Sigma_k) = 0 = \mathrm{Ext}^1_\Lambda(T/Q_k, \Sigma_k)$$

and the support of Σ_k is contained in the support of T. Therefore, $D(\Sigma_k)$ is the unique wall that contains all vertices of the region $R(T)$ not equal to Q_k. Since $\mathrm{Hom}_\Lambda(Q_k, \Sigma_k) \neq 0$, the wall is "red", i.e., the corresponding c-vector is negative.

If P is a nonsplit projective summand of T or $P = P_i[1]$, the wall opposite $g(P)$ in $R(T)$ is $D(M)$ for some M. However, $D(M)$ contains all of the split projectives in $Gen(T)$. Therefore, no object of $Gen(T)$ can map nontrivially to M. In particular, P cannot map nontrivially to M. So, $D(M)$ is a "green" wall of $R(T)$. □

Recall that a support tilting module T is a *green mutation* of T' if either $T = T' \oplus A$ for some exceptional module A or $T = Q' \oplus B$, $T' = Q' \oplus B$ where Q, Q' are exceptional modules so that $\mathrm{Ext}^1_\Lambda(Q, Q') = 0$. We say that T is a mutation of T' *through* Σ if Σ is the unique exceptional module with support in the support of T so that $\mathrm{Hom}_\Lambda(B, \Sigma) = 0 = \mathrm{Ext}^1_\Lambda(B, \Sigma)$ where B is the largest common summand of T, T'. In that case, the c-vector of the mutation is $\underline{\dim}\, \Sigma$ and $D(\Sigma)$ is the wall that separates the regions $R(T)$ and $R(T')$ in the cluster fan as shown in Figure 3. (See [18].)

We recall [20] that $Gen(\mathcal{W}(T)) = Gen(T)$.

Lemma 4.10. *Let $M_1, \cdots, M_m$ be a maximal forward hom-orthogonal sequence of Schurian modules. Then, for every $0 \le k \le m$ there is a support tilting module T_k satisfying the following.*

(a) *$M_1, \cdots, M_k \in Gen(T_k)$.*
(b) *$\mathrm{Hom}_\Lambda(T_k, M_\ell) = 0$ for all $\ell > k$. (So, $T_0 = 0$.)*
(c) *$M_{k+1}, \cdots, M_m$ form a sequence of green mutations from T_k to Λ. (So, $T_m = \Lambda$.)*
(d) *M_k is a simple object of the wide subcategory $\mathcal{W}(T_k)$.*
(e) *$T_{k-1} \mapsto T_k$ is a green mutation through $D(M_k)$. (So, M_k is exceptional.)*

For $k = 0$ we see that (c) implies that $\underline{\dim}\, M_1, \cdots, \underline{\dim}\, M_m$ are the c-vectors of a maximal green sequence which completes the proof of Theorem 4.8. Therefore, it suffices to prove this lemma.

Proof. By downward induction on k. When $k = m$ we have $T_m = \Lambda$ and $(a), (b), (c)$ hold trivially. So, suppose that $k \le m$ and that $(a)_k, (b)_k, (c)_k$ hold as well as $(c)_{k+1}, (d)_{k+1}$ (which we take to be vacuous when $k = m$). It follows from $(a)_k, (b)_k$ and the maximality of (M_i) that $M_1, \cdots, M_k$ is a maximal forward hom-orthogonal sequence of exceptional modules in $Gen(T_k)$.

