4. LECTURE ON NONCROSSING PARTITIONS

4.1. **Some background.** When we were studying link invariant [6], Kent Orr and I used the homology of torsion free nilpotent groups. I will explain a basic classical example.

Let $U_n(A)$ be the ring of unipotent $n \times n$ upper triangular matrices with coefficients in the ring A:

$$U_3(A) := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in A \right\}, \text{ unipotent since } \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}^3 = 0_3$$

(Unipotent means identity matrix plus a nilpotent matrix.)

I am interested in $U_n(\mathbb{Z})$. This is a discrete subgroup of the contractible group $U_n(\mathbb{R}) \cong \mathbb{R}^{\binom{n}{2}}$. Take the orbit space $M = U_n(\mathbb{R})/U_n(\mathbb{Z})$.

(1) M is a smooth compact manifold of dimension $\binom{n}{2}$.
(2) $M = K(\pi, 1)$ where $\pi = U_n(\mathbb{Z})$ since its universal cover $U_n(\mathbb{R})$ is contractible.
(3) $H^k(M) = H^k(U_n)$.

Theorem 4.1 (Bott,Kostant,Nomizu)**.** *(1950) Computed cohomology of $U_n(\mathbb{Z})$ with rational coefficients*

n	$\binom{n}{2}$	$rk\,H^0$	$rk\,H^1$	$rk\,H^2$	$rk\,H^3$	$rk\,H^4$	$rk\,H^5$	$rk\,H^6$
1	0	1	0					
2	1	1	1					
3	3	1	2	2	1			
4	6	1	3	5	6	5	3	1
5	10	1	4	9	$\cdots$			
6	15	1	5	14	$\cdots$			
7	21	1	6	20	$\cdots$			
8	28	1	7	27	$\cdots$			

The sum of ranks is always $n!$. The general theorem is: $rk\,H^k(Nilp(\Delta))$ is the number of elements of the Weyl group $W(\Delta)$ of length k for any Dynkin diagram Δ.

For example, there are 6 permutations of 4 letters of length 3 meaning product of 3 simple reflections. One in particular is

$$c = (34)(23)(12)$$

This is called the *coxeter*. Nonstandard choices of the coxeter, e.g. $c' = (23)(12)(34)$ give nonstandard definitions of "noncrossing partitions".

Proposition 4.2. *$U_n(\mathbb{Z})$ has presentation:*
Generators: x_{ij} where $1 \le i < j \le n$
Relations:

(1) $[x_{ij}, x_{jk}] = x_{ik}$
(2) $[x_{ij}, x_{k\ell}] = 1$ *if* $j \neq k, i \neq \ell$.

$U_n(\mathbb{Z})$ is the *nilpotent group of type A_{n-1}*.

4.2. **Theorem of I-Orr-Todorov-Weyman.**

Definition 4.3. The "picture group" $G(A_{n-1})$ is the group with presentation:
Generators: x_{ij} where $1 \le i < j \le n$
Relations:

(1) $[x_{ij}, x_{jk}] = x_{ik}$
(2) $[x_{ij}, x_{k\ell}] = 1$ if $(i,j),(k,\ell)$ are noncrossing: the closed intervals $[i,j],[k,\ell]$ are either disjoint or one is contained in the interior of the other.

$$[x, y] := y^{-1}xyx^{-1}$$

Note:

$$G(A_{n-1}) \twoheadrightarrow U_n(\mathbb{Z})$$

Theorem 4.4 (IOTW)**.** [10] *The cohomology of $G(A_{n-1})$ is free in every degree with ranks:*

n	$\lfloor \frac{n}{2} \rfloor$	$rk\,H^0$	$rk\,H^1$	$rk\,H^2$	$rk\,H^3$	$rk\,H^4$	$rk\,H^5$	$rk\,H^6$
1	0	1	0					
2	1	1	1					
3	1	1	2					
4	2	1	3	2				
5	2	1	4	5				
6	3	1	5	9	5			
7	3	1	6	14	14			
8	4	1	7	20	28	14		
9	4	1	8	27	48	42		

These are "ballot numbers"

$$rk\,H^k(G(A_{n-1})) = b(n-k, k)$$

$b(y, n)$ is defined to be the number of ways in which y "yes" votes and n "no" votes can be cast in an ordered sequence in such a way that the number of "yes" votes is always greater than or equal to the number of "no" votes. So, $b(n-k, k) = 0$ if $2k > n$.

