PERIODIC TREES AND PROPICTURES

KIYOSHI IGUSA, MOSES KIM, GORDANA TODOROV, AND JERZY WEYMAN

Abstract. Periodic tree are combinatorial structures which are in bijection with cluster tilting objects of affine type $\widetilde{A}_{n-1}$. The internal edges of the tree encode the c-vectors corresponding to the cluster tilting object. (See [10].) In this paper we assign a unipotent matrix to each tree, giving the relationship between cluster tilting objects and (pro)pictures for torsion-free (pro)nilpotent groups.

Contents

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Introduction

The goal of this paper is to extend the theory of pictures for nilpotent groups from [7] to residually torsion free nilpotent groups and show that the cluster diagram for $\widetilde{A}_{n-1}$ is such a generalized picture. We call this a propicture since it is an inverse limit of pictures for an inverse system of groups. Every cluster tilting object gives an element of this residually nilpotent group and we use noncommutative c-vectors and periodic trees to give a formula for this element. A simple example of a cluster tilting object of type $\widetilde{A}_3$ shows that the standard c-vectors in commutative root space $\mathbb{Z}^n$ gives the wrong element of the group. The idea comes from the theory of noncommutative cluster variables recently introduced by Berenstein and Retakh [2]. Picture for other root systems will be fully explained in another paper.

We begin by redefining mixed cobinary trees and extending to the n-periodic case. In the new definition, we do not assume that these trees are planar. We define an n-periodic tree $\mathcal{T}$ abstractly and show that $\mathcal{T}$ admits an embedding into the plane and the set of embeddings is a convex open subset $R(\mathcal{T})$ of Euclidean space $\mathbb{R}^{n+1}$. After projecting to $\mathbb{R}^n$ and intersecting with the unit sphere, we get open sets which are identical to the top dimensional simplices of the cluster triangulation derived in [8]. The proof uses the stability theorem for virtual semi-invariants: We show that, after a linear change of coordinates, the complement of the union of the sets $R(\mathcal{T})$ becomes the union of the supports $D(\sigma)$ of virtual semi-invariants for the quiver $\widetilde{A}^{\varepsilon}_{n-1}$.

1. Period trees

We review the definition of a periodic trees. The main property is that each periodic tree admits a linear embedding into the xy plane so that projection to the second coordinate is order preserving.

2. Nilpotent group elements associated to trees

Classically, every finite root system Φ has an associated nilpotent group $N(\Phi)$ given by Chevalley generators and relations. When the orientation of the quiver is changes, we choose different generators for the same group. For the cases A_n and $\widetilde{A}_{n-1}$, the groups are matrix groups. In this section we will associate an element of this group to each cluster tilting object and describe how the element changes when the cluster tilting object is mutated.

2.1. **Finite case.** As an example, take the case of the algebra KA_n^ε where A_n^ε is a quiver of type A_n with vertices $1, \cdots, n$ and arrows given by the sign sequence $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_{n-1})$ where ε_i gives the direction of the arrow i —— $i+1$. ($\varepsilon_i = +$ if $i \leftarrow i+1$, $\varepsilon_i = -$ if $i \to i+1$.) The positive roots of the root system are given by $\beta_{ij} = e_{i+1} + e_{i+2} + \cdot + e_j$ where $0 \le i < j \le n$. A cluster is given by a collection of n linearly independent roots called *edge vectors* which label the edges of an admissible tree. The nodes of the tree are $t_0, \cdots, t_n$ and we recall that the edge vector which labels an edge with endpoints t_i, t_j with $i < j$ is $\pm\beta_{ij}$ where the sign is equal to the sign of the slope of the edge. We recall that an admissible tree corresponds to a cluster tilting object in the cluster category of KA_n^ε and the c-vectors of the cluster tilting object are the negatives of the edge vectors of the tree. (This is the opposite sign convention from the one in [9] where the initial cluster tilting object is taken to be the algebra itself instead of $\Lambda[1]$ as in [10].)

Admissibility of trees is as defined in [9] which is analogous to the infinite case defined in [10]. We note that $\varepsilon_0, \varepsilon_n$ are irrelevant for questions of admissibility of trees. We refer to $\varepsilon_1, \cdots, \varepsilon_{n-1}$ as the *internal signs*.

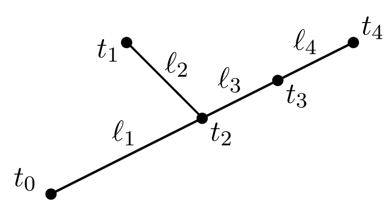


Figure 1. An admissible tree with internal signs $-, -, +$ attached to vertices t_1, t_2, t_3 and edge vectors $\gamma_1 = \beta_{02}$, $\gamma_2 = -\beta_{12}$, $\gamma_3 = \beta_{23}$, $\gamma_4 = \beta_{34}$ giving the four edges and their slopes.

In this case the nilpotent group associated to KA_n^ε is the group $N_{n+1}(\mathbb{Z})$ of unipotent upper triangular $(n+1) \times (n+1)$ integer matrices. For each cluster tilting object, the associated tree $\mathcal{T}$ gives the entries $g(\mathcal{T})_{ij}$ for $0 \le i, j \le n$ of the corresponding matrix $g(\mathcal{T})$ by the following formula where $\varepsilon_0 = +$ by default and γ_{ij} denotes the unique path from t_i

to t_j in $\mathcal{T}$.

$$g(\mathcal{T})_{ij} = \begin{cases} 1 & \text{if } i = j \\ \varepsilon_i & \text{if } i < j \text{, the path } \gamma_{ij} \text{ is monotonically increasing and} \\ & \ \gamma_{ij} \text{ does not pass through any } t_k \text{ for } i < k < j \text{ with } \varepsilon_k = - \\ 0 & \text{otherwise} \end{cases}$$

We will say that the path γ_{ij} is $\mathcal{T}$*-stable* if the above conditions holds making $g(\mathcal{T})_{ij} = \varepsilon_i$. For the example in Figure 1 this matrix is:

$$g(\mathcal{T}) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The nonzero entries are: $g_{01} = g_{02} = \varepsilon_0 = +$ since the path from t_0 to t_1 and to t_2 are always going up. $g_{03} = g_{04} = 0$ since the path from t_0 to t_3 and to t_4 pass through t_2 which is negative. $g_{23} = g_{24} = \varepsilon_2 = -$ since the path from t_2 to t_3 and to t_4 are ascending and the second path is not blocked at t_3 since $\varepsilon_3 = +$. Similarly, $t_{34} = \varepsilon_3 = +$.

Theorem 2.1. *Under mutation of the cluster tilting object, the matrix changes by an elementary operation:*

$$g(\mu_k(\mathcal{T})) = g(\mathcal{T})x(\gamma_k)^{-1} = g(\mathcal{T})x(c_k)$$

where $x(\gamma_k) = x(c_k)^{-1}$ *is the elementary matrix*

$$x(\gamma_k) = x(\delta_k\beta_{ij}) = x_{ij}(\delta_k\varepsilon_i)$$

if the edge vector of the k*-th edge of* $\mathcal{T}$ *is* $\gamma_k = -c_k = \delta_k\beta_{ij}$.

In the example given above, we have $x(c_1) = x_{02}(-1)$ which subtracts Column 0 from Column 2, $x(c_2) = x_{12}(-1)$ which subtracts Column 1 from Column 2 and $x(c_3) = x_{23}(1)$, $x(c_4) = x_{34}(1)$ are both column additions.

Proof of Theorem 2.1. We give a short visual proof. A detailed independent proof of the $\widetilde{A}_{n-1}$ case, which includes the A_n case as a truncation, is given later.

Suppose that $\mathcal{T}$ is an admissible tree with c-vectors $c_i(\mathcal{T})$. We consider the mutation $T^* = \mu_k(\mathcal{T})$ through the kth wall. Then T^* is obtained from $\mathcal{T}$ by changing the sign δ_k of the slope of the kth edge $\ell_k = (t_i, t_j)$ corresponding to the edge vector $\gamma_k(\mathcal{T}) = \delta_k\beta_{ij}$. By symmetry, we assume that $\delta_k = +1$. I.e., ℓ_k is sloped up in $\mathcal{T}$ and sloped down in T^*. There are exactly four cases corresponding to the possible signs for $\varepsilon_i, \varepsilon_j$.

In all four cases, the figures show that any path γ_{ab} for $b \neq j$ will be stable in $\mathcal{T}$ if and only if it is stable in T^*. So, $g(\mathcal{T})_{ab} = g(T^*)_{ab}$ if $b \neq j$. In the first two cases, when $\varepsilon_i = +$, there are stable paths in $\mathcal{T}$ which end in t_j which come from T^* stable paths ending in t_i. So, $g(\mathcal{T})_{aj} = g(T^*)_{aj} + g(T^*)_{ai}$. In the last two cases, when $\varepsilon_i = -$, ascending paths in $\mathcal{T}$ ending in t_j are unstable if they come from ascending paths in T^* ending in t_i. So, $g(\mathcal{T})_{aj} = g(T^*)_{aj} - g(T^*)_{ai}$. $\square$

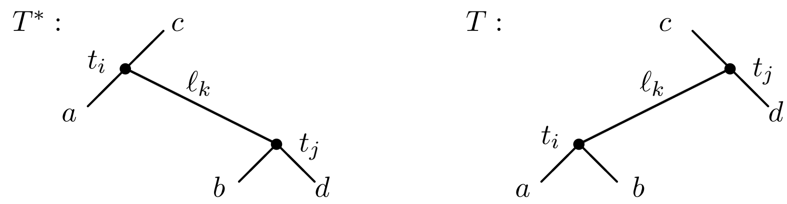


FIGURE 2. *Case* $(+,+)$. We need to show that $g(\mathcal{T}) = g(T^*)x_{ij}$. Ascending paths ac, bc, dc are the same on both sides, e.g., for any nodes t_a, t_c in a, c respectively, γ_{ac} is T^*-stable iff it is $\mathcal{T}$-stable. On the right tree $\mathcal{T}$ the only new ascending paths are t_i to t_j which is $\mathcal{T}$-stable and a to t_j which is $\mathcal{T}$-stable iff a to t_i is T^*-stable. So, $g(\mathcal{T})_{aj} = g(T^*)_{aj} + g(T^*)_{ai}$.

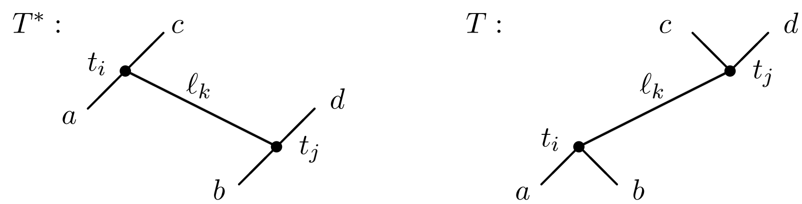


FIGURE 3. *Case* $(+,-)$. The claim here is $g(\mathcal{T}) = g(T^*)x_{ij}$. Ascending paths ac, bc, bd are the same on both sides. The new path ad on the right is not stable. So, the only new stable ascending paths in $\mathcal{T}$ are t_i to t_j and a to t_j assuming a to t_i is stable in T^*. So, $g(\mathcal{T})_{aj} = g(T^*)_{aj} + g(T^*)_{ai}$.

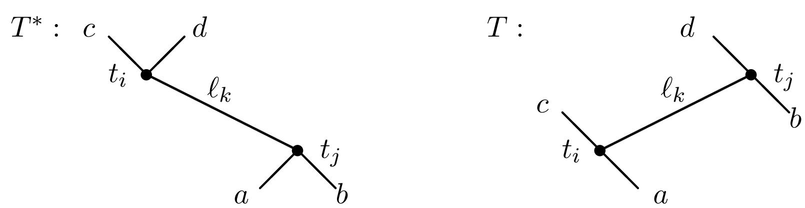


FIGURE 4. *Case* $(-,+)$. Here we need to show that $g(\mathcal{T}) = g(T^*)x_{ij}^{-1}$. Ascending paths ac, ad, bd are the same on both sides. The path bc always goes left so does not count. When a is left of t_i, the path at_j is allowed in T^* but not in $\mathcal{T}$. And $\mathcal{T}$ has the new path t_i to t_j. So, $g(\mathcal{T})_{aj} = g(T^*)_{aj} - g(T^*)_{ai}$.

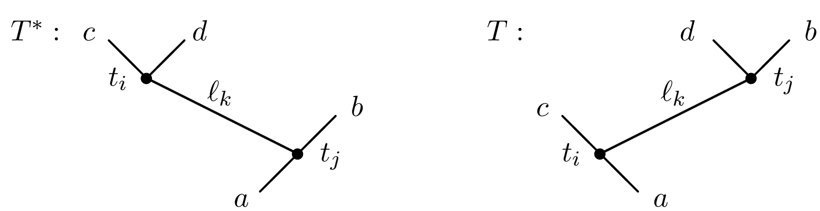


FIGURE 5. *Case* $(-,-)$. We need to show that $g(\mathcal{T}) = g(T^*)x_{ij}^{-1}$. Ascending paths ac, ad, ab are the same on both sides. When a is left of t_i, the path at_j is allowed in T^* but not in $\mathcal{T}$. And $\mathcal{T}$ has the new path t_i to t_j. So, $g(\mathcal{T})_{aj} = g(T^*)_{aj} - g(T^*)_{ai}$.