$(d)_k$: Let Σ_i be the simple objects of the wide subcategory $\mathcal{W}(T_k)$. For each i there is a $j_i \le k$ so that $\mathrm{Hom}(M_{j_i}, \Sigma_i) \neq 0$, otherwise, we could insert Σ_i after M_k in the sequence $M_1, \cdots, M_k$ contradicting its maximality. Let $f_i : M_{j_i} \to \Sigma_i$ be nonzero. Since $M_{j_i} \in Gen(T_k) = Gen(\mathcal{W}(T_k))$, the image of f_i is in $\mathcal{W}(T_k)$. Since Σ_i is simple, f_i must be onto. But $M_k \in Gen(\mathcal{W}(T_k))$. So, there is a nonzero map $\Sigma_i \to M_k$ for some i. The composition $M_{j_i} \twoheadrightarrow \Sigma_i \to M_k$ is nonzero. So, we must have $M_{j_i} = M_k$. Since M_k is Schurian, the composition $M_k \twoheadrightarrow \Sigma_i \to M_k$ must be an isomorphism. So, $\Sigma_i \cong M_k$, proving $(d)_k$.

$(e)_k$: By Lemma 4.9, we can let T_{k-1} be the support tilting module obtained by mutation of T_k though M_k.

$(c)_{k-1}$ holds by $(c)_k$ and the construction of T_{k-1}.

$(b)_{k-1}$: Let B be the common summand of T_k, T_{k-1} so that $T_k = B \oplus Q$ and $T_{k-1} = B \oplus Q'$. Since B is a summand of T_k, $\operatorname{Hom}_\Lambda(B, M_\ell) = 0$ for all $\ell > k$. Since the g-vectors of the components of B lie on $D(M_k)$ we also have $\operatorname{Hom}_\Lambda(B, M_k) = 0$. By Lemma 4.9, Q' is a nonsplit projective in $Gen(T_{k-1})$ and B is a generator of $Gen(T_{k-1})$. Therefore, $\operatorname{Hom}_\Lambda(T_{k-1}, M_\ell) = 0$ for all $\ell \geq k$.

$(a)_{k-1}$: Since Q' is a nonsplit projective in $Gen(T_{k-1})$, $Gen(T_{k-1}) = Gen(B) \subseteq Gen(T_k)$. To prove that $M_1, \cdots, M_{k-1}$ lie in $Gen(B)$, suppose not. Let j be minimal so that M_j is not in $Gen(B)$. Let $X \subset M_j$ be the trace of B in M_j, i.e., the largest submodule of M_j which lies in $Gen(B)$. Since this is a torsion class, $\operatorname{Hom}_\Lambda(B, M_j/X) = 0$. But $M_j/X \in Gen(T_k)$ and B is Ext-projective in this category. Therefore, $\operatorname{Ext}^1_\Lambda(B, M_j/X) = 0$. This implies that the g-vector of B lies in $D(M_k)$. So, $M_j/X \in add(M_k)$. So, $\operatorname{Hom}_\Lambda(M_j, M_k) =\neq 0$ which is not possible for $j < k$. This proves $(a)_{k-1}$.

By induction on $m - k$, the proof of the Lemma is complete. Theorem 4.8 follows from $(c)_0$. □

5. Proofs

5.1. Proof that $\mathcal{W}(S)$ is a wide subcategory.

This is well-known.

Definition 5.1. For any subset S of $\mathbb{R}^n$ let $\mathcal{W}(S)$ be the full subcategory of *nmod-*$\mathcal{M}$ of all nilpotent modules M so that $S \subseteq D(M)$.

It is well-known and easy to see that $\mathcal{W}(S)$ is a *wide subcategory* of *nmod-*$\mathcal{M}$, in other words, it is closed under summands, extensions, kernels and cokernels. We review the proof in our setting using equivalent statements about the sets $D(M)$.

Lemma 5.2. *For any $S \subseteq \mathbb{R}^n$, $\mathcal{W}(S)$ is closed under extensions, cokernels of monomorphisms, kernels of epimorphisms. Equivalently,*

$$D(A) \cap D(B) \subseteq D(E)$$
$$D(A) \cap D(E) \subseteq D(B)$$
$$D(E) \cap D(B) \subseteq D(A)$$

for any short exact sequence $A \to E \to B$.

