CATEGORIES OF REPRESENTATIONS OF CYCLIC POSETS

KIYOSHI IGUSA, BRANDEIS UNIVERSITY

0.1. **Abstract.** This is joint work with Gordana Todorov. Let $R = K[[t]]$ where K is any field. Given a "recurrent" cyclic poset X and "admissible automorphism" ϕ, we construct an R-linear Frobenius category $\mathcal{F}_\phi(X)$. I will go over the definition of a Frobenius category and indicate why our construction satisfies each condition. By a well-known result of Happel, the stable category $\mathcal{C}_\phi(X)$ will be a triangulated category over K. In each example in the chart below, $\mathcal{C}_\phi(X)$ will be a cluster category:

cyclic poset	automorphism	cluster category	comments
X	ϕ	$\mathcal{C}_\phi(X)$	
Z_n $1 < 2 < \cdots < n < \sigma 1$	$\phi(i) = i+1$	$\mathcal{C}(A_{n-3})$	2-CY
$\mathbb{Z}$ (with cyclic order)	$\phi(i) = i+1$	$\mathcal{C}(A_\infty)$ infinity-gon	2-CY
S^1	id	$\mathcal{C}$ continuous cluster category	not 2-CY but has clusters $Y[1] \cong Y$
$S^1 * \mathbb{Z}$	id $\phi(x,i) = (x, i+1)$	$\tilde{\mathcal{C}}$ $\tilde{\mathcal{C}}'$	not 2-CY ($Y[1] \cong Y$) 2-CY
$Z_m * \mathbb{Z}$	$\phi(i,j) = (i+1, j)$ $\phi(m,j) = (1, j+1)$	contains m-cluster category of type A_∞	$(m+1)$-CY
$\mathcal{P}(1)/3\mathbb{Z} * \mathbb{Z}$	$\phi^3(x,i) = (x, i+1)$	$\begin{pmatrix}\text{3-cluster category} \\ \text{of type } A_\infty\end{pmatrix}^3$	4-CY

I will go over some of the easier examples of this construction. CY means *Calabi-Yau*.

1.1. **Cyclic poset.** is same as periodic poset $\tilde{X}$. i.e. $\exists$ poset automorphism $\sigma : \tilde{X} \to \tilde{X}$ so that $x < \sigma x$ for all x. Also:

- $(\forall x, y \in \tilde{X})\ x \leq \sigma^j y$ for some $j \in \mathbb{Z}$.

  (1) Z_n: $\tilde{X} = \mathbb{Z}$, $\sigma(x) = x + n$ (n fixed).
  (2) $X = S^1$. Then $\tilde{X} = \mathbb{R}$ with $\sigma(x) = x + 2\pi$.
  (3) $\tilde{X} * \mathbb{Z}$ means $\tilde{X} \times \mathbb{Z}$ with lexicographic order (from van Roosmalen).

Let X = set of σ orbits. How to describe cyclic poset structure just in terms of X?

Following, van Roosmalen 1011.6077, p.10 and Drinfeld 0304064, p.5, (who refers to Besser and Greyson), this structure is equivalent to an $\mathbb{N}$-category structure on X.

Definition 1.1.1. An $\mathbb{N}$-*category* is a category $\mathcal{X}$ with the property that the additive monoid $\mathbb{N}$ acts freely on every Hom set

$$\mathbb{N} \times \mathcal{X}(x, y) \to \mathcal{X}(x, y)$$

so that composition satisfies:

$$nf \circ mg = (n + m)fg : x \to z$$

(*Acting freely* means Hom sets are disjoint unions of copies of $\mathbb{N}$: $\mathcal{X}(x, y) = \coprod \mathbb{N} f_i$.)

Proposition 1.1.2. *A cyclic poset structure on a set X is the same as an $\mathbb{N}$ category $\mathcal{X}$ with object set X so that every Hom set $\mathcal{X}(x, y)$ is freely generated by one morphism f_{xy}.*

So, given three objects, $x, y, z \in X$, we have

$$f_{yz} f_{xy} = n f_{xz} \tag{1.1}$$

for some $n \in \mathbb{N}$.

1.2. **Linearized cyclic poset.** We write: $\mathcal{X} = (X, c)$.

Definition 1.2.1. For any field $\Bbbk$, the (completed) *linearization* $\widehat{\Bbbk\mathcal{X}}$ of $\mathcal{X}$ is defined to be the category with object set X and morphism sets

$$\widehat{\Bbbk\mathcal{X}}(x, y) = \Bbbk^{\mathcal{X}(x,y)} \cong \Bbbk[[t]]$$

composition is given by

$$(r f_{yz}) \circ (s f_{xy}) = r s t^n f_{xz}$$

for any $r, s \in R := \Bbbk[[t]]$ where n is given by (1.1).