2.2. **Noncommutative root space.** To give the formula for $g(\mathcal{T})$ in the affine case, we use a noncommutative root space. For convenience of notation we will take two copies of the free nonabelian group on n generators: $F_n(\mathcal{T})$ and $\widetilde{\mathbb{Z}}^n$. The first will depend on a choice of periodic tree $\mathcal{T}$ and the second will be independent of $\mathcal{T}$.

Given an n-periodic tree $\mathcal{T}$, let $F_n(\mathcal{T})$ denote the free group generated by the set of edges ℓ_k of $\mathcal{T}$. Let $F_n^+(\mathcal{T})$ be the free monoid generated the edges of $\mathcal{T}$ and let $F_n^-(\mathcal{T})$ be the set of all inverses of elements of $F_n^+(\mathcal{T})$.

Let $\widetilde{\mathbb{Z}}^n$ denote the free group generated by $x_1, x_2, \cdots, x_n$ together with the epimorphism $\widetilde{\mathbb{Z}}^n \to \mathbb{Z}^n$ sending x_i to e_i. For every $i < j$ let $\tilde{\beta}_{ij}$ be the lifting of $\beta_{ij} \in \mathbb{Z}^n$ to $\widetilde{\mathbb{Z}}^n$ given by

$$\tilde{\beta}_{ij} := x_{i+1}x_{i+2}\cdots x_j \tag{2.1}$$

where the subscripts are taken modulo n. We also use the convention that $\tilde{\beta}_{ii}$ is the identity for all i and $\tilde{\beta}_{ji} = \tilde{\beta}_{ij}^{-1}$. Then $\tilde{\beta}_{ij}\tilde{\beta}_{jk} = \tilde{\beta}_{ik}$ for all integers i, j, k. For each multiple $k\eta$ of the null root $\eta = \beta_{0n}$ we take n distinct liftings $\tilde{\eta}_i^k = (\tilde{\beta}_{i,n+i})^k = \tilde{\beta}_{i,kn+i}$. But every other root has a uniquely chosen lifting (given by (2.1)) which we call the *standard lifting*.

An element $g \in F_n$ will be called *sign coherent* if g lies in F_n^+ or F_n^-. Similarly for elements of $\widetilde{\mathbb{Z}}^n$.

We define the *noncommutative* C-matrix of $\mathcal{T}$ to be the homomorphism

$$\widetilde{C}_T : F_n(\mathcal{T}) \to \widetilde{\mathbb{Z}}^n$$

which sends the generator $\ell_k = (p_i, p_j)$, $i < j$, to

$$\widetilde{C}_T(\ell_k) = \tilde{\beta}_{ij}^{-\delta_k}$$

where $\delta_k = \pm 1$ is the sign of the slope of ℓ_k. Note that $\widetilde{C}_T$ is *sign coherent* by definition since it sends each generator ℓ_k of F_n to a product of generators of $\widetilde{\mathbb{Z}}^n$ (such as $\tilde{\beta}_{ij}$) or a product of inverse generators of $\widetilde{\mathbb{Z}}^n$ (such as $\tilde{\beta}_{ij}^{-1}$).

Definition 2.2. For any periodic tree $\mathcal{T}$ and any i, j let γ_{ij} be the unique path in $\mathcal{T}$ from p_i to p_j. Let $\tilde{\gamma}_{ij}$ be the element of the free group $F_n(\mathcal{T})$ given by $\tilde{\gamma}_{ij} = \ell_{k_1}^{u_1} \cdots \ell_{k_s}^{u_s}$ where $\ell_{k_1}, \cdots, \ell_{k_s}$ are the edges in the path γ_{ij} in the order they are traversed and $u_t = \pm 1$ is positive or negative depending on whether γ_{ij} goes down or up along the edge ℓ_{k_t}, resp.

Lemma 2.3. *$\widetilde{C}_T$ sends $\tilde{\gamma}_{ij}$ to $\tilde{\beta}_{ij}$ for all i, j.*

Proof. The equation $\widetilde{C}_T(\ell_k) = \tilde{\beta}_{ij}^{-\delta_k}$ is equivalent to $\widetilde{C}_T(\ell_k^{-\delta_k}) = \tilde{\beta}_{ij}$. In the formula $\tilde{\gamma}_{ij} = \ell_{k_1}^{u_1} \cdots \ell_{k_s}^{u_s}$, the exponents can be rewritten as: $u_t = -\delta_t v_t$ where $v_t = \pm 1$ is positive or negative depending on whether the path γ_{ij} goes right or left along the edge ℓ_t. The reason is that, when $\delta_t = +$, right is the same as up on ℓ_t and, when $\delta_t = -$, right is the same as down on ℓ_t. This implies $\widetilde{C}_T$ takes the last step $\ell_s^{u_s}$ (from t_r to t_j) of $\tilde{\gamma}_{ij}$ to $\widetilde{C}_T(\ell_{k_s}^{u_s}) = \tilde{\beta}_{pq}^{v_s}$. When $v_s = +$ this is $\tilde{\beta}_{pq} = \tilde{\beta}_{rj}$. When $v_s = -$ this is $\tilde{\beta}_{pq}^{-1} = \tilde{\beta}_{jr}^{-1} = \tilde{\beta}_{rj}$. Thus $C_T(\tilde{\gamma}_{rj}) = C_T(\ell_{k_s}^{u_s}) = \tilde{\beta}_{rj}$. So, the lemma holds for $s = 1$. For $s \geq 2$ we get $C_T(\tilde{\gamma}_{ij}) = C_T(\tilde{\gamma}_{ir}\ell_{k_s}^{u_s}) = C_T(\tilde{\gamma}_{ir})C_T(\ell_{k_s}^{u_s}) = \tilde{\beta}_{ir}\tilde{\beta}_{pq}^{v_s} = \tilde{\beta}_{ir}\tilde{\beta}_{rj} = \tilde{\beta}_{ij}$. □

Theorem 2.4. *$\widetilde{C}_T$ is a sign coherent isomorphism of groups and its inverse $\widetilde{C}_T^{-1} : \mathbb{Z}^n \to F_n(\mathcal{T})$ is also sign coherent.*

Proof. Since $\tilde{\gamma}_{ij}\tilde{\gamma}_{jk} = \tilde{\gamma}_{ik}$, the homomorphism $f : \widetilde{\mathbb{Z}}^n \to F_n(\mathcal{T})$ sending x_i to $\tilde{\gamma}_{i-1,i}$ will send $\tilde{\beta}_{ij}$ to $\tilde{\gamma}_{ij}$ for all $i < j$. So, f is the inverse of $\widetilde{C}_T$.

It was shown in the proof of Proposition 2.1.4 in [10] that each $\gamma_{i-1,i}$ is either monotonically increasing or monotonically decreasing. Therefore, each $\tilde{\gamma}_{i-1,i}$ is sign coherent which implies that $\widetilde{C}_T^{-1}$ is sign coherent. □

Example 2.5. In Figure 1 in [10] we have

(1) $\ell_1 = (p_1, p_5)$ with negative slope. So, $\widetilde{C}_T(\ell_1) = \tilde{\beta}_{15}$.
(2) $\ell_2 = (p_2, p_3)$ with negative slope. So, $\widetilde{C}_T(\ell_2) = \tilde{\beta}_{23}$.
(3) $\ell_3 = (p_1, p_8) = (p_{-2}, p_5)$ with positive slope. So, $\widetilde{C}_T(\ell_3) = \tilde{\beta}_{18}^{-1}$.

As an example of the lemma, the path from p_2 to p_6 gives $\tilde{\gamma}_{26} = \ell_1^{-1}\ell_3^{-1}\ell_2$ since we go up from p_2 to p_{-2} to p_5, then down to p_6. Application of $\widetilde{C}_T$ gives:

$$\widetilde{C}_T(\tilde{\gamma}_{26}) = \widetilde{C}_T(\ell_1^{-1}\ell_3^{-1}\ell_2) = \tilde{\beta}_{15}^{-1}\tilde{\beta}_{18}\tilde{\beta}_{23} = \tilde{\beta}_{15}^{-1}\tilde{\beta}_{15}\tilde{\beta}_{58}\tilde{\beta}_{89} = \tilde{\beta}_{59} = \tilde{\beta}_{26}$$

The path $\ell_1^{-1}\ell_3^{-1}\ell_2$ has *length* $4 + 7 + 1 = 12$. However, it has *word-length* 3 since it is a product of three edges and inverse edges. The inverse of $\widetilde{C}_T$ is given by:

(1) $\widetilde{C}_T^{-1}(x_1) = \tilde{\gamma}_{01} = \ell_2^{-1}\ell_1^{-1}\ell_3^{-1}\ell_1^{-1}\ell_3^{-1}\ell_1^{-1}$
(2) $\widetilde{C}_T^{-1}(x_2) = \tilde{\gamma}_{12} = \ell_1\ell_3\ell_1$
(3) $\widetilde{C}_T^{-1}(x_3) = \tilde{\gamma}_{23} = \ell_2$

2.3. **Admissible and stable roots.**

Definition 2.6. We say that $\tilde{\beta}_{ij}$ is *$\mathcal{T}$-admissible* if $\widetilde{C}_T^{-1}(\tilde{\beta}_{ij}) \in F_n^-$ or, equivalently, the path γ_{ij} from p_i to p_j in $\mathcal{T}$ is monotonically increasing. It is possible that some liftings of the null root are admissible and some not. However, if $\tilde{\eta}_j$ is admissible, then so are all of its powers $\tilde{\eta}_j^k$.

Proposition 2.7. *There is a $\mathcal{T}$-admissible lifting $\tilde{\eta}_j = \tilde{\beta}_{j,j+n}$ of the null root if and only if the infinite path in $\mathcal{T}$ is monotonically increasing and passes through the node p_j.*

Proof. If $\tilde{\eta}_j$ is $\mathcal{T}$-admissible then the path $\gamma_{j,j+n}$ is monotonically increasing in $\mathcal{T}$ and repetition of this path gives an infinite, monotonically increasing path which passes through p_j. The converse is also very easy. □

Definition 2.8. We say $\tilde{\beta}_{ab}$ is a *subroot* of $\tilde{\beta}_{ij}$ and we write $\tilde{\beta}_{ab} \subseteq \tilde{\beta}_{ij}$ if there exists an integer s so that the following hold:

(1) $i + sn \le a < b \le j + sn$.
(2) Either $i + sn = a$ or $\varepsilon_a = -$.
(3) Either $j + sn = b$ or $\varepsilon_b = +$.

Similarly, $\tilde{\beta}_{ab}$ is a *quotient root* of $\tilde{\beta}_{ij}$ and we write $\tilde{\beta}_{ij} \twoheadrightarrow \tilde{\beta}_{ab}$ if for some s we have:

(1) $i + sn \le a < b \le j + sn$.
(2) Either $i + sn = a$ or $\varepsilon_a = +$.
(3) Either $j + sn = b$ or $\varepsilon_b = -$.

Lemma 2.9. *Suppose that $\tilde{\beta}_{ab} \subseteq \tilde{\beta}_{ij}$ and $\tilde{\beta}_{ab}$ is $\mathcal{T}$-admissible (equivalently, $\tilde{\gamma}_{ab} \in F_n^-(\mathcal{T})$). Then γ_{ij} contains γ_{ab}. Equivalently, $\tilde{\gamma}_{ab}$ is a subword of $\tilde{\gamma}_{ij}$. Similarly, if $\tilde{\beta}_{ij} \twoheadrightarrow \tilde{\beta}_{ab}$ and $\tilde{\gamma}_{ab} \in F_n^+(\mathcal{T})$ then $\tilde{\gamma}_{ab}$ is a subword of $\tilde{\gamma}_{ij}$.*

Proof. We prove the first statement, the second being equivalent to the first under reflection through the x-axis and reversal of all signs ε_k.

Claim 1: The paths γ_{ab}, γ_{ij} intersect.

If they do not intersect, then $i < a < b < j$ and $\varepsilon_a = -, \varepsilon_b = +$. Since γ_{ab} is monotonically increasing, it must stay to the right of p_a and to the left of p_b. So, γ_{ab} together with the ascending vertical line from p_a and descending vertical line from p_b divide the plane into two parts with p_i in the left part and p_j in the right part. To get from one side to the other, γ_{ij} must meet γ_{ab} at some point.

Let p_k be the lowest node in the intersection $\gamma_{ab} \cap \gamma_{ij}$.

Claim 2: $k = a$.

Otherwise, p_a lies above γ_{ij}. Let t be any point in γ_{ij} which lies below the point p_a. Then, by Corollary 1.2.11 in [10], the path from t to p_a must be monotonically increasing. It must also pass through the node p_k and it must be decreasing after that. This is a contradiction. So, $k = a$ and p_a lies on γ_{ij}.

Similarly, p_b lies on γ_{ij} forcing γ_{ab} to be contained in γ_{ij} since $\mathcal{T}$ is a tree. $\square$

Definition 2.10. We say that $\tilde{\beta}_{ij}$ is $\mathcal{T}$*-stable* if it is $\mathcal{T}$-admissible and has no $\mathcal{T}$-admissible subroot $\tilde{\beta}_{kj}$ where $i < k < j$. This is equivalent to saying that γ_{ij} is monotonically increasing and does not pass through any p_k with $\varepsilon_k = -$ and $i < k < j$.