Proof. By definition of $\mathcal{W}(S)$, $A, B \in \mathcal{W}(S)$ iff $S \subseteq D(A) \cap D(B)$. We want to prove that $E \in \mathcal{W}(S)$ which is equivalent to $S \subseteq D(E)$. So, the two formulations of first statement in the lemma are equivalent. We prove the second. Let $x \in D(A) \cap D(B)$. We want to show that $x \in D(E)$. The first step is easy:

$$x \cdot \underline{\dim}\, E = x \cdot \underline{\dim}\, A + x \cdot \underline{\dim}\, B = 0$$

For any $C \subseteq E$, we have a short exact sequence $A \cap C \to C \to D \subseteq B$. So,

$$x \cdot \underline{\dim}\, C = x \cdot \underline{\dim}(A \cap C) + x \cdot \underline{\dim}\, D \leq 0.$$

So, $x \in D(E)$ proving that $\mathcal{W}(S)$ is closed under extensions. The proofs of the other two statements are similar. □

Lemma 5.3. *Let $f : A \to B$ be a morphism of nilpotent modules with image C. Then $D(A) \cap D(B) \subseteq D(C)$. Equivalently, $\mathcal{W}(S)$ is closed under images.*

Proof. Let $x \in D(A) \cap D(B)$. Since $C \subseteq B$ we have $x \cdot \underline{\dim} C' \leq 0$ for any $C' \subseteq C$. Since C is a quotient module of A we have $x \cdot \underline{\dim} C \geq 0$. Therefore $x \in D(C)$. □

Lemma 5.4. *$D(A \oplus B) = D(A) \cap D(B)$. So, $\mathcal{W}(S)$ is closed under summands.*

Proof. By Lemma 5.2, $D(A) \cap D(B) \subseteq D(A \oplus B)$. By Lemma 5.3, $D(A \oplus B) \subseteq D(A)$ (and $D(A \oplus B) \subseteq D(B)$) since A is the image of an endomorphism of $A \oplus B$. □

Theorem 5.5. *For any $S \subseteq \mathbb{R}^n$, $\mathcal{W}(S)$ is a wide subcategory of nmod-$\mathcal{M}$.*

Proof. Any summand of $M \in \mathcal{W}(S)$ lies in $\mathcal{W}(S)$ by Lemma 5.4. $\mathcal{W}(S)$ is closed under extensions by Lemma 5.2. Given any morphism $f : A \to B$ where $A, B \in \mathcal{W}(S)$, the exact sequences $\ker f \hookrightarrow A \twoheadrightarrow \operatorname{im} f$ and $\operatorname{im} f \hookrightarrow B \twoheadrightarrow \operatorname{coker} f$ show that $\ker f, \operatorname{coker} f \in \mathcal{W}(S)$ using Lemmas 5.2 since $\operatorname{im} f \in \mathcal{W}(S)$ by Lemma 5.3. □

5.2. **HN stratification.** In this subsection we show that a green nonlinear stability function $Z_\bullet$ gives an HN-stratification of any nilpotent module X. This detailed proof is based on the half-page proof in [28]. The proof in [3] is also very short and elegant. For just the idea of the proof, the reader should go to these original sources.

Definition 5.6. For any nilpotent module M, the set of values of $\mu_t(M)$ is bounded. So, $t > \mu_t(M)$ for $t >> 0$ and $t < \mu_t(M)$ for $t << 0$. So, the set of all $t \in \mathbb{R}$ for which $\mu_t(M) = t$ is closed, bounded and nonempty. Let $t_0(M)$ be the smallest element of this set. Since $t_0(M)$ depends only on the dimension vector of M, there are only finitely many values of $t_0(M')$ for all submodules $M' \subseteq M$. Let $t_1(M)$ be the smallest value of $t_0(M')$ for all nonzero $M' \subseteq M$.