Theorem 4.5 (IOTW/I-Todorov/I)**.** [10][11] *There is a $CAT(0)$ ($\Rightarrow$ contractible) space on which $G(A_{n-1})$ acts freely. The quotient space $X(A_{n-1})$ is a finite CW-complex with C_n cells where C_n are the* Catalan numbers

$$C_n := \frac{1}{n+1}\binom{2n}{n}$$

$X(A_{n-1})$ is analogous to the classical nilmanifold and $H^k(X(A_{n-1})) = H^k(G(A_{n-1}))$.

Remark 4.6. This is a theorem with three proofs. But with $CAT(0)$ replaced by simply "contractible" in the first two proofs. The first proof in [10] used semi-invariants and pictures following the ideas in the first papers by [IOTW] and [6]. The second proof [11] generalized this to any modulated quiver of finite representation type by using "signed exceptional sequences". In the present setting an exceptional sequence is a sequence of n transpositions whose product is equal to the coxeter $c = (n-1, n)\cdots(23)(12)$ which gives 2^{n-1} morphisms given by the signed exceptional sequences:

$$[\pm\beta_{n-1,n}, \cdots, \pm\beta_{23}, \pm\beta_{12}] : \{V\} \to \Omega.$$

4.3. **Def of noncrossing partitions [Becker, *Planar rhyme schemes* (1952)].** A *partition* $\mathcal{P}$ of a finite set X is a disjoint union $X = \coprod S_i$ of X into nonempty subsets S_i which are called the *parts* of $\mathcal{P}$.

Definition 4.7. A *noncrossing partition* (ncp) of a totally ordered set $V = \{v_1, v_2, \cdots, v_n\}$ is a partition $\mathcal{P}$ of V so that $\nexists a < b < c < d$ where a, c lie in one part and b, d lie in another part of the partition.

Another useful characterization is: Two finite sets X, Y of integers are *noncrossing* if, considering their supports $supp\, X, supp\, Y$ (defined to be the smallest closed intervals in $\mathbb{R}$ containing the sets X, Y) one of the sets is disjoint from the support of the other.

Theorem 4.8. *There are C_n ncp's of n.*

Ncp's are ordered by refinement. $\mathcal{Q}$ is a *refinement* of $\mathcal{P}$ if every part of $\mathcal{Q}$ is contained in a (unique) part of $\mathcal{P}$. This gives a set mapping $\pi : \mathcal{Q} \twoheadrightarrow \mathcal{P}$. We say $\mathcal{Q} < \mathcal{P}$.

The largest part if $\{V\}$ (with one part), the smallest is the partition Ω of V into n one-point sets. Then $\mathcal{P} > \mathcal{Q} \Rightarrow rk\,\mathcal{P} > rk\,\mathcal{Q}$ where

$$rk\,\mathcal{P} = n - |\mathcal{P}|$$

4.4. **Adjacent parts and the set of edge vectors.** Suppose that $\mathcal{P} > \mathcal{Q}$ and $rk\,\mathcal{P} - rk\,\mathcal{Q} = 1$. This means there is one part of $\mathcal{P}$ which contains two parts of $\mathcal{Q}$. Or, said the other way: There are two parts of $\mathcal{Q}$ which have merged into one part of $\mathcal{P}$.

Definition 4.9. Two parts of $\mathcal{Q}$ will be called *adjacent* if merging them produces another ncp $\mathcal{P}$.

Problem: Find all pairs of adjacent parts of $\mathcal{Q}$.

Answer: They are parallel or one covers the other.

The *covering order* of the parts of a partition is given by containment of support. We say X *covers* Y if $supp\, Y \subset supp\, X$ and there is nothing between them. Two parts are adjacent if either

1. one is the parent of the other in the covering order (one part covers the other)
2. the two parts have the same parent
3. both are maximal in support order (so, have no parent)

In cases (2),(3) we say the parts are *parallel*. A *parallel set* is the set of all parts parallel to one given part.

Example 4.10. Take the partition of $V = \{0, 1, 2, 3, 4, 5, 6, 7\}$ into five parts $A = \{0, 5\}, B = \{1, 3\}, C = \{2\}, D = \{4\}, E = \{6\}, F = \{7\}$. This is a noncrossing partition.

$$A : 0 \text{ ———————————— } 5 \quad E : 6 \quad F : 7$$

$$B : 1 \text{ ———— } 3 \quad D : 4$$

$$C : 2$$

There are three parallel sets: $\{A, E, F\}$, $\{B, D\}$, $\{C\}$.

$\mathcal{Q}$ is a disjoint union of parallel sets.