Proposition 2.11. *There are only finitely many $\mathcal{T}$-stable roots $\tilde{\beta}_{ij}$ (including all roots and all powers of liftings of null roots).*

Proof. Suppose not. Then the length of a $\mathcal{T}$-stable root is unbounded. Let m be the length of the longest edge in $\mathcal{T}$ and take a stable root $\tilde{\beta}_{ij}$ with length $j - i > 3m^2(n+1)$. Then $\gamma = \widetilde{C}_T^{-1}(\tilde{\beta}_{ij}) \in F_n^-$ has word-length $> 3m(n+1)$. So, γ is a product of 3 elements $\gamma = \gamma_1\gamma_2\gamma_3$ in F_n^- each of word-length $> mn$. Then each γ_s repeats some edge m times. This implies that $\mathcal{T}$ has an infinite monotonically increasing path which necessarily contains a node with sign -1. Since each γ_s contains $m-1$ fundamental domains of the infinite ascending path, γ_s must pass through $m-1$ translates of this negative node. Since each translate is n units to the right of the previous one, γ_s moves $mn-n$ units to the right. This is greater than the largest amount a monotonically increasing path can move to the left. Therefore, the middle path γ_2 passes through a negative node p_k with $i < k < j$ making $\tilde{\beta}_{ij}$ unstable. $\square$

The formula for $g(\mathcal{T})$ can now be given in terms of an n-periodic unipotent upper triangular integer matrix $\tilde{g}(\mathcal{T})$ with entries

$$\tilde{g}(\mathcal{T})_{ij} = \begin{cases} 1 & \text{if } i = j \\ \varepsilon_i & \text{if } \tilde{\beta}_{ij} \text{ is } \mathcal{T}\text{-stable} \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.12 (Lam-Pylyavskyy[13])**.** Given any n-periodic unipotent integer matrix g with entries g_{ij}, let $\overline{g}$ be the $n \times n$ matrix $\overline{g}$ with coefficients in $\mathbb{Z}[[t]]$ whose entries are

$$\overline{g}_{ij} = t^{j-i} \sum_{k=0}^{\infty} a_k t^{kn}$$

where $a_k = g_{i,j+kn}$. We call $\overline{g}$ the *folding* of g.

Lemma 2.13. [13] *Folding is a monomorphism from the group of n-periodic unipotent integer matrices into $SL_n(\mathbb{Z}[[t]])$.*

Let $g(\mathcal{T})$ be the folding of $\tilde{g}(\mathcal{T})$.

Theorem 2.14. *For a mutation $T^* = \mu_k(\mathcal{T})$ in the direction of $\ell_k = (p_a, p_b)$ with positive slope in $\mathcal{T}$, we have:*

$$g(T^*) = g(\mathcal{T})x(\tilde{\beta}_{ab})^{-1}$$

where $x(\tilde{\beta}_{ab})$ is equal to the identity matrix I_n with $\varepsilon_a t^{b-a}$ added to the $(\overline{a}, \overline{b})$ entry where the overline denotes reduction modulo n.

Note that $x(\tilde{\beta}_{ab})$ is the folding of the n-periodic matrix which is the product of the infinite elementary matrices $x_{a+kn,b+kn}(\varepsilon_a)$ which commute since n does not divide $b-a$.

Proof. The theorem follows from the following three statements which are proved in the lemmas below.

1. For $j \not\equiv b$ module n, $\tilde{\beta}_{ij}$ is $\mathcal{T}$-stable if and only if it is T^*-stable.
2. If $\tilde{\beta}_{ia}$ is not $\mathcal{T}$-stable (and therefore not T^*-stable) then $\tilde{\beta}_{ib}$ is also not $\mathcal{T}$-stable and not T^*-stable.
3. If $\tilde{\beta}_{ia}$ is $\mathcal{T}$-stable (and T^*-stable) then $\tilde{g}(\mathcal{T})_{ib} = \tilde{g}(T^*)_{ib} + \varepsilon_a\varepsilon_i$.

Indeed, the first statement implies that $\tilde{g}(\mathcal{T})_{ij} = \tilde{g}(T^*)_{ij}$ for $j \neq b$ and the other two statements imply that $\tilde{g}(\mathcal{T})_{ib} = \tilde{g}(T^*)_{ib} + \varepsilon_a\tilde{g}(\mathcal{T})_{ia}$ proving the theorem after folding. $\square$

Lemma 2.15. *Any T^*-admissible root $\tilde{\beta}_{ij}$ is also $\mathcal{T}$-admissible.*

Proof. We can reduce to the case of A_n where $\mathcal{T}$ is a finite tree since any counterexample will persist in some truncation of the infinite tree.

In that case, the equation $C_T^{-1} = -\Gamma_T^{-1} = -V_T^t E_\varepsilon$ (Theorem 2.3.12 in [10]) implies that the matrices C_T^{-1} and $C_{T^*}^{-1}$ differ only in the kth row. If β_{ij} is T^*-admissible but not $\mathcal{T}$-admissible then the path γ_{ij} in $\mathcal{T}$ must be increasing except along the edge ℓ_k where it is decreasing. This implies that γ_{ij} must rise up to p_b from the right and rise up from p_a to the left. So, β_{ab} is a subroot of β_{ij} and, by the lemma above, γ_{ij} contains γ_{ab} which is ℓ_k going up. This is a contradiction. So, $\tilde{\beta}_{ij}$ is $\mathcal{T}$-admissible. $\square$

Lemma 2.16. *Suppose $\tilde{\beta}_{ij}$ is $\mathcal{T}$-admissible but not T^*-admissible. Then $\tilde{\beta}_{ij} \twoheadrightarrow \tilde{\beta}_{ab}$ and $\tilde{\beta}_{ic}, \tilde{\beta}_{dj}$ are T^*-admissible (and thus also $\mathcal{T}$-admissible) where $c \geq i$ is minimal so that $c \equiv a$ modulo n and $d \leq j$ is maximal so that $d \equiv b$. When $b \not\equiv j$ this implies that $\tilde{\beta}_{ij}$ is not $\mathcal{T}$-stable.*

Proof. As in the previous lemma, we reduce to the finite case and we get that the path γ_{ij} in T^* is increasing except for the edge ℓ_k^* where it is decreasing. In the periodic tree T^*, the path γ_{ij} is increasing except at the translates of the edges ℓ_k^*. So, the path rises to p_a and falls to p_b and rises again (or stops if $b = j$). In any case, $\tilde{\beta}_{ij} \twoheadrightarrow \tilde{\beta}_{ab}$. The initial rising part of γ_{ij} is γ_{ic} and the final rising part is γ_{dj}.

When $b \not\equiv j$ then $d < j$ making $\tilde{\beta}_{dj}$ a $\mathcal{T}$-admissible subroot of $\tilde{\beta}_{ij}$ making the latter $\mathcal{T}$-unstable. $\square$

We can now prove the three lemmas which give the theorem.

Lemma 2.17. *If $j \not\equiv b$ then $\tilde{\beta}_{ij}$ is $\mathcal{T}$-stable iff it is T^*-stable.*

Proof. Suppose that $\tilde{\beta}_{ij}$ is $\mathcal{T}$-stable. Then $\tilde{\beta}_{ij}$ is T^*-admissible by Lemma 2.16 since $j \not\equiv b$. If $\tilde{\beta}_{ij}$ is not T^*-stable then there is $\tilde{\beta}_{kj} \subseteq \tilde{\beta}_{ij}$ which is T^*-admissible. But then $\tilde{\beta}_{kj}$ is also $\mathcal{T}$-admissible by Lemma 2.15 contradicting the assumption that $\tilde{\beta}_{ij}$ is $\mathcal{T}$-stable.

Suppose that $\tilde{\beta}_{ij}$ is T^*-stable. Then it is $\mathcal{T}$-admissible by Lemma 2.15. If it is not $\mathcal{T}$-stable then there is $\tilde{\beta}_{kj} \subseteq \tilde{\beta}_{ij}$ which is $\mathcal{T}$-admissible. By choosing k to be maximal we may assume that $\tilde{\beta}_{kj}$ is $\mathcal{T}$-stable. Then, by the previous paragraph, $\tilde{\beta}_{kj}$ is also T^*-stable contradicting the assumption that $\tilde{\beta}_{ij}$ is T^*-stable. $\square$

Lemma 2.18. *If $i < a$ and $\tilde{\beta}_{ia}$ is $\mathcal{T}$-unstable and T^*-unstable then $\tilde{\beta}_{ib}$ is also $\mathcal{T}$-unstable and T^*-unstable.*

Proof. Suppose that $\tilde{\beta}_{ib}$ is T^*-stable. Then γ_{ib} is monotonically increasing in T^*. So, $\gamma_{ia} = \gamma_{ib}\gamma_{ab}^{-1}$ is also monotonically increasing. So, $\tilde{\beta}_{ia}$ is T^*-admissible. Since $\tilde{\beta}_{ia}$ is unstable by assumption, the path γ_{ia} passes through a negative vertex p_k where $i < k < a$. But p_k cannot lie on the edge γ_{ab}^{-1} since $k < a < b$. So, p_k lies on γ_{ib} and $\tilde{\beta}_{ib}$ is T^*-unstable.

Suppose that $\tilde{\beta}_{ib}$ is $\mathcal{T}$-stable. Then γ_{ib} ends in the edge $\ell_k = (p_a, p_b)$. Otherwise, γ_{ib} ascends to p_b from some vertex p_c where $c > b$ making p_b a positive vertex. The vertical line descending from p_b would then separate p_i from p_c giving a contradiction. So, $\gamma_{ib} = \gamma_{ia}\gamma_{ab}$. But $\tilde{\beta}_{ia}$ being $\mathcal{T}$-unstable means that γ_{ia} passes through a negative vertex p_k with $i < k < a$. So, $\gamma_{ib} = \gamma_{ia}\gamma_{ab}$ also passes through p_k and $i < k < b$. So $\tilde{\beta}_{ib}$ is also $\mathcal{T}$-unstable. $\square$

Lemma 2.19. *If $\tilde{\beta}_{ia}$ is $\mathcal{T}$ and T^*-stable then $\tilde{g}(\mathcal{T})_{ib} = \tilde{g}(T^*)_{ib} + \varepsilon_a\varepsilon_i$.*

Proof. We are given that γ_{ia} is monotonically increasing in both $\mathcal{T}$ and T^*. Now consider two cases: $\varepsilon_a = +$ and $\varepsilon_a = -$.

Suppose $\varepsilon_a = +$. Then γ_{ia} cannot pass through p_b in either $\mathcal{T}$ or T^*. So, $\gamma_{ib} = \gamma_{ia}\gamma_{ab}$ in both cases. This implies that $\tilde{\beta}_{ib}$ is $\mathcal{T}$-admissible but not T^*-admissible. Also $\tilde{\beta}_{ib}$ is $\mathcal{T}$-stable since, if γ_{ib} passes through a negative vertex, then so does γ_{ia}, p_a being positive. So, in this case we get

$$\tilde{g}(T^*)_{ib} = 0, \quad \tilde{g}(\mathcal{T})_{ib} = \varepsilon_i = 0 + \varepsilon_a\varepsilon_i$$

Now suppose that $\varepsilon_a = -$. Then, in $\mathcal{T}$, $\gamma_{ia}\gamma_{ab}$ is monotonically increasing and therefore equal to γ_{ib}. Since p_a is negative and $i < a < b$, this makes $\tilde{\beta}_{ib}$ not $\mathcal{T}$-stable. In T^*, the path γ_{ia} must end in $\ell_k^* = \gamma_{ab}^{-1}$ since this is the only edge which ascends to p_a. So, $\gamma_{ia} = \gamma_{ib}\gamma_{ab}^{-1}$. Since this is monotonically increasing, γ_{ib} must also be monotonically increasing. If this path goes through a negative node p_k then we must have $k < a$. Otherwise, p_k would be underneath the edge γ_{ab} which is not allowed. Since $\tilde{\beta}_{ia}$ is T^*-stable, this cannot happen. So, $\tilde{\beta}_{ib}$ is also T^*-stable. This gives

$$\tilde{g}(T^*)_{ib} = \varepsilon_i, \quad \tilde{g}(\mathcal{T})_{ib} = 0 = \varepsilon_i + \varepsilon_a\varepsilon_i$$

In both cases we have $\tilde{g}(\mathcal{T})_{ib} = \tilde{g}(T^*)_{ib} + \varepsilon_a\varepsilon_i$. $\square$

This concludes the proof of Theorem 2.14.

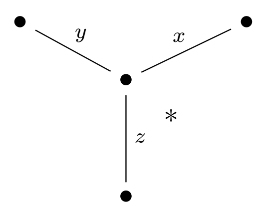
2.4. **Pictures in finite case.** In this subsection we define "pictures", give some examples, especially of the cluster picture of type A_n. These are labels with positive roots, finite trees and unipotent matrices. We explain the cluster picture for all finite type quivers, including type A_n in another paper.

2.4.1. 2-*dimensional pictures.*

Definition 2.20. Suppose that G is a group with presentation $G = \langle \mathcal{X} \,|\, \mathcal{Y} \rangle$. Then a *picture* for G is a planar graph with edges labeled by generators and vertices labeled with relations or their inverses in such a way that the labels on the edges at any vertex, when read counterclockwise, give the relation at that vertex. (Edges are normally oriented: i.e., the dual graph is a directed graph called a *spherical diagram.*) In addition, when a relation, such as x^n, is equal to a cyclic permutation of itself, as part of the data we indicate the preferred starting point of the relation by choosing one angle at each vertex.

By a *planar graph* we mean a subcomplex of the 1-skeleton of a finite simplicial decomposition of the 2-sphere. In other words, we have a finite simplicial complex K and a homeomorphism $|K| \cong S^2$. To avoid wild topological embedding we will assume that this homeomorphism is *linear* in the sense that it is an embedding $|K| \hookrightarrow \mathbb{R}^3\backslash 0$ which is linear on each simplex composed with the *normalization map* $\mathbb{R}^3\backslash 0 \to S^2$ sending v to $\frac{v}{||v||}$.

For example, if we have a relation $xy^{-1}z$, then we could have a vertex:

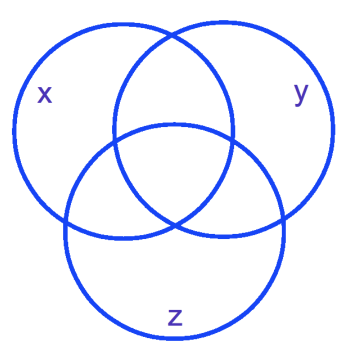


The asterisque $*$ indicates the starting point of the relation. This is usually uniquely determined and we can delete it from the figure. For example, here is a picture for the group

$$G = \mathbb{Z}^3 = \langle x, y, z \,|\, [x, y], [y, z], [z, x] \rangle$$

where

$$[x, y] := y^{-1}xyx^{-1}$$



This is a linear triangulation of the sphere since it is the projection to $\mathbb{R}^2$ of three great circles in S^2 which form the normalization of an octahedron.

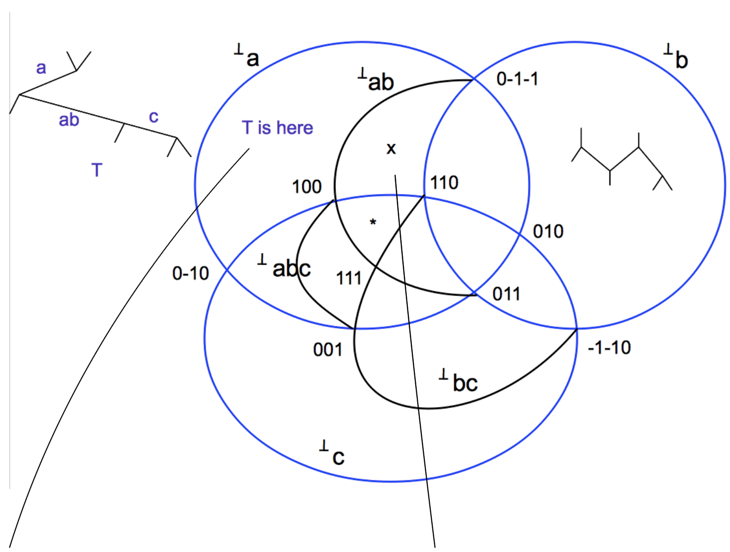
Proposition 2.21. *Given any picture for G, each region R can be labeled with an element of the group $g(R) \in G$ in such a way that the group label changes by right multiplication by the edge label when passing through an edge:*

$$g(R^*) = g(\mathcal{T})x(\ell)$$

$$R \quad \Big|_{\,\ell} \quad R^*$$

$$g(R) \quad \Big|_{\,x(\ell)} \quad \begin{matrix} g(R^*) = \\ g(R)x(\ell) \end{matrix}$$

Start with $g(R_0) = e$ for the unique unbounded region R_0.



T has c-vector $(ab)^{-1} = -\beta_{02}$, x has c-vector $ab = \beta_{02}$

2.4.2. *d-dimensional pictures.*

Definition 2.22. By a *d-dimensional picture* for $G = \langle \mathcal{X} \,|\, \mathcal{Y} \rangle$ we mean a subcomplex L of the $d-1$ skeleton of a finite linear triangulation of the d-sphere $S^d \subseteq \mathbb{R}^{d+1}$ together with the following additional numbered data satisfying the lettered conditions below.

(1) Each $d-1$ simplex τ in L has a normal orientation and a label $x(\tau) \in \mathcal{X}$.

(2) Each $d-2$ simplex ρ in L has a normal orientation and a label $y(\rho)$ which is either an element of $\mathcal{Y} \cup \mathcal{Y}^{-1}$ or an unreduced word of the form xx^{-1} for some $x \in \mathcal{X}$.

(a) L is the union of its $d-1$ simplices.

(b) Around each $d-2$ simplex ρ, let $\tau_1, \cdots, \tau_n$ be the $d-1$ simplices of L which have ρ as a face ordered cyclically around ρ according to the given normal orientation of ρ. Then,

$$y(\rho) = x(\tau_1)^{\delta_1} x(\tau_2)^{\delta_2} \cdots x(\tau_n)^{\delta_n}$$

where the sign δ_i is $+1$ or -1 depending on whether the given normal orientation of τ_i agrees with the cyclic orientation about ρ, i.e., the given normal vector points from τ_i to τ_{i+1} iff $\delta_i = +1$.

Proposition 2.23. *Suppose $d \geq 2$ and L be a d-dimensional picture for $G = \langle \mathcal{X} \mid \mathcal{Y} \rangle$ then each connected component R of the complement of L in S^d can be labelled with an element $g(R) \in G$ so that if R, R' contain open d-simplices on either side of the $d-1$ simplex τ in L with R' on the positive side of τ then*

$$g(R') = g(R)x(\tau)$$

Proof. Following one of the standard proofs of Poincaré duality for the manifold S^d, we take the 1-skeleton DK^1 of the dual cell decomposition DK of the triangulation K of the sphere given as part of the data of the picture L. The vertices of DK are the barycenters of the d-simplices of K and 1-cells of DK are transverse to the $d-1$ simplices of K and connect the barycenters of the two adjacent d-simplices. Start with any d-simplex σ_0 in K and let $g(\sigma_0) = e \in G$. For any other d-simplex σ, choose a path in DK^1 from the barycenter of σ_0 to the barycenter of σ. Let $\tau_1, \cdots, \tau_n$ be the $d-1$ simplices that we cross along this path. Then we define $g(\sigma)$ to be the product

$$g(\sigma) = x(\tau_1)^{\delta_1} x(\tau_2)^{\delta_2} \cdots x(\tau_n)^{\delta_n}$$

where δ_i is $+1$ or -1 depending on whether we traverse τ_i in the given normal direction or in the opposite direction. If τ_i is not in L then define $x(\tau_i) = e$. Since $d \geq 2$, any two paths are homotopic and the homotopy goes through 2-cells in the dual complex DK. These are transverse to the $d-2$ simplices of K. Every time the path crosses a codimension 2 simplex ρ, the product $\prod x(\tau_i)^{\delta_i}$ changes by the relation $y(\rho)$. Therefore, the element $g(\sigma)$ remains unchanged. So, $g(\sigma)$ is well defined and it satisfies the proposition by construction. □

Definition 2.24. Suppose that G, H are groups with presentations $G = \langle \mathcal{X} \mid \mathcal{Y} \rangle$, $H = \langle \mathcal{X}' \mid \mathcal{Y}' \rangle$. Then a group homomorphism $f : G \to H$ is *covered by a map of presentations* if the following hold.

(1) There is a mapping $f_1 : \mathcal{X} \to \mathcal{X}' \cup \mathcal{X}'^{-1} \cup \{e\}$ sending each generator of G to a generator of H or its inverse or e.

(2) The mapping f_1, when applied to a relator $r \in \mathcal{Y}$ which is a word $r = w(x_1, \cdots, x_n)$ in x_i and x_i^{-1}, gives a word $f_1(r) := w(f_1(x_1), \cdots, f_1(x_n))$ which is required to be either:
   (a) A reduced word in $\mathcal{Y} \cup \mathcal{Y}'$ with some e's inserted or
   (b) The trivial word (product of e's) or
   (c) The product of e's plus two cancelling terms xx^{-1} where $x \in \mathcal{X}'$.

Proposition 2.25. *A group homomorphism $f : G \to H$ which is covered by a map of presentations will sends pictures L for G to pictures $f(L)$ for H by relabeling the $d-1$*

dimensional faces of L and removing faces with labels in the kernel of f. Furthermore, f will map the G labels of the regions complementary to L to the H labels of the corresponding (possibly large) region complementary to $f(L) \subseteq L$ in S^d.

Proof. Suppose that L is a d-dimensional picture for G with labels $x(\tau)$, $y(\rho)$. Then let $f(L)$ be the closure of the set of all $d-1$ simplices τ of L for which $f_1(x(\tau)) \neq e$. If $f_1(x(\tau)) = x'$ then we give $\tau \subset f(L)$ the new label x'. If $f_1(x(\tau)) = x'^{-1} \in \mathcal{X}'^{-1}$ then we reverse the orientation of the simplex τ and label it with x'. The restrictions on what f_1 does to the relations of G insure that the $d-2$ simplices of $f(L)$ have unique labels and orientations to make it a picture. In particular note that such unreduced words like $f_1(r) = x_1 x_1^{-1} x_2 x_2^{-1}$ are not allowed since this is not an allowable set of labels for codimension one faces coming together along any codimension two face. $\square$

3. Propictures in affine case

- Def of projective presentation of a progroup with easy examples.
- Def of propicture for progroup with easy examples (?)
- Progroup associated to quiver of type $\widetilde{A}_{n-1}$.
- Perpendicular category of the null root.
- Periodic trees with zero slope.
- Propictures for cluster tilting objects of type $\widetilde{A}_{n-1}$.

3.1. **Pro-finite presentation of a progroup.** By a *progroup* we mean the inverse limit of an inverse system of groups

$$\cdots \twoheadrightarrow G_n \twoheadrightarrow G_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow G_2 \twoheadrightarrow G_1$$

Typically, this comes from a filtration of a group by a descending sequence of normal subgroups: $G \rhd N_1 \rhd N_2 \rhd \cdots$. Then

$$\hat{G} = \lim G/N_k$$

is a progroup. We will consider the case when each G_n is finitely presented.

Definition 3.1. By a *pro-finite presentation* of a progroup $\hat{G} = \lim G_n$ we mean the following.

(1) A generating set $\mathcal{X}$ together with a positive grading given by a function $\deg : \mathcal{X} \to \mathbb{Z}_{>0}$ so that there are only finitely many generators in each degree.
(2) A set $\mathcal{Y}$ of relations w which are totally ordered products of generators and inverse generators:
$$w = \prod_{\lambda \in \Lambda} x_\lambda$$
with $x_\lambda \in \mathcal{X} \cup \mathcal{X}^{-1}$, Λ totally ordered, having only finitely many generators in each degree. (The indexing set is only well-defined up to isomorphism of totally ordered countable sets.) We assume that each relator w is reduced and cyclically reduced, i.e., cannot be written in the form aba^{-1}.
(3) We define the *degree* of a word w to be the smallest degree of any of its letters. Then we assume that $\mathcal{Y}$ has only finitely many words in each degree.

The group G_n is defined to be the group with generating set $\mathcal{X}_n = \{x \in \mathcal{X} \mid \deg x \le n\}$ and relation set $\mathcal{Y}_n$ given by taking each $w \in \mathcal{Y}$ and deleting all letters of degree $> n$. Then each G_n is a finitely presented group. So, $\hat{G} = \lim G_n$ is a pro-finitely presented group with *pro-finite presentation* $\hat{G} = \langle \mathcal{X} \mid \mathcal{Y} \rangle^\wedge$. We say that $\hat{G}$ is *tame* if it has only finitely many infinite relations w.

One example of a progroup with pro-finite presentation is the additive group of p-adic integers. Written multiplicatively, this has pro-finite generating set $\mathcal{X} = \{g_1, g_2, \cdots\}$ with $\deg g_n = n$ and relation set $\mathcal{Y} = \{r_1, r_2, \cdots\}$ where $r_n = g_n^{-1} g_{n+1}^p$. Since all relations are finite words, $\hat{G}$ is *tame*.

Note that, given a pro-finite presentation of a progroup, the epimorphism $G_{n+1} \twoheadrightarrow G_n$ is covered by a map of presentations $\mathcal{X}_{n+1} \to \mathcal{X}_n \cup \{e\}$, $\mathcal{Y}_{n+1} \to \mathcal{Y}_n \cup \{e\}$ and therefore any picture for G_{n+1} gives a picture for G_n.

3.2. **Propictures for progroups.** Given a progroup $\hat{G} = \lim G_n$ with pro-finite presentation as above and any positive integer d, we define a d-dimensional *propicture* for $\hat{G}$ to be a sequence of d-dimensional pictures $L_n \subseteq S^d$ for $G_n = \langle \mathcal{X}_n \,|\, \mathcal{Y}_n \rangle$ with the property that L_{n+1} *reduces* to L_n in the sense that the epimorphism $G_{n+1} \to G_n$ maps L_{n+1} to L_n. (But note that $L_n \subseteq L_{n+1}$ as sets.) We say that a component of the complement of L_n in S^d is *stable* if it does not meet L_m for any $m > n$. We say that a propicture $\{L_n\}$ for a tame progroup $\hat{G}$ is *tame* if the number of unstable regions in the complement of L_n is bounded with total volume going to zero as $n \to \infty$.

It follows from the last proposition that each stable region in the complement of $L = \bigcup L_n$ has a well-defined group label in $\hat{G}$ provided that a base point is chosen in some fixed stable region. (We take $g(*) = e$ for the base point region.)

3.3. **Trees with slope zero.** Before constructing the cluster propicture associated to a quiver of type $\widetilde{A}_{n-1}$ we need to analyze periodic trees with zero slope. This will lead to the "null generators" of our groups.

Recall [10] that a periodic tree $\mathcal{T}$ has *zero slope* if it has a periodic function $\psi \in R(\mathcal{T})$ with zero slope. Then $\psi(p_i) = \psi(p_{i+kn})$ for any k and therefore ψ takes at most n values. Since $R(\mathcal{T})$ is an open region, it must also contain periodic functions of positive and negative slope.