Lemma 5.7. *If $\mu_t(M) < t$ then $t_0(M) < t$.*

Proof. Since $\mu_{t'}(M) > t'$ for $t' << 0$, $\exists t'' < t$ so that $\mu_{t''}(M) = t'' \geq t_0(M)$. □

Lemma 5.8. *For all $M' \subseteq M$ and $t < t_1(M)$ we have*

$$\mu_t(M') > t.$$

By continuity, we also have $\mu_{t_1(M)}(M') \geq t_1(M)$.

Proof. If $\mu_t(M') \leq t$ then, since $|\mu_t(M')|$ is bounded, there must be some $t' \leq t$ so that $\mu_{t'}(M') = t'$. Then $t_1(M) \leq t' \leq t$ by definition of $t_1(M)$, proving the lemma. □

Proposition 5.9. *Suppose $Z_\bullet$ is green for Λ and $M \in$ mod-Λ is Z_t-semistable for some $t \in \mathbb{R}$. Then*

$$t = t_0(M) = t_1(M)$$

Conversely, if $t_0(M) = t_1(M)$, then M is Z_{t_0}-semistable for $t_0 = t_0(M)$.

Proof. Suppose $t_0 = t_0(M) = t_1(M)$. Then $\mu_{t_0}(M) = t_0$ and, by Lemma 5.8 above, $\mu_{t_0}(M') \geq t_0$ for all $M' \subseteq M$. So, M is Z_{t_0}-semistable.

Conversely, suppose that M is Z_t-semistable and $Z_\bullet$ is green. Then

$$t \geq t_0(M) \geq t_1(M).$$

So, it suffices to show that $t \leq t_1(M)$. To prove this, suppose not. Then $t > t_1 = t_1(M)$. By definition of $t_1(M)$ there is $0 \neq M' \subseteq M$ so that $\mu_{t_1}(M') = t_1 < t$.

Let $M' \subseteq M$ be minimal with the property that $\mu_{t'}(M') = t'$ for some $t' < t$. Taking t' minimal we may assume $t' = t_0(M') < t$. By minimality of M' we have $t_0(M'') \geq t > t'$ for all $M'' \subsetneq M'$. By Lemma 5.8 this implies $\mu_{t'}(M'') > t'$. So, M' is $Z_{t'}$-stable. Since $Z_\bullet$ is *green for* Λ, any $t'' > t'$ sufficiently close to t' has the property that $\mu_{t''}(M') < t''$. Since $t' < t$, we can also take $t'' < t$. But $\mu_t(M'') \geq t$ by the assumption that M is Z_t-semistable. By the intermediate value theorem in calculus, there must be a point t_* with $t'' < t_* \leq t$ so that $\mu_{t_*}(M'') = t_*$. Taking t_* minimal we must have:

$$\frac{d}{dt}\mu_t(M'')\bigg|_{t=t_*} \geq 1$$

So, M'' cannot be Z_{t_*}-semistable. So, there is some $M_* \subsetneq M''$ so that $\mu_{t_*}(M_*) < t_*$. By Lemma 5.7 there is some $t'_* < t_*$ so that $\mu_{t'_*}(M_*) = t'_* < t_* < t$ contradicting the minimality of M'. So, $t \leq t_0 \leq t_1 \leq t$, showing that the three numbers are equal. □

Remark 5.10. For $Z_\bullet$ green, we say M is *$Z_\bullet$-stable/semistable* if M is Z_t-stable/semistable for some t. Proposition 5.9 implies that the value of t is uniquely determined. It also implies that M is $Z_\bullet$-semistable if and only if $t_0(M) = t_1(M)$.