This is an *R-category*: Hom sets are R-modules and composition is R-bilinear.

Definition 1.2.2. A *representation* of $\mathcal{X}$ is defined to be an R-linear functor

$$M : \widehat{\Bbbk\mathcal{X}} \to R\text{-}mod$$

Definition 1.2.3. Let $\mathcal{P}(\mathcal{X})$ be the category of all finitely generated projective representations of $\mathcal{X}$.

Proposition 1.2.4 (Yoneda)**.** $\mathcal{P}(X) \cong add\,\widehat{\Bbbk\mathcal{X}}^{\mathrm{op}}$

Let P_x be the projective representation of X generated at the point $x \in X$.

1.3. **Frobenius category.**

Definition 1.3.1. Let $\mathcal{F}(X)$ denote the category of all pairs (P, e) where $P \in \mathcal{P}(X)$ and $e : P \to P$ so that $e^2 = \cdot t$ (mult by t). Morphism $f : (P, e) \to (Q, e)$ are maps $f : P \to Q$ so that $ef = fe$.

Lemma 1.3.2. *The functor $G : \mathcal{P}(X) \to \mathcal{F}(X)$ given by*

$$GP := \left(P \oplus P, \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix}\right) : \qquad P \underset{id}{\overset{\cdot t}{\rightleftarrows}} P$$

is both left and right adjoint to the forgetful functor $F : \mathcal{F}(X) \to \mathcal{P}(X)$.

Theorem 1.3.3. *For any cyclic poset X, $\mathcal{F}(X)$ is a Frobenius category where a sequence*

$$(A, e) \rightarrowtail (B, e) \twoheadrightarrow (C, e)$$

is defined to be exact in $\mathcal{F}(X)$ if $A \rightarrowtail B \twoheadrightarrow C$ is (split) exact in $\mathcal{P}(X)$. GP are the projective injective objects.

Proof. We can easily verify each step in the definition of a Frobenius category, namely, a *Frobenius category* is an exact category which has enough projectives so that all projective objects are injective. (E.g., $\Bbbk G$-*mod* for any finite group G.) An *exact category* is an additive category with a collection of exact sequences

$$\mathcal{E} = \{A \rightarrowtail_f B \twoheadrightarrow_g C\}$$

so that

(1) $A = ker\, g$ and $C = coker\, f$
(2) $0 \rightarrowtail 0 \twoheadrightarrow 0 \in \mathcal{E}$
(3) Given $A \rightarrowtail_{f_1} B \twoheadrightarrow coker\, f_1$, $B \rightarrowtail_{f_2} C \twoheadrightarrow coker\, f_2$ in $\mathcal{E}$, then

$$A \xrightarrow{f_2 f_1} C \twoheadrightarrow coker\, f_2 f_1 \quad \in \mathcal{E}$$

(4) The pushout of any $A \rightarrowtail B \twoheadrightarrow C \ \in \mathcal{E}$ along any $f : A \to A'$ is in $\mathcal{E}$.
(5) Dually for pull backs along $C' \to C$.

Finally, GP is projective since G is left adjoint to F and GP is injective because G is right adjoint to F. So, $\mathcal{F}(\mathcal{X})$ is Frobenius. □

1.4. **Twisted version.** An automorphism ϕ of X is *admissible* if:

$$x \le \phi(x) \le \phi^2(x) \le \sigma x$$

for all $x \in \tilde{X}$. In $\widehat{\Bbbk\mathcal{X}}$ this gives

$$P_x \xrightarrow{\eta_x} \phi P_x = P_{\phi(x)} \xrightarrow{\xi_x} P_x$$

giving natural transformations

$$P \xrightarrow{\eta_P} \phi P \xrightarrow{\xi_P} P$$

Definition 1.4.1. Let $\mathcal{F}_\phi(X)$ be the full subcategory of $\mathcal{F}(X)$ of all (P, e) where e factors through $\eta_P : P \to \phi P$.

Theorem 1.4.2. *$\mathcal{F}_\phi(X)$ is a Frobenius category with projective-injective objects*

$$G_\phi P := \left(P \oplus \phi P, \begin{bmatrix} 0 & \xi_P \\ \eta_P & 0 \end{bmatrix} \right) : \qquad P \underset{\eta_P}{\overset{\xi_P}{\rightleftarrows}} \phi P$$

2. Cluster categories

Definition 2.0.3. The *stable category* $\underline{\mathcal{F}}$ of a Frobenius category $\mathcal{F}$ has the same set of objects as $\mathcal{F}$ with morphism sets:

$$\underline{\mathcal{F}}(A, B) = \frac{\mathcal{F}(A, B)}{A \to P \to B, P \text{ proj-inj}}$$

Theorem 2.0.4 (Happel)**.** *The stable category of a Frobenius category is triangulated.*

Definition 2.0.5. Let $\mathcal{C}(X) = \underline{\mathcal{F}}(X)$ and $\mathcal{C}_\phi(X) = \underline{\mathcal{F}}_\phi(X)$.

Theorem 2.0.6. *In all examples on page 1, $\mathcal{F}(X)$ is Krull-Schmidt R-category with indecomposable objects:*

$$M(x, y) := \left(P_x \oplus P_y, \begin{bmatrix} 0 & \beta \\ \alpha & 0 \end{bmatrix} \right) : \qquad P_x \underset{\alpha}{\overset{\beta}{\rightleftarrows}} P_y$$

with $x, y \in X$.

Corollary 2.0.7. *$\mathcal{C}_\phi(X)$ is Krull-Schmidt $\Bbbk$-category with indecomposable objects $M(x, y)$ where $y \not\cong \phi x, \phi^{-1} x$.*

Remark 2.0.8. *Cluster categories* were first constructed by Buan-Marsh-Reineke-Reiten-Todorov (0402054) as orbit categories. This construction is an alternate construction in type A.

2.1. **Example** $X = Z_n, \phi(i) = i+1$**.** The cyclic poset has n elements in a circle. Indecomposable objects are $M(x,y)$ where x,y are at least two steps apart (because $M(i,i+1)$ is projective-injective). This is the well-known CCS model (0401316) for the cluster category of type A_{n-3}. (But they did not give the triangulated structure of the category.)

2.2. **Example** $X = \mathbb{Z}, \phi(i) = i+1$**.** Indecomposable objects are $M(x,y)$ where $x, y \in \mathbb{Z}$, $|y-x| \geq 2$. This is the ∞-gon (0902.4125).

2.3. **Example** S^1 **with** $\phi = id$**.** The objects are $M(x,y)$ where x,y are distinct points on the circle. This is the continuous cluster category (1209.1879).

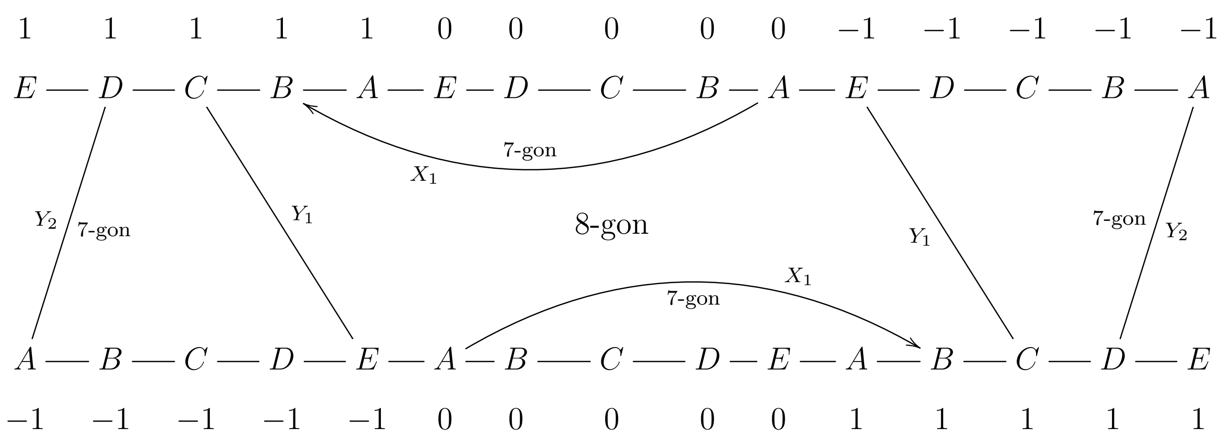
2.4. **Example** $X = Z_5 * \mathbb{Z}$**.**

$$\phi(x,i) = \begin{cases} (x+1,i) & \text{if } 1 \leq x < 5 \\ (1,i+1) & \text{if } x = 5 \end{cases}$$

Then $\mathcal{C}_\phi(X)$ is 6-CY.

Theorem 2.4.1. *Maximal compatible sets of* 6 *rigid objects correspond to 2-periodic partitions of the doubled* ∞*-gon into* 7*-gons (except for the one in the middle).*

Example of a maximal compatible set of 6-rigid objects in $\mathcal{C}_\phi(Z_5 * \mathbb{Z})$. $M(x,y)$ is arc from x to y (horizontal if standard, vertical if nonstandard). Compatible arcs do not cross. There is 8-gon in center. Other regions have 7 sides.



Standard: $X_1 = M(A_0, B_1)$ (horizontal).
$Y_1 = M(C_1, E_{-1})$, $Y_2 = M(A_{-1}, D_1)$ are nonstandard but $(m+1)$-rigid (vertical).
Notation: $(1,j) = A_j, (2,j) = B_j$, etc.