We recall that there is a corresponding cluster tilting object V for the quiver $\widetilde{A}^{\varepsilon}_{n-1}$ and $E^t_\varepsilon R(V) = FR(\mathcal{T})$. So, $\psi \in R(\mathcal{T})$ corresponds to $v = (E^t_\varepsilon)^{-1} F\psi$. The slope of ψ is $\frac{1}{n}\langle v, \eta\rangle$. Therefore, ψ has zero slope if and only if v lies on the hyperplane H given by $\langle v, \eta\rangle = 0$. We call H the *horizon* and we say that a point $v \in \mathbb{R}^n$ lies above, on or below the horizon if $\langle v, \eta\rangle$ is positive, zero or negative respectively.

We have the following easy observation.

Proposition 3.2. *Suppose that β is a real Schur root. Then*
1. $\langle \beta, \eta\rangle = 1$ *if β is preprojective,*
2. $\langle \beta, \eta\rangle = -1$ *if β is preinjective and*
3. $\langle \beta, \eta\rangle = 0$ *if β is regular.*

Furthermore, $\langle \eta, v\rangle + \langle v, \eta\rangle = 0$ for all v.

Corollary 3.3. *The periodic function π_η corresponding to the vector η is $\pi_\eta(i) = -\varepsilon_i/2$.*

Proof. Since the form $\langle\cdot,\cdot\rangle$ is nondegenerate, π_η is uniquely determined by the equation
$$\langle \eta, \beta_{ij}\rangle = \pi_\eta(j) - \pi_\eta(i) = (\varepsilon_i - \varepsilon_j)/2$$
This gives the formula in the Corollary. $\square$

Lemma 3.4. *Suppose that $\mathcal{T}$ is a periodic tree with slope zero. Then there exist nodes p_a, p_b which are positive and negative respectively so that $p_a > p_b$.*

Proof. Let γ be the unique doubly infinite path in $\mathcal{T}$. Then γ cannot be monotonically increasing or decreasing. So, it reaches a local maximum at say p_a and a local minimum at say p_b. Then p_a must be a positive node and p_b must be negative. Taking $b - a$ to be minimal, the path from p_a to p_b is monononically decreasing. So, $p_a > p_b$ as claimed. $\square$

Lemma 3.5. *Suppose that v lies on the horizon H and $\pi_v : \mathbb{Z} \to \mathbb{R}$ is a corresponding n-periodic function of slope zero. Then the following are equivalent.*
1. $\langle v, \beta_{ab}\rangle > 0$ *for some preprojective root β_{ab}.*

(2) *There is a finite set of cluster tilting objects V_i so that some neighborhood of v lies in the union of the sets $\overline{R}(V_i)$.*
(3) *There is a set of cluster tilting objects V_α so that some neighbrohood of v lies in the union of the sets $\overline{R}(V_\alpha)$.*
(4) *There is a finite set of trees T_i with the property that some neighborhood of $E_\varepsilon^t v$ is contained in the union of the sets $F\overline{R}(T_i)$.*
(5) *There is a set of trees T_α with the property that some neighborhood of $E_\varepsilon^t v$ is contained in the union of the sets $F\overline{R}(T_\alpha)$.*
(6) *$\pi_v(a) < \pi_v(b)$ for some a, b with $\varepsilon_a = -$ and $\varepsilon_b = +$.*

Proof. It is easy to see that (1) and (6) are equivalent since $\langle v, \beta_{ab}\rangle = \pi_v(b) - \pi_v(a)$. Also (2) and (4) are equivalent by what we proved earlier and similarly (3),(5) are equivalent. The implication (2) $\Rightarrow$ (3) is immediate. So, it suffices to show that (3) implies (1) and (6) implies (4). We now show that (6) $\Rightarrow$ (4).

Since π_v has slope zero, we may choose a, b so that $a < b < a + n$ and $\pi_v(a) < \pi_v(b) > \pi_v(a+n)$. Take the neighborhood of π_v consisting of all π' so that $\pi'(a) < \pi'(b) > \pi'(a+n)$. Then for such π' and any periodic tree $\mathcal{T}$ with height function π', the lengths of the edges of $\mathcal{T}$ are bounded by $2n$. The reason is that any edge $\ell = (p_i, p_j)$ starting at $a \le i < a + n$ must end at $j < a + 2n$ since the edge cannot get past the vertical walls ascending from $(a+n, \pi'(a+n))$ and $(a+2n, \pi'(a+2n))$ without passing through the vertical wall descending from $(b + n, \pi'(b + n))$. But there are only finitely many trees, us to isomorphism, with edges bounded in length by $2n$. A general π' near π_v lies in $R(\mathcal{T})$ for one of these trees. So, the union of the closures $\overline{R}(\mathcal{T})$ will contain a neighborhood of π_v.

Conversely, suppose that (1) does not hold. Then $\langle v, \beta_{ab}\rangle \le 0$ for all preprojective roots β_{ab}. Then any neighborhood of v will contain a vector $w = (E_\varepsilon^t)^{-1} F\pi'$ on the horizon so that $\langle w, \beta_{ab}\rangle < 0$ for all preprojective β_{ab}. Equivalently, the minimum value of $\pi'(a)$ for all negative nodes a is greater than the maximum value of $\pi'(b)$ for all positive nodes b. By the lemma above, such a function π' cannot be the height function for any periodic tree $\mathcal{T}$. Therefore, no such w lies in $R(V)$ for any cluster tilting object V. Also, w will not in general lie on any $D(\beta)$ since the intersection $D(\beta) \cap H$ has codimension at least one in H and there are only countably many roots β. Therefore, w will, in general, not lie in $\overline{R(V)}$ for any cluster tilting object V. This proves that (3) implies (1) as claimed. $\square$

Note that, by what was proved earlier, (2) and (3) are equivalent for all $v \in \mathbb{R}^n$, not just those on the horizon. Let $D(\eta)$ denote the complement of the set of all $v \in \mathbb{R}^n$ satisfying (2) and (3) in the lemma above. By the lemma above, $D(\eta)$ is the set of all vectors $v \in \mathbb{R}^n$ so that no neighborhood of v is contained in a finite union of closed sets $\overline{R}(V)$ for cluster tilting objects V. We call $D(\eta)$ the *support of the null root.*

Proposition 3.6. *A vector $v \in \mathbb{R}^n$ lies in $D(\eta)$ if and only if it satisfies the following.*
(1) *$\langle v, \eta\rangle = 0$, i.e., $v \in H$ and*
(2) *$\langle v, \beta\rangle \le 0$ for all preprojective roots β. I.e., none of the equivalent conditions in the lemma above hold.*

Also, $D(\eta)$ is the closure of the set of all $w \in H$ so that w does not lie in $\overline{R}(V)$ for any cluster tilting object V.

Proof. This follows from the lemma and the observation that vectors not on the horizon do not lie in $D(\eta)$. To see this, take any v above the horizon and let $\pi_v : \mathbb{Z} \to \mathbb{R}$ be a corresponding periodic map. Then π_v has positive slope, say m/n. If $\pi' : \mathbb{Z} \to \mathbb{R}$ is another periodic so that $|\pi'(i) - \pi_v(i)| < \frac{m}{3n}$ then, for each i there are at most n integers j so that $\pi_v(i) - \pi_v(j)$ and $\pi_v(i) - \pi_v(j)$ have opposite sign (since this can happen for at most one j in every coset of $n\mathbb{Z}$ in $\mathbb{Z}$.) Therefore, there are only finitely many isomorphism classes of periodic monomorphisms π' near π_v when the slope of π_v is positive. The case of negative slope is similar.

The last statement was proved in the last paragraph of the proof of the lemma above. $\square$

Corollary 3.7. *$D(\eta)$ is a convex polytope in H with finitely many sides. The boundary of $D(\eta)$ is the set of all $v \in D(\eta)$ so that $\langle v, \beta \rangle = 0$ for some preprojective root β. In particular, η lies in the interior of $D(\eta)$.*

Proof. Since $\langle \eta, \eta \rangle = 0$ and $\langle \eta, \beta \rangle = -1 < 0$ for all preprojective β, η lies in the interior of $D(\eta)$ since $\langle v, \beta \rangle < 0$ for all preprojective β is an open condition. Conversely, suppose that $\langle v, \beta \rangle = 0$ for some preprojective β. Then, for any $t > 0$ we have

$$\langle v - t\eta, \beta \rangle = \langle v, \beta \rangle - t \langle \eta, \beta \rangle = 0 + t = t > 0$$

Therefore, $v \in \underset{\cdot}{D}(\eta)$ if and only if $v \in H$ and $\langle v, \beta \rangle = 0$ for some preprojective β. $\square$

Let $\pi_v : \mathbb{Z} \to \mathbb{R}$ be a periodic map with slope 0 corresponding to $v \in D(\eta)$. Then Condition (2) in the Proposition above is equivalent to the condition that $\pi_v(a) \geq \pi_v(b)$ for all a, b with $\varepsilon_a = -, \varepsilon_b = +$. For any $1 \leq a, b \leq n$ with $\varepsilon_a = -, \varepsilon_b = +$ let $D_{ab}(\eta)$ denote the set of all $v \in D(\eta)$ with the property that, for any corresponding periodic function π_v, $\pi_v(a) \leq \pi_v(i)$ for all i with $\varepsilon_i = -$ and $\pi_v(b) \geq \pi_v(j)$ for all j with $\varepsilon_j = +$. Then we obtain a finite decomposition:

$$D(\eta) = \bigcup D_{ab}(\eta) \tag{3.1}$$

where the union is over all $1 \leq a, b \leq n$ with $\varepsilon_a = -, \varepsilon_b = +$. The definition of $D_{ab}(\eta)$ can be rephrased as follows.

Proposition 3.8. *Let k be any integer so that $a + kn > b + n$. Then $D_{ab}(\eta)$ is the set of all $v \in D(\eta)$ so that $\langle v, \beta' \rangle \leq 0$ for all proper subroots $\beta' \subsetneq \beta_{b,a+kn}$.*

Proof. Proper subroots of $\beta_{a,b+kn}$ are either preprojective or regular. But the condition $\langle v, \beta' \rangle \leq 0$ for preprojective roots holds for all $v \in D(\eta)$. So, it suffices to consider the regular subroots β' of $\beta_{b,a+kn}$. These are β_{bd} where $b < d < b + n$ with $\varepsilon_d = +$ and β_{ca} where $a - n < c < a$ with $\varepsilon_c = -$. But the condition $\langle v, \beta_{bd} \rangle = \pi_v(d) - \pi_v(b) \leq 0$ for all such d is equivalent to the maximality of $\pi_v(b)$ and the condition $\langle v, \beta_{ca} \rangle = \pi_v(a) - \pi_v(c) \leq 0$ for all c with $\varepsilon_c = -$ is equivalent to the minimality of $\pi_v(a)$. $\square$

3.4. **Noncrossing roots.**

Definition 3.9. Two real Schur roots $\beta_{ij}, \beta_{k\ell}$ will be called *noncrossing* if they are edge vectors of two disjoint edges of the same n-periodic tree $\mathcal{T}$.

Lemma 3.10. *Two real Schur roots $\beta_{ij}, \beta_{k\ell}$ so that $j - i \leq \ell - k$ are noncrossing if and only if the following conditions hold.*

1. *i, j, k, ℓ are distinct modulo n.*
2. *If $k < i + rn < j + rn < \ell$ for some integer r then $\varepsilon_i = \varepsilon_j$.*
3. *If $i + sn < k < j + sn < \ell$ for some integer s then $\varepsilon_k \neq \varepsilon_j$.*
4. *If $k < i + tn < \ell < j + tn$ for some integer t then $\varepsilon_i \neq \varepsilon_\ell$.*

Proof. It is clear that these conditions are necessary. For (1), n does not divide $j-i$ and $\ell-k$ since $\beta_{ij}, \beta_{k\ell}$ are real Schur roots. Also, i, j cannot be congruent to k, ℓ since the corresponding edges ℓ_1, ℓ_2 are disjoint. For (2), ℓ_1 must lie either over or under ℓ_2. In the first case we must have $\varepsilon_i = \varepsilon_j = -$ and in the second case $\varepsilon_i = \varepsilon_j = +$. For (3), the right end of ℓ_1 lies either over or under the left end of ℓ_2. In the first case $j = -$ and $k = +$. In the second case $j = +, k = -$. (4) is similar.

Conversely, suppose that these conditions are satisfied. Then we will construct an n-periodic tree two of whose edge vectors are β_{ij} and $\beta_{k\ell}$.

First, suppose both (2), (3) hold. Then $s < r$. So, we would have $k < j + sn < j + rn < \ell$ making $\ell - k > n$. Then $\beta_{k\ell}$ cannot be a regular root. So, $\varepsilon_k \neq \varepsilon_\ell$ which forces $\varepsilon_i = \varepsilon_\ell$. So, (4) cannot hold in this case. By symmetry we assume $\varepsilon_i = \varepsilon_j = \varepsilon_\ell = +$ and $\varepsilon_k = -$. Let $\psi : \mathbb{Z} \to \mathbb{R}$ be the n-periodic function with slope 1 given by

$$\psi(m) = \begin{cases} m - k & \text{if } \varepsilon_m = - \\ m + (j-i) - \ell - \frac{2}{3} & \text{if } m \text{ is congruent to } i \text{ mod } n \\ m - \ell - \frac{1}{3} & \text{otherwise.} \end{cases}$$

Since ψ is a monomorphism, there is a unique n-periodic tree $\mathcal{T}$ with ψ as periodic morphism by Theorem 1.5.1 in [10]. $\psi(i) < \psi(j)$ are consecutive values of ψ as are $\psi(k) > \psi(\ell)$. When $i < m < j$ with $\varepsilon_m = +$ we have $m \leq j - 1$. So,

$$\psi(m) = m - \ell - \tfrac{1}{3} < j - \ell - \tfrac{2}{3} = \psi(i) < \psi(j).$$

If $k < i + rn < m < j + rn < \ell$ with $\varepsilon_m = -$ we have

$$\psi(m) = m - k > 0 > j + rn - \ell - \tfrac{1}{3} = \psi(j + rn) > \psi(i + rn).$$

Therefore, by Corollary 1.2.13 in [10], the unique periodic tree $\mathcal{T}$ with n-periodic morphism ψ has β_{ij} as edge vector. A similar calculation using the fact that (4) does not occur shows that $-\beta_{k\ell}$ is an edge vector of $\mathcal{T}$. Indeed, if $k < m < \ell$ and $\varepsilon_m = -$ then $\psi(m) = m - k > 0 = \psi(k) > \psi(\ell) = -\frac{1}{3}$. If $k < m < \ell$ and $\varepsilon_m = +$ then $\psi(m) = m - \ell - \frac{1}{3} < -\frac{1}{3} = \psi(\ell)$ unless $m \equiv i$ in which case $m - i + j < \ell$ by not(4). So, $\psi(m) = m + (j-i) - \ell - \frac{2}{3} < -\frac{2}{3} < \psi(\ell) = -\frac{1}{3}$ in that case as well. So, the lemma holds in this case.

Next, suppose that both (3), (4) hold. Then (2) does not hold by the discussion above. There are two cases: $\varepsilon_i = \varepsilon_j$ and $\varepsilon_i \neq \varepsilon_j$. Take the first case. By symmetry assume $\varepsilon_i = \varepsilon_j = -$ and $\varepsilon_k = \varepsilon_\ell = +$. Taking s maximal in (3) and t minimal in (4) we may assume $s = 0, t = 1$. So, $j - n < i < k < j < \ell < j + n$. Let $\psi : \mathbb{Z} \to \mathbb{R}$ be the n-periodic monomorphism of slope 1 given by

$$\psi(m) = \begin{cases} m - j + \frac{1}{4} & \text{if } m \text{ is congruent to } j \text{ mod } n \\ m - i & \text{if } \varepsilon_m = - \text{ and } m \text{ is not congruent to } j \text{ mod } n \\ m - n - k - \frac{1}{2} & \text{if } m \text{ is congruent to } k \text{ mod } n \\ m - n - \ell - \frac{1}{4} & \text{otherwise} \end{cases}$$

Then $\psi(i) < \psi(j)$ and $\psi(k) < \psi(\ell)$ are consecutive values of ψ and the unique n-periodic tree with n-periodic morphism ψ has β_{ij} and $\beta_{k\ell}$ as two of its edge vectors.

In the second case we assume by symmetry that $\varepsilon_i = \varepsilon_k = -$, $\varepsilon_j = \varepsilon_\ell = +$ and $j - i \le \ell - k$. Then we use the n-periodic monomorphism given by

$$\psi(m) = \begin{cases} m & \text{if } \varepsilon_m = - \\ m - (j-i) + \frac{1}{3} & \text{if } m \text{ is congruent to } j \text{ mod } n \\ m - (\ell - k) + \frac{2}{3} & \text{otherwise} \end{cases}$$

Then $\psi(i) < \psi(j)$ and $\psi(k) < \psi(\ell)$ are consecutive values of ψ and the unique n-periodic tree with periodic morphism ψ has edge vectors β_{ij} and $\beta_{k\ell}$.

The case when (2), (4) both hold is analogous to the case when (2), (3) both hold. The case when only one or none of the conditions (2), (3), (4) hold is easier to verify. So, the lemma holds in all cases. □

Theorem 3.11. *Suppose that β_{ij} and $\beta_{k\ell}$ are real Schur roots of $K\widetilde{A}^\varepsilon_{n-1}$. Then $\beta_{ij}, \beta_{k\ell}$ are noncrossing if and only if the corresponding exceptional $K\widetilde{A}^\varepsilon_{n-1}$-modules are hom-orthogonal and do not extend each other.*

Proof. If the modules corresponding to $\beta_{ij}, \beta_{k\ell}$ are not hom-orthogonal then there is an integer p so that the intersection of closed intervals $[i+pn, j+pn] \cap [k, \ell]$ supports a module which is a quotient of one and a submodule of the other. But conditions (1), (2), (3), (4) in the lemma do not allow this for any value of p.

To see that the modules do not extend each other, we compute the pairings $\langle \beta_{ij}, \beta_{k\ell} \rangle = 0 = \langle \beta_{k\ell}, \beta_{ij} \rangle$. These pairings are the sums over all integers p of the pairings of $\beta_{i+pn,j+pn}$ with $\beta_{k\ell}$ in the infinite covering quiver. And each of these is zero by the lemma. □

3.5. **Propresentation of progroup for A^ϵ_{n-1}.** Recall that a quiver of finite type with n vertices gives an $n-1$ dimensional picture giving a simplicial decomposition of S^{n-1}. The codimension 1 faces are labeled with generators $x(\beta)$ where β are the positive roots of the Dynkin diagram associated to the quiver. The union of the faces with label $x(\beta)$ is equal to the support $D(\beta)$ of the associated determinantal semi-invariant c_β. The support of β is a subset of the hyperplane consisting of all v so that $\langle v, \beta \rangle = 0$. These hyperplanes are all distinct.

For propictures for the progroup associated to a quiver of type $\widetilde{A}_{n-1}$ the support $D(\eta)$ of the null root η breaks up into a finite union $D(\eta) = \bigcup D_{ab}(\eta)$ which all lie on the same hyperplane H. But the $D_{ab}(\eta)$ will be labeled with distinct generators of our progroup.

Given a quiver $\widetilde{A}^\varepsilon_{n-1}$ of type $\widetilde{A}_{n-1}$ with sign function ε, we will get a tame propicture for a tame progroup $\hat{G}(\widetilde{A}^\varepsilon_{n-1})$. This tame progroup has two kinds of generators:

(1) a generator $x(\beta)$ for every real Schur root β. We call these *real generators.*
(2) a generator η_{ab} for every $1 \le a, b \le n$ where $\varepsilon_a = -$ and $\varepsilon_b = +$. We call these *null generators.*

The *cluster propicture* will be defined to be the union of the sets $D(\beta)$ with labels $x(\beta)$ and the sets $D_{ab}(\eta)$ with labels η_{ab}. This will be an inverse system of pictures for our progroup.

To make sense of the relations, we will work over a more familiar progroup, namely

$$SL_n(\mathbb{Z}[[t]]) = \lim SL_n(\mathbb{Z}[t]/(t^k))$$

The real generator $x(\beta_{ij})$ lies over the elementary matrix $E_{ij}(\varepsilon_i t^{j-i})$ which is equal to the identity matrix I_n except for its ij entry (indices taken modulo n) which is equal to $\varepsilon_i t^{j-i}$.

This is a unimodular matrix since $j-i$ is not divisible by n. The null generator η_{ab} will lie over the diagonal matrix D_{ab} with diagonal entries

$$d_i = \begin{cases} (1-t^n)^{-1} & \text{if } i \equiv a (\text{mod } n) \\ 1-t^n & \text{if } i \equiv b (\text{mod } n) \\ 1 & \text{otherwise} \end{cases}$$

Here $(1-t^n)^{-1} = 1 + t^n + t^{2n} + t^{3n} + \cdots \in \mathbb{Z}[[t]]$. So, D_{ab} is also a unimodular $n \times n$ matrix with coefficients in $\mathbb{Z}[[t]]$.

Definition 3.12. Given any n-periodic sign function ε we define $\hat{G}(\widetilde{A}_{n-1}^{\varepsilon})$ to be the progroup with pro-finite presentation given as follows. The generators are

(a) $x(\beta_{ij})$, with degree $j-i$, for all real Schur roots β_{ij}.
(b) η_{ab}, with degree n, for all $1 \le a, b \le n$ with $\varepsilon_a = -, \varepsilon_b = +$.

The relations are

(1) $x(\beta_{ij})$ and $x(\beta_{ab})$ commute if β_{ij}, β_{ab} are noncrossing.
(2) $x(\beta_{ij})x(\beta_{jk}) = x(\beta_{jk})x(\beta_{ik})x(\beta_{ij})$ if i, j, k are distinct modulo n and $\varepsilon_j = +$.
(3) $x(\beta_{ij})x(\beta_{ik})x(\beta_{jk}) = x(\beta_{jk})x(\beta_{ij})$ if i, j, k are distinct modulo n and $\varepsilon_j = -$.
(4) $x(\beta_{ij})$ and η_{ab} commute if β_{ij} and $\beta_{a,b+kn}$ are noncrossing for all positive integers k. (In particular, β_{ij} is regular.)
(5) $x(\beta_{ij})\eta_{kj}x(\beta_{j,i+n}) = x(\beta_{j,i+n})\eta_{ki}x(\beta_{ij})x$ if $i < j < i+n$ are positive and k is negative.
(6) $x(\beta_{ij})\eta_{ik}x(\beta_{j,i+n}) = x(\beta_{j,i+n})\eta_{jk}x(\beta_{ij})x$ if $i < j < i+n$ are negative and k is positive.
(7) $x(\alpha)x(\beta) = x(\beta)x(\beta+\eta)x(\beta+2\eta)\cdots\eta_{ab}\cdots x(\alpha+2\eta)x(\alpha+\eta)x(\alpha)$ for any $a < b < a+n$ with $\varepsilon_a = -, \varepsilon_b = +$ where we use the notation $\alpha = \beta_{ab}$ and $\beta = \beta_{b,a+n}$. This relation is equivalent to

$$\eta_{ab}^{-1} = \cdots x(\alpha+2\eta)x(\alpha+\eta)x(\alpha)x(\beta)^{-1}x(\alpha)^{-1}x(\beta)x(\beta+\eta)x(\beta+2\eta)\cdots$$

It is easy to verify that these relations holds in the matrix group $SL_n(\mathbb{Z}[[t]])$. For example, we check relation (7). Since only the indices a, b are involved, we may assume $n = 2, a = 1, b = 2$. Then

$$\prod_{k=0}^{\infty} E_{21}(-t^{2k+1})E_{12}(-t)E_{21}(t)\prod_{k=0}^{\infty} E_{12}(t^{2k+1})$$

$$= E_{21}(-t(1-t^2)^{-1})E_{12}(-t)E_{21}(t)E_{12}(t(1-t^2)^{-1}) = D_{12}^{-1}$$

By this last relation, η_{ab} is an (infinite) product of the other generators. So, η_{ab} is not needed to generate the progroup $\hat{G}(\widetilde{A}_{n-1}^{\varepsilon})$. However, we need this for the cluster propicture since η_{ab} is the label for the set $D_{ab}(\eta)$. The following two figures are examples of "null pictures" which we will define in general later. These are pictures for the group $\hat{G}(\widetilde{A}_{n-1}^{\varepsilon})$ in the usual sense and, being pictures, they use only finitely many generators and use only the finite relations among these generators. Figures 6 and 7 show the null pictures for two possible orientations of $\widetilde{A}_3$.

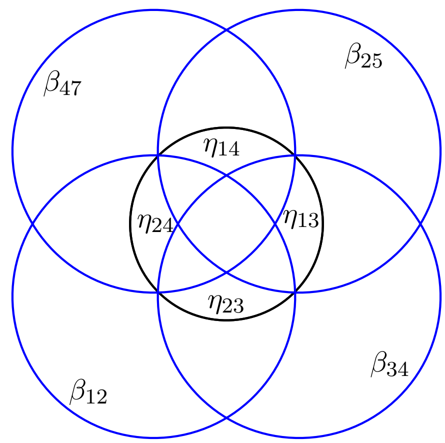


FIGURE 6. Example of picture for $\widetilde{A}_3$ with sign function $(+, +, -, -)$. All four null generators $\eta_{13}, \eta_{14}, \eta_{23}, \eta_{24}$ appear. All four regular real Schur roots appear as the four blue circles labeled $x(\beta_{ij})$. And these commute. So there are no five-valent vertices representing relations (2),(3). All labels are on the positive side of each edge. Relations (1), (5) and (6) are used in the picture.

In Figure 7, the matrix at (1) is:

$$\begin{bmatrix} (1-t^4)^{-1} & 0 & 0 & t^3 \\ t^3(1-t^4)^{-1} & 1 & 0 & t^2 \\ t^2(1-t^4)^{-1} & 0 & 1 & t \\ 0 & 0 & 0 & 1-t^4 \end{bmatrix}$$

When we cross the black arc labelled η_{14}, we right multiply by D_{14}^{-1}. So the matrix at (2) in Figure 7 is

$$\begin{bmatrix} 1 & 0 & 0 & t^3(1-t^4)^{-1} \\ t^3 & 1 & 0 & t^2(1-t^4)^{-1} \\ t^2 & 0 & 1 & t(1-t^4)^{-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When we go from region (2) to region (3) we do the column operations $E_{21}(-t^3)$, $E_{31}(-t^2)$, $E_{12}(t)$, $E_{23}(t)$ to get the following matrix in region (3):

$$\begin{bmatrix} 1 & t & t^2 & t^3(1-t^4)^{-1} \\ 0 & 1 & t & t^2(1-t^4)^{-1} \\ 0 & 0 & 1 & t(1-t^4)^{-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.6. **Cluster propicture for A^{ϵ}_{n-1}.** The cluster propicture is a sequence of pictures $L_k \subset S^{n-1}$ for the inverse system of groups G_k where G_k is the group $\hat{G}(\widetilde{A}^{\varepsilon}_{n-1})$ modulo all generators of degree $> k$.

Definition 3.13. Given a quiver $\widetilde{A}^{\varepsilon}_{n-1}$ of type $\widetilde{A}_{n-1}$ with sign ε and $k \geq 1$, the *cluster picture of level* k, denoted L_k is defined as follows. As a set, $L_k \subseteq S^{n-1}$ is the union of the following sets.

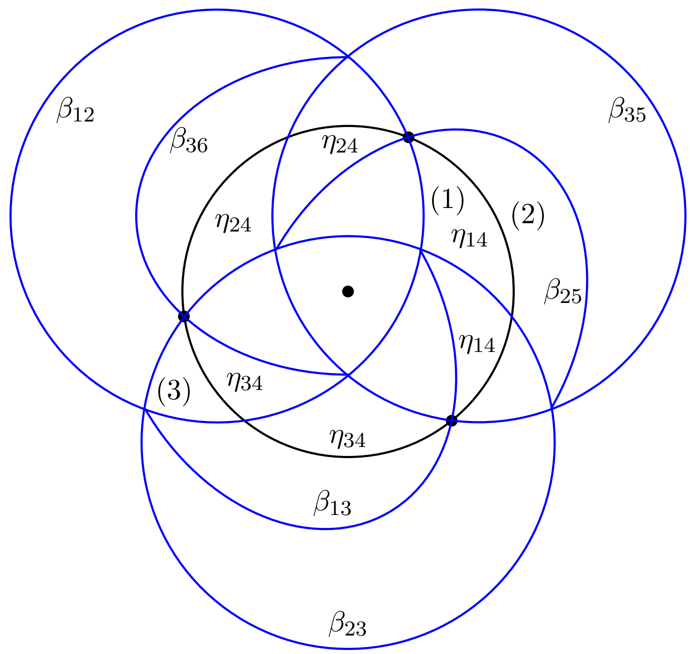


FIGURE 7. Example of picture for $\widetilde{A}_3$ with sign function $(+,+,+,-)$. All three null generators $\eta_{14}, \eta_{24}, \eta_{34}$ appear as do all six regular real Schur roots. Each 5-valent vertex is Relation (2). Every 4-valent vertex is Relation (4) and every 6-term relation is Relation (5).

(1) $D(\beta_{ij}) \cap S^{n-1}$ where β_{ij} is a real Schur root of length $j - i \le k$. (Recall that $D(\beta)$ is the support of β and is given by the virtual stability theorem.)
(2) $D(\eta) \cap S^{n-1}$ if $k \ge n$.

The sets $D(\beta)$ are labeled with the real generator $x(\beta)$ and oriented in the direction of the root β. The subset $D_{ab}(\eta)$ of $D(\eta)$ is labeled η_{ab} and oriented in the direction of preprojective roots. We will show that this defines a picture for G_k.

Example 3.14. Consider the quiver Q_{++-} of type $\widetilde{A}_2$ with sign $(+,+,-)$ at level 3. This means we mod out all generators of degree ≥ 4. There are 6 real Schur roots of length ≥ 3:

(1) Two preprojective roots β_{01}, β_{02}.
(2) Two preinjective roots β_{13}, β_{23}.
(3) Two regular roots β_{12}, β_{24}

Also, there are two null generators η_{13}, η_{23}. Figure 3.6 gives the cluster picture for the group G_3. The center vertex is the location of the null root η and the 6 regions which touch the null root are unstable. The other regions are stable and the image in $SL_3(\mathbb{Z}[t]/(t^4))$ of the group labels are typed in.

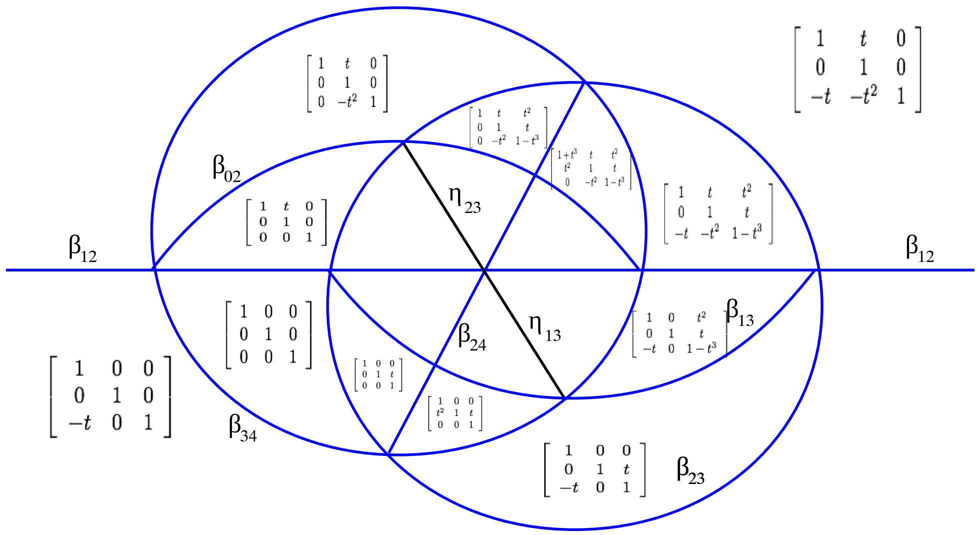


FIGURE 8. Cluster propicture for $\widetilde{A}_2$, level 3 with $SL_3(\mathbb{Z}[t]/(t^4))$ labels in stable regions. All edges are oriented away from the negative projective cluster tilting object (labeled with identity matrix).

We need to analyze and describe the cluster propicture one piece at a time using the codimension 2 simplices ρ as organizational tools. These come in three types: They are either infinite or finite and the finite ones are either stable or unstable.

(1) Infinite ρ. There simplices are the boundaries of infinitely many codimension 1 simplices τ_i. We will show that there are a finite number of such ρ and the link $Lk(\rho)$ of these simplices give Relation (7). (Recall that $Lk(\rho)$ is the poset of simplices in S^{n-1} which contain $Lk(\rho)$ considered as a simplicial complex.)
(2) Finite stable ρ. These simplices are the boundaries of finitely many codimension 1 simplices and the regions between these simplices are eventually all stable. The link of ρ gives one of the Relations (1), (2) or (3).
(3) Finite unstable ρ. These simplices are the boundaries of finitely many codimension 1 simplices but the regions between these simplices remain unstable. The link of ρ in this case gives Relations (4), (5) or (6).

3.6.1. *Codimension 2 simplices of infinite type.* Suppose that ρ is a codimension 2 simplex of L_k so that the image of ρ in L_∞ lies in the intersection of infinitely many $D(\tau)$. Then, there exist integers a, b so that ρ is contained in $D(\beta_{a,b+jn})$ for infinitely many positive integers j. Also, since there are only finitely many regular Schur roots, $\beta_{a,b+jn}$ must be either proprojective or preinjective. I.e., $\varepsilon_a \neq \varepsilon_b$.

For any $v \in \rho$, the stability conditions give:

(1) $\langle v, \beta_{a,b+jn}\rangle = 0$.
(2) $\langle v, \beta\rangle \leq 0$ for all $\beta \subseteq \beta_{a,b+jn}$.

Since $\beta_{a,b+jn} = \beta_{ab} + j\eta$, having (1) hold for more than one value of j implies that $\langle v, \eta \rangle = 0$. I.e., v lies on the horizon H. Since every preprojective root is a subroot of $\beta_{a,b+jn}$ for a sufficiently large j, having (2) hold for infinitely many positive j implies that $v \in D(\eta)$. By Corollary 3.7, v lies in the boundary of $D(\eta)$. Thus ρ is a subset of $\partial D(\eta)$.

This implies that $\pi_v(i) \geq \pi_v(j)$ for all i, j so that $\varepsilon_i = -, \varepsilon_j = +$ where π_v is any periodic function corresponding to v. But (2) implies $\langle v, \beta_{a,b+jn} \rangle = \pi_v(b) - \pi_v(a) = 0$. Thus $\pi_v(a) = \pi_v(b)$. Since a, b have opposite sign, we have either $v \in D_{ab}(\eta)$ or $v \in D_{ba}(\eta)$ depending on whether $\varepsilon_a = +$ or $\varepsilon_b = +$ respectively.

The converse of this statement is also true:

Proposition 3.15. *Let $1 \leq a, b \leq n$ with $\varepsilon_a = -, \varepsilon_b = +$. Then*

$$D_{ab}(\eta) \cap \partial D(\eta) = \bigcap D(\beta_{b,a+jn}) = \bigcap D(\beta_{a,b+kn})$$

where the first intersection is over all integers j so that $b+jn > a$ and the second intersection is over all k so that $a + kn > b$. Furthermore, no point outside of $D_{ab}(\eta) \cap \partial D(\eta)$ lies in infinitely many of the sets $D(\beta_{a,b+jn})$ and $D(\beta_{b,a+kn})$. Thus, $\partial D(\eta) = \bigcup (D_{ab}(\eta) \cap \partial D(\eta))$ is the set of all vectors in $\mathbb{R}^n$ which lie in infinitely many $D(\beta)$.

Proof. The "furthermore" part is what we already proved. Thus, we know that both infinite intersections are contained in $D_{ab}(\eta) \cap \partial D(\eta)$. Conversely, suppose $v \in D_{ab}(\eta) \cap \partial D(\eta)$. Then, for all i, j with $\varepsilon_i = +, \varepsilon_j = -$ we have by definition of $D_{ab}(\eta)$ that

$$\pi_v(i) \leq \pi_v(b) \leq \pi_v(a) \leq \pi_v(j).$$

By Corollary 3.7, $v \in \partial D(\eta)$ implies $\pi_v(i) = \pi_v(j)$ for some i, j. So, we must have $\pi_v(a) = \pi_v(b)$. Thus $\langle v, \beta \rangle = 0$ for all β in the two infinite intersections.

If $\beta = \beta_{a,b+kn}$ is preprojective then any $\beta' \subseteq \beta_{a,b+kn}$ is also preprojective. So, $\langle v, \beta' \rangle \leq 0$ since $v \in D(\eta)$. Therefore, v lies in every term $D(\beta_{a,b+kn})$ in the second infinite intersection.

Suppose $\beta = \beta_{b,a+jn}$. Then any proper subroot of β is either preprojective or has the form β_{bi} or β_{ja}. In the second case, $\langle v, \beta_{bi} \rangle = \pi_v(i) - \pi_v(b) \leq 0$ since $\pi_v(i) \leq \pi_v(b)$. Similarly, $\langle v, \beta_{ja} \rangle = \pi_v(a) - \pi_v(j) \leq 0$. So, v lies in every term $D(\beta_{b,a+jn})$ in the first infinite intersection. So, the three sets are equal. ☐

Corollary 3.16. *Let ρ be a codimension 2 simplex in $D_{ab}(\eta) \cap \partial D(\eta)$. Then $\rho \subseteq D(\beta)$ if and only if β is equal to one of the preprojective roots $\beta_{a,b+kn}$ or preinjective roots $\beta_{b,a+jn}$ in the proposition above.*

Proof. Let $v \in \rho$ be a general point. Then v and thus π_v are constrained by at most two linearly independent equations, namely, the slope of π_v is zero and $\pi_v(a) = \pi_v(b)$. So, for general v we have

$$\pi_v(i) < \pi_v(b) \leq \pi_v(a) < \pi_v(j).$$

for all $i \neq a$ with $\varepsilon_i = +$ and all $j \neq b$ with $\varepsilon_j = -$. If $v \in D(\beta_{ij})$ then $\pi_v(i) = \pi_v(j)$. So, we must have $i, j = a, b$ modulo n. The roots listed are the only ones having this property. ☐

Lemma 3.17. *Let ρ be a codimension 2 simplex in $D_{ab}(\eta) \cap \partial D(\eta)$ and suppose that $\rho \subset D(\beta)$. Then ρ lies on the boundary of $D(\beta)$ if and only if β has length $> n$.*

Proof. Assume for notational simplicity that $a < b < a + n$. If β has length $> n$ then $D(\beta)$ has no points below the horizon for β preprojective and no points above the horizon for β preinjective. Conversely, if β has length $< n$ then either $\beta = \beta_{ab}$ or $\beta_{b,a+n}$. We can choose a periodic function π with zero slope taking n distinct values so that

$$\pi(i) < \pi(a) < \pi(b) < \pi(j).$$

for all $i \neq a$ with $\varepsilon_i = +$ and all $j \neq b$ with $\varepsilon_j = -$. By Lemma 3.5 (6), the vector $\pi = \pi_v$ lies in $\overline{R}(\mathcal{T})$ for some periodic tree $\mathcal{T}$. But, π cannot lie on the boundary of $R(\mathcal{T})$ since $\pi(i) \neq \pi(j)$ for i, j not congruent modulo n. So, $\pi \in R(\mathcal{T})$. Let V be the corresponding cluster tilting object. Since there are no values of π between $\pi(a)$ and $\pi(b)$, two of the edge vectors of $\mathcal{T}$ are β_{ab} and $-\beta_{b,a+n}$. So, $\partial\overline{R}(V)$ contains a piece of $D(\beta_{ab})$ below the horizon and a piece of $D(\beta_{b,a+n})$ above the horizon showing that the interior of ρ is contained in the interiors of both of these supports. Without loss of generality, suppose W is cluster tilting object such that $R(W)$ has a common face (namely $\beta_{b,a+n}$) with $R(V)$. Let $u = b_\rho + \epsilon t$, where $b_\rho \in \rho$, $\epsilon > 0$ and $t \in \mathbb{Z}$. Then, $u \in R(W)$. Moreover, $\pi_u = \pi_\eta + \epsilon\pi_t$, where $\pi_t(k) = k$ is a periodic morphism satisfying $\pi_u(i) < \pi_u(a)$ for $a < i < a+n$, $\pi_u(a+n) < \pi_u(j)$ for $a < j < a+n$, and $\pi_u(a) < \pi_u(b) < \pi_u(a+n)$. Thus, $\text{int}\rho \subset \text{int}D(\beta_{b,a+n})$ above the horizon. Similar argument for $\epsilon < 0$. □

Theorem 3.18. *Suppose that ρ is an $n-3$ simplex of $L_k \subseteq S^{n-1}$ which is contained in an unbounded number of $n-2$ simplices of L_s as $s \geq k$ goes to infinity. Then the labels of the $n-2$ simplices of L_k in cyclic order around ρ give the word*

$$x(\beta)^{-1}x(\alpha)^{-1}x(\beta)x(\beta+\eta)x(\beta+2\eta)\cdots\eta_{ab}\cdots x(\alpha+2\eta)x(\alpha+\eta)x(\alpha)$$

where $\alpha = \beta_{b,a+n}$ and $\beta = \beta_{ab}$ for some $a < b < a+n$ where $\varepsilon_a = -, \varepsilon_b = +$.

Proof. It remains to show that the sets $D(\beta + j\eta)$ and $D(\alpha + k\eta)$ are in the stated cyclic order with the stated normal orientations. To prove this, we project along the simplex ρ onto the plane as follows.

Let $f : \mathbb{R}^n \to \mathbb{R}^2$ be the linear function given by

$$f(v) = (\langle v, \beta_{ab}\rangle, \langle v, \eta\rangle) = (\pi_v(b) - \pi_v(a), \pi_v(n) - \pi_v(0)).$$

Then $D_{ab}(\eta) \cap \partial D(\eta)$ lies in the kernel of f. The inverse image of the x-axis is the horizon H and $y > 0$, $y < 0$ give points above and below the horizon. The point $v = \eta$ goes to $f(\eta) = (1, 0)$. And the dimension vector $P = \sum \underline{\dim} P_i$ of the sum of all projective module goes to $f(P) = (b-a, n)$. For long preprojective roots $\beta_{a,b+jn}$, $j > 0$, the support $D(\beta_{a,b+jn})$ maps to a ray going from the origin into the region between $f(\eta)$ and $f(P)$ and is oriented counterclockwise towards $f(P)$ since $D(\beta_{a,b+jn})$ lies above the horizon and $\langle P, \beta_{a,b+jn}\rangle = b - a + jn > 0$ and $\langle \eta, \beta_{a,b+jn}\rangle = -1 < 0$. Similarly, $D(\beta_{b,a+kn})$ for $k \geq 2$ maps to a ray going from the origin into the region between $-f(P)$ and $f(\eta)$ and is also oriented counterclockwise. The supports of the short roots β_{ab} and $\beta_{b,a+n}$ extend to both sides of the origin and are oriented towards $f(P)$.

The support $D(\eta)$ of the null vector goes to the positive x-axis since it contains η. So, it lies in the correct place in cyclic order. To see that the other supports are in correct cyclic order, note that, e.g.,

(1) $\langle P, \beta_{a,b+jn}\rangle = b - a + jn$
(2) $\langle \eta, \beta_{a,b+jn}\rangle = 1$.

So, the ratio between the distance from $D(\beta_{a,b+jn})$ to η and its distance to P is strictly decreasing as j increases. In other words, the slope of the line $f(D(\beta_{a,b_jn})$ is strictly increasing (decreasing in absolute value) as j increases. Similarly, the slop of $f(D(\beta_{b,a+kn})$ is strictly decreasing as k increases. This shows that these supports lie in the correct order with the correct orientations to form the stated relation. □

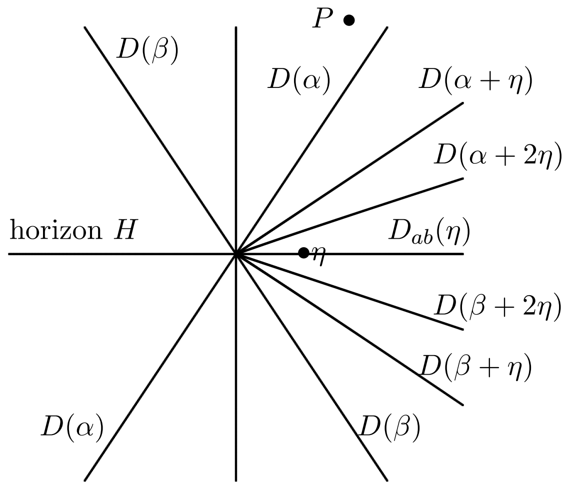


FIGURE 9. Image in $\mathbb{R}^2$ of the supports of $\alpha + k\eta$ and $\beta + j\eta$. All supports are oriented towards P since P is projective.

3.6.2. *Codimension 2 simplices of finite stable type.* Suppose that ρ is an $n-3$ simplex in L_k so that ρ lies in $\overline{R}(V_i)$ for finitely many cluster tilting objects V_i. Then ρ lies in the $n-3$ simplex spanned by the dimension vectors of the objects in a unique partial cluster tilting object M_ρ with $n-2$ summands and the right perpendicular category of this cluster tilting object $M_\rho^\perp := \{X \in mod\, K\widetilde{A}^\varepsilon_{n-1} \mid \mathrm{Hom}(M_\rho, X) = 0 = \mathrm{Ext}(M_\rho, X)\}$ is an abelian category of finite type. So, $M_\rho^\perp$ is of type $A_1 \times A_1$ or A_2. In the first case there are exactly two indecomposable objects in $M_\rho^\perp$ which do not extend each other. These are M_α and M_β.

Lemma 3.19. *ρ lies in $D(\beta)$ if and only if M_β lies in $M_\rho^\perp$. Furthermore, ρ lies in the boundary of $D(\beta)$ if and only if M_β is not a simple object in $M_\rho^\perp$.*

Proof. The first statement is by definition of $D(\beta)$. For the second statement, suppose that M_β is not simple. Then M_β contains a subobject M_α. Since this lies in $M_\rho^\perp$, we have that $\langle v, \alpha\rangle = 0$ for all $v \in \rho$. Since α, β are linearly independent, $D(\beta)$ contains points w so that $\langle w, \alpha\rangle < 0$. So, ρ lie in $\partial D(\beta)$. Conversely, suppose that M_β is simple. Then $\langle v, \alpha\rangle < 0$ for all $\alpha \subsetneq \beta$ since otherwise M_α lies in $M_\rho^\perp$ contradicting the assumption that M_β is simple. This is an open condition on v. So, ρ cannot lie on the boundary of $D(\beta)$. □

Theorem 3.20. *Suppose that ρ is an $n-3$ simplex in L_k which lies in finitely many sets $\overline{R}(V)$. Then either*

(0) *ρ lies in exactly one support $D(\beta)$ and ρ is not in the boundary of $D(\beta)$. So, the word around ρ is $x(\beta)x(\beta)^{-1}$ which is the trivial relation.*
(1) *ρ lies in exactly two supports $D(\alpha), D(\beta)$ so that α, β do not extend each other and so that ρ does not lie on the boundary of either. So, the word around ρ is a commutator of $x(\alpha)$ and $x(\beta)$.*
(2) *ρ lies in exactly three supports $D(\beta_{ab}), D(\beta_{bc})$ and $D(\beta_{ac})$ and ρ lies on the boundary of $D(\beta_{ac})$. The word in this case is*

$$x(\beta_{ab})^{-1}x(\beta_{bc})^{-1}x(\beta_{ab})x(\beta_{ac})x(\beta_{bc})$$

if $\varepsilon_c = -$ and

$$x(\beta_{bc})^{-1}x(\beta_{ab})^{-1}x(\beta_{bc})x(\beta_{ac})x(\beta_{ab})$$

if $\varepsilon_c = +$.

In both nontrivial cases, the word given by reading the labels of the $n-2$ simplices containing ρ in cyclic order is one of the first three relations in the progroup of $\widetilde{A}^{\varepsilon}_{n-1}$.

Proof. This basically follows from the lemma. The only thing we need to check is that the supports $D(\beta)$ are in the correct order with the correct orientation. The only questionable case is the third case where the position of the commutator is not obvious.

The two cases of the last case can be combined using the following notation. Let $\beta = \beta_{ac}$ be the extension of the other two roots. Let α be the subroot and γ the quotient root. Then the statement is that the word order is:

$$x(\gamma)^{-1}x(\alpha)^{-1}x(\gamma)x(\beta)x(\alpha)$$

To see this, we apply the linear projection $f : \mathbb{R}^n$ to $\mathbb{R}^2$ whose kernel contains ρ. Then $D(\alpha)$ and $D(\gamma)$ maps to lines through the origin. Placing $f(P)$ at the top, these lines are oriented upward. Since $\beta = \alpha + \gamma$, the line $D(\beta)$ has slope between the slopes of $D(\alpha)$ and $D(\gamma)$. But $D(\beta)$ must be on the positive side of $D(\gamma)$ and the negatives side of $D(\alpha)$ by the stability condition that $\langle v, \alpha\rangle \leq 0$ for all $\alpha \subseteq \beta$. The picture must therefore be as in Figure 10 and the theorem follows. $\square$

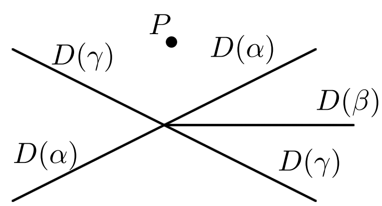


FIGURE 10. $D(\beta)$ cannot extend to the positive side of $D(\alpha)$ (on the left in the figure) since these are the points v where $\langle v, \alpha\rangle > 0$.

3.6.3. *Codimension 2 simplices of finite unstable type.* Finally, suppose that ρ is an open codimension 2 simplex in L_ℓ (with $\ell \geq n$) of finite type for which one of the adjacent regions is unstable. There are two case.

(a) ρ lies in $D(\eta)$.
(b) ρ does not lie in $D(\eta)$.

Since $D(\eta) = \bigcup D_{ab}(\eta)$ and these are subcomplexes, in Case (a), ρ is contained in at least one $D_{ab}(\eta)$. So, Case (a) has subcases depending on how many of these subcomplexes ρ lies in. But $D(\eta)$ has codimension 1 with the linear condition of having zero slope. So, the intersection $D_{ac}(\eta) \cap D_{bd}(\eta)$ will in general have codimension 3 with two more linear conditions given by $\pi(a) = \pi(b)$ and $\pi(c) = \pi(d)$. So, in order for such a intersection to have codimension ≤ 2 we must have either $a = b$ or $c = d$. So, Case 1 has exactly three subcases:

(a1) ρ lies in only one $D_{ab}(\eta)$
(a2) ρ lies in $D_{ac}(\eta) \cap D_{bc}(\eta)$
(a3) ρ lies in $D_{ac}(\eta) \cap D_{ad}(\eta)$

In Subcases (a1) and (a2), the situation is given by the following lemma.

Lemma 3.21. *Suppose that $\varepsilon_a = \varepsilon_b = -$ and $a < b < a+n$. Then*

$$D(\beta_{ab}) \cap D(\beta_{b,a+n}) = \bigcup D_{ac}(\eta) \cap D_{bc}(\eta)$$

where the union is over all c with $\varepsilon_c = +$. Similarly, if $\varepsilon_c = \varepsilon_d = +$ and $c < d < c+n$ then

$$D(\beta_{cd}) \cap D(\beta_{d,c+n}) = \bigcup D_{ac}(\eta) \cap D_{ad}(\eta)$$

where the union is over all a with $\varepsilon_a = -$. Furthermore, these sets are convex polytopes of codimension 2 and the interior of the first set lies in the interior of both $D(\beta_{ab})$ and $D(\beta_{b,a+n})$ and the interior of the second set lies in the interior of both $D(\beta_{cd})$ and $D(\beta_{d,c+n})$.

Proof. We verify only the first statement since the second is completely analogous. Consider a vector $v \in D(\beta_{ab}) \cap D(\beta_{b,a+n})$. Let $\pi_v : \mathbb{Z} \to \mathbb{R}$ be a corresponding periodic function. Then

$$\langle v, \beta_{ab}\rangle = \pi_v(b) - \pi_v(a)$$

So, the conditions $\langle v, \beta_{ab}\rangle = 0$ and $\langle v, \beta_{b,a+n}\rangle = 0$ imply that $\pi_v(a) = \pi_v(b) = \pi_v(a+n)$. These are two linear conditions which imply in particular that π_v has zero slope. Equivalently, $\langle v, \eta\rangle = 0$.

Claim 1: $\pi_v(j) \le \pi_v(b)$ whenever $\varepsilon_j = +$.

To see this note that either $a < j < b$ or $b < j < a+n$ (or $j = a$ or b) up to addition of a multiple of n to j. In the first case, we have $\beta_{aj} \subseteq \beta_{ab}$ and the stability condition implies $\