Lemma 5.11. *Let $M_1 \subseteq M$ be maximal so that $t_0(M_1) = t_1(M)$. Then M_1 contains all $M' \subseteq M$ having $t_0(M') = t_1(M)$. In particular, M_1 is unique.*

Proof. Suppose M_1 does not have this property. Then, there is $M'_1 \subsetneq M$ with $t_0(M'_1) = t_1$ so that M_1, M'_1 do not contain each other. Also, $M_1 + M'_1$ properly contains M_1. So, $t_0(M_1 + M'_1) > t_1$ by maximality of M_1. This implies $\mu_{t_1}(M_1 + M'_1) > t_1$. Since $\mu_{t_1}(M_1) = t_1$ we must have

$$\mu_{t_1}\left(\frac{M_1 + M'_1}{M_1}\right) = \mu_{t_1}\left(\frac{M'_1}{M_1 \cap M'_1}\right) > t_1.$$

Since $\mu_{t_1}(M'_1) = t_1$ we conclude that $\mu_{t_1}(M_1 \cap M'_1) < t_1$ contradicting the definition of $t_1 = t_1(M)$. This proves the lemma. □

Lemma 5.12. *Let $Z_\bullet$ be green. Let $M_1 \subseteq M$ be the maximal submodule so that $t_0(M_1) = t_1 = t_1(M)$. If $M_1 \neq M$ then $t_1(M/M_1) > t_1$.*

Proof. Suppose not. Then the set

$$S = \{s \leq t_1 \,|\, \mu_s(M'/M_1) = s \text{ for some } M_1 \subsetneq M' \subseteq M\}$$

is closed and nonempty. Also, $t_1 \notin S$ since that would imply $\mu_{t_1}(M') = t_1$ contradicting the maximality of M_1. Let $m < t_1$ be the maximal element of S and let $M'' \supsetneq M_1$ be minimal so that $\mu_m(M'/M_1) = m$.

<u>Claim</u>: M'/M_1 is Z_m-semistable.

Proof: Suppose not. Then there exists $M_1 \subsetneq M'' \subsetneq M'$ so that $\mu_m(M''/M_1) < m$. But, as we observed in the proof of Lemma 5.11, $\mu_{t_1}(M''/M_1) > t_1$ by maximality of M_1. This implies that $\mu_s(M''/M_1) = s$ for some $m < s < t_1$ contradicting the maximality of m. So, the Claim holds.

Since $Z_\bullet$ is green, we have $\mu_s(M'/M_1) < s$ for $s > m$ close to m, in particular, for some $m < s < t_1$. But $\mu_{t_1}(M'/M_1) > t_1$. This leads to the same contradiction as in the proof of the Claim above. Therefore, the Lemma holds. □

We are now ready to prove the main theorem about green nonlinear stability functions.

Theorem 5.13 (HN-filtration)**.** *Let $Z_\bullet$ be a nonlinear stability function for the modulated quiver $\mathcal{M}$ which is green for $\Lambda = T\mathcal{M}/J$ and let $M \in mod$-Λ. Then there exist a unique sequence of real numbers $t_1 < t_2 < \cdots < t_m$ and a unique filtration*

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_m = M$$

with the property that M_i/M_{i-1} is Z_{t_i}-semistable for $i = 1, 2, \cdots, m$. Furthermore, $t_1 = t_1(M)$.

The sequence of submodules $M_1, M_2, \cdots, M_m$ will be called the *Harder-Narasimhan (HN) filtration* of M with respect to $Z_\bullet$.

Proof. (Existence) We first show the existence of the sequence $(M_1, t_1), \cdots, (M_m, t_m)$ by induction on the size of $M \in mod$-Λ. If M is simple then we let $t_1 = t_0(M) = t_1(M)$. Then $M_1 = M$ is Z_{t_1}-stable and we are done.

If M is not simple, let $t_1 = t_1(M)$ be as given in Definition 5.6. Let M_1 the unique largest submodule of M with $t_0(M_1) = t_1(M)$. Then M_1 is Z_{t_1}-semistable by Proposition 5.9. By induction on the size of M, the module M/M_1 has a unique HN-filtration

$$0 = M_1/M_1 \subsetneq M_2/M_1 \subsetneq \cdots \subsetneq M_m/M_1 = M/M_1$$

and there are $t_2 < t_3 < \cdots < t_m$ so that $(M_k/M_1)/(M_{k-1}/M_1) \cong M_k/M_{k-1}$ is Z_{t_k}-semistable for $k = 2, 3, \cdots, m$. Furthermore, $t_2 = t_1(M/M_1)$ by induction and this is $> t_1$ by Lemma 5.12. So, $M_1, M_2, \cdots, M_m$ is an HN-filtration of M.

(Uniqueness) If (M_1', t_1') is the beginning of another HN-filtration of M then the key step is to show that $t_1' = t_1$. This follows from the fact that

$$M_1/(M_1 \cap M_1') \cong (M_1 + M_1')/M_1'.$$

Since the theorem holds for M/M_1', we have $t_2' = t_1(M/M_1')$ and:

$$t_1 \geq t_0(M_1/(M_1 \cap M_1')) = t_0((M_1 + M_1')/M_1') \geq t_1(M/M_1') = t_2' > t_1'$$

By definition of $t_1(M)$ we have $t_1 \leq t_1'$. So, $t_1' = t_1$.

Next we show that $M_1' = M_1$. By Lemma 5.11, $M_1' \subseteq M_1$. If they are not equal then M_1/M_1' is nonzero with $\mu_{t_1}(M_1/M_1') = t_1$. This contradicts the induction hypothesis that $t_2' = t_1(M/M_1') > t_1' = t_1$. So, $M_1' = M_1$. The rest of the filtration is the unique filtration of M/M_1. So, the HN-filtration of M is unique. □

This theorem has many well-known consequences and reformulations. The most obvious, given by uniqueness of the HN-filtration of $A \oplus B$ is the following:

Corollary 5.14. *If A, B are $Z_\bullet$-semistable with $t_0(A) < t_0(B)$ then* $\text{Hom}(A, B) = 0$.

5.3. **Proof of Lemma A.** We prove Lemma A (Corollary 5.18 below) as a corollary of Proposition 5.17. Here Λ is a finite dimensional hereditary algebra over a field K.

Lemma 5.15. *Let X, Y be objects of $nmod$-$\mathcal{M}$ so that the interiors of $D(X), D(Y)$ meet as some point x_0. Then X, Y are hom orthogonal.*

Proof. By assumption the linear stability function whose corresponding path goes through x_0 has perturbations which go through $D(X), D(Y)$ in either order. So, X, Y are hom-orthogonal. □

Lemma 5.16. *Let σ be a codimension k simplex in the cluster complex of Λ. Then $\mathcal{W}(\sigma)$ is equal to $\mathcal{W}_0(\sigma)$, the wide subcategory of mod-Λ generated by the exceptional objects of $\mathcal{W}(\sigma)$.*

Proof. Since $codim\,\sigma = k$, $\mathcal{W}_0(\sigma)$ has k simple objects $S_1, \cdots, S_k$. Let $M \in \mathcal{W}(\sigma)$, $M \notin \mathcal{W}_0(\sigma)$. Consider first the case when M is a subobject of some $X \in \mathcal{W}_0(\sigma)$.

The condition $\sigma \subseteq D(M)$ implies that $\underline{\dim}\, M = \sum \lambda_i S_i$ for some $\lambda \in \mathbb{Z}$. Since $\underline{\dim}\, M$ has nonnegative entries, at least one λ_i is positive. Then $\langle \underline{\dim}\, P_i, \underline{\dim}\, M\rangle = \lambda_i \langle \underline{\dim}\, P_i, \underline{\dim}\, S_i\rangle > 0$ where P_i is the projective cover of S_i in $\mathcal{W}_0(\sigma)$. So, there is a nonzero morphism $f : P_i \to M$ with image $Y \subset M \subset X$. Since P_i, X both lie in $\mathcal{W}_0(\sigma)$, so do Y and X/Y. Then the exact sequence $Y \to M \to M/Y \subset X/Y$ shows, by induction on the size of M that $M \in \mathcal{W}_0(\sigma)$.

Now consider a general element $M \in \mathcal{W}(\sigma)$. As before there is a nonzero map $f : P_i \to M$. By the previous case, $\ker f \in \mathcal{W}_0(\sigma)$. So, the image of f also lies in $\mathcal{W}_0(\sigma)$. But $\operatorname{coker} f$ is an object of $\mathcal{W}(\sigma)$ which is smaller than M. So, it also lies in $\mathcal{W}_0(\sigma)$. Since $\mathcal{W}_0(\sigma)$ is closed under extension, M lies in $\mathcal{W}_0(\sigma)$ as claimed. □

Proposition 5.17. *If $D(M)$ meets the interior of a cluster simplex σ then $D(M)$ contains σ. Equivalently, $M \in \mathcal{W}(\sigma)$.*

Proof. Suppose not. Let M, σ form a counterexample so that $(\dim M, \dim \sigma)$ is minimal in lexicographic order. Let $x_0 \in D(M) \cap int\,\sigma$. Equivalently, $M \in \mathcal{W}(x_0)$.

Claim 1: x_0 lies in the interior of $D(M)$. Equivalently, M is a simple object of $\mathcal{W}(x_0)$.

Proof: If not, $x_0 \in D(M')$ for some $0 \neq M' \subseteq M$. Then $M', M/M'$ lie in $\mathcal{W}(x_0)$ and thus also in $\mathcal{W}(\sigma)$ by induction on $\dim M$. Since $\mathcal{W}(\sigma)$ is closed under extension, $M \in \mathcal{W}(\sigma)$ and M is not a counterexample.

Claim 1 implies that $F_M = \operatorname{End}(M)$ is a division algebra. So, M is indecomposable.

Claim 2: $\dim \sigma = 1$. So, $codim\,\sigma = n - 2$.

Proof: Since any $x_0 \in D(M) \cap int\,\sigma$ lies in the interior of $D(M)$, $D(M) \cap \sigma = H(M) \cap \sigma$. The hyperplane $H(M)$ cuts σ into two parts. Taking two vertices of σ on opposite sides of $H(M)$ we get an edge of σ whose interior meets $D(M)$. By minimality of $\dim \sigma$ this edge is equal to σ.

Since σ is an edge in the cluster complex of Λ, there are two ext-orthogonal exceptional objects T_1, T_2 of the cluster category of Λ so that their g-vectors $E^t\, \underline{\dim}\, T_i$ form the endpoints of σ. Let $T = T_1 \oplus T_2$. Then $T^\perp$, the full subcategory of mod-Λ of all X so that $\operatorname{Hom}_\Lambda(T, X) = 0 = \operatorname{Ext}^1_\Lambda(T, X)$ is the wide subcategory $\mathcal{W}_0(\sigma)$ of mod-Λ of rank $n-2$ spanned by $n-2$ simple objects $S_1, \cdots, S_{n-2}$ and generated by the corresponding projective objects $P_1, \cdots, P_{n-2}$. By Lemma 5.16, $\mathcal{W}_0(\sigma) = \mathcal{W}(\sigma)$.

Claim 3: There is an extension $P \hookrightarrow E \twoheadrightarrow M$ in $\mathcal{W}(x_0)$ with the following properties.

(1) P is a projective object of $\mathcal{W}(\sigma) = \mathcal{W}_0(\sigma)$.
(2) $E^\perp$ contains $\mathcal{W}(\sigma)$. Equivalently, E is in the wide subcategory spanned by T.
(3) E is indecomposable.

Suppose for a moment that Claim 3 holds. Let $\mathcal{W}_0$ be the wide subcategory of mod-Λ spanned by T. If $D(E) \cap \sigma$ contains only x_0 then, up to reordering, we must have $\operatorname{Hom}_\mathcal{D}(T_1, E) \neq 0$ and $\operatorname{Ext}^1_\mathcal{D}(T_2, E) \neq 0$ where $\mathcal{D}$ is the bounded derived category of $\mathcal{W}_0$. But this is not possible since T_1, T_2 are consecutive objects in the Auslander-Reiten sequence of the cluster category of the rank 2 hereditary abelian category $\mathcal{W}_0$. Thus Claim 3 leads to a contradiction proving the proposition.

Proof of Claim 3: The extension $P \to E \to M$ is a lifting of the universal extension of M by the simple objects of $\mathcal{W}(\sigma)$. More precisely, for each i, let $e_{ij} \in \mathrm{Ext}^1_\Lambda(M, S_i)$, $j = 1, \cdots, m_i$ form a basis of $\mathrm{Ext}^1_\Lambda(M, S_i)$ over the division algebra $F_M = \mathrm{End}_\Lambda(M)$. Since Ext^1 is right exact, e_{ij} lift to elements $\tilde{e}_{ij} \in \mathrm{Ext}^1_\Lambda(M, P_i)$. Let $P = \coprod P_i^{m_i}$ and let $P \to E \to M$ be the extension of M by P corresponding to the element of $\mathrm{Ext}^1_\Lambda(M, P) = \coprod \mathrm{Ext}^1_\Lambda(M, P_i)^{m_i}$ with ij term $\tilde{e}_{ij}$. Then for any simple object S_i of $\mathcal{W}(\sigma)$ the connecting homomorphism in the six term sequence:

$$0 \to (M, S_i) \to (E, S_i) \to (P, S_i) \xrightarrow{\cong} \mathrm{Ext}^1_\Lambda(M, S_i) \to \mathrm{Ext}^1_\Lambda(E, S_i) \to \mathrm{Ext}^1_\Lambda(P, S_i) \to 0$$

is an isomorphism where (X, Y) is short for $\mathrm{Hom}_\Lambda(X, Y)$. Since $\mathrm{Hom}_\Lambda(M, S_i) = 0$ by Claim 1 and Lemma 5.15 and $\mathrm{Ext}^1_\Lambda(P, S_i) = 0$ since P is projective, we conclude that $E^\perp$ contains each S_i and therefore all of $\mathcal{W}(\sigma)$ proving (2).

Since M is hom-orthogonal to all S_i, $\mathrm{Hom}_\Lambda(M, P) = 0$. So, $\mathrm{Hom}_\Lambda(E, M) \cong \mathrm{Hom}_\Lambda(M, M)$ is one-dimensional over F_M. Therefore, E has one component which maps onto M and any other component of E must lie in P. But $\mathrm{Hom}_\Lambda(E, P) = 0$ by (2). So, E has only one component. This proves (3) in Claim 3 and completes the proof of the proposition. □

Corollary 5.18 (Lemma A)**.** *Given a hereditary algebra* Λ *and a nonrigid module* M*, the semistability set* $D(M)$ *does not meet the interior of any simplex of the cluster complex of codimension* 0 *or* 1*.*

Proof. Since $D(A \oplus B) = D(A) \cap D(B)$ (Lemma 5.4), we may assume that M is indecomposable. If $D(M)$ meets the interior of a simplex σ then, by Proposition 5.17, $D(M)$ contains σ. Equivalently, $M \in \mathcal{W}(\sigma)$. This is not possible if σ has full dimension since $D(M)$ lies in a hyperplane. So, $\mathit{codim}\,\sigma = 1$ and $\sigma \subseteq D(M_\beta)$ for some exceptional module M_β. By Lemma 5.16, $\mathcal{W}(\sigma) = \mathcal{W}_0(\sigma) = \mathit{add}\, M_\beta$. So, $M \cong M_\beta$ is rigid, contrary to assumption. □

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