The category of ncp's is a category whose objects are the ncp's of V. Morphisms are given by binary forests (disjoint unions of binary trees, one on every "relatively parallel" set))

4.5. **Def of (rooted) binary tree.** A binary root structure on V is a set of $n-1$ directed edges and one vertex $v_k \in V$ called the root given recursively as follows.

$n = 1$. There is a unique binary tree structure on $V = \{v_1\}$ with root v_1.

$n > 1$. A binary tree structure is given by choosing a root v_k and binary tree structures on $V' = \{v_1, \cdots, v_{k-1}\}$ and $V'' = \{v_{k+1}, \cdots, v_n\}$ with the roots v_i, v_j of V', V'' replaced with edges $v_k - v_i, v_k - v_j$.

Definition 4.11. A set of edge vectors $v_j - v_i$ is called *compatible* if it forms a subset of a binary tree.

Theorem 4.12. *Compatible sets of edges form a flag complex. (A set is compatible if it is pairwise compatible.)*

Proof. Putnam level question. Find the root. Then use induction on n.

It follows from this theorem that the classifying space of the category of noncrossing partitions is locally $CAT(0)$. The corresponding statement is not true in type $\tilde{A}_n$ because of the quasi-simple modules.

However, there is a chance that it holds for finite Dynkin quivers.

Conjecture 4.13. *A set of n roots of the root system Φ of a modulated Dynkin quiver Q form the c-vectors of a cluster if and only if every pair of them occurs as two c-vectors of some cluster depending on the pair.*

The classifying space of the cluster morphism category of Q is locally $CAT(0)$ if and only if this conjecture holds.

4.6. **The category of noncrossing partitions.** Given a totally ordered finite set V, the category $\mathcal{NP}(V)$ has the ncp's $\mathcal{P}$ as objects. Morphisms $[T] : \mathcal{P} \to \mathcal{Q}$ exist if $\mathcal{Q}$ is a refinement of $\mathcal{P}$.

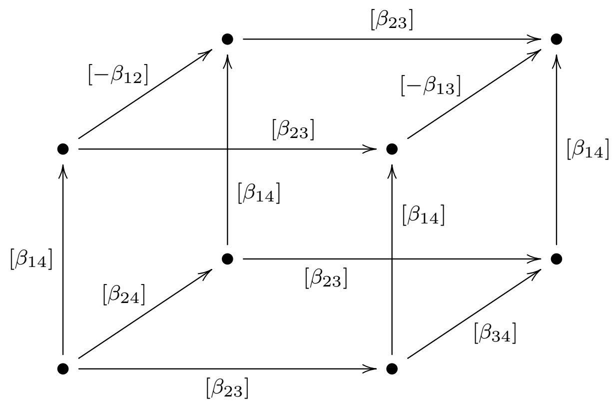
A morphism $[T] : \mathcal{P} \to \mathcal{Q}$ is given by choosing a rooted binary tree T_α for every relative parallel set V_α in $\mathcal{Q}$ relative to $\mathcal{P}$ defined as follows.

For each part P_i of $\mathcal{P}$, the set $\mathcal{Q}_i = \pi^{-1}(P_i)$ is a noncrossing partition of P_i considered as a subset of V. The relative parallel sets V_α are the parallel sets in each $\mathcal{Q}_i$. Note: the entire set V is a disjoint union of these relative parallel sets.

"There is a unique sensible way to compose morphisms."

4.7. **CAT(0) spaces and the theorem of Gromov.** This category is *cubical.* Every morphism of rank k has exactly $k!$ factorizations into morphisms of rank 1. This means the geometric realization of the category is a union of cubes. In the universal cover these cubes meet on faces. (In the proof I cut each k-cube into 2^k cubes of half-size in every direction. These small cubes meet on their faces in the universal cover.)

Theorem 4.14 (Gromov)**.** *A cubical simply connected space is $CAT(0)$ (and thus contractible) if and only if the link of every vertex is a flag complex.*



In a cubical category, the link of an object is the join of the incoming link and the outgoing link. The incoming links are flag by the thm about compatible sets in trees. The outgoing links are flags by cluster theory. (The refinements of $\mathcal{P}$ into one more part are in bijection with the objects in the corresponding cluster category. Objects form a cluster=morphism iff they are maximal pairwise compatible sets.)

Since morphisms are given by clusters, we call them *cluster morphisms*.

Finally, the fundamental group of the category is not hard to compute.

Theorem 4.15. *$B\mathcal{NP}(V)$ is a $K(\pi,1)$ with $\pi = G(A_{n-1})$.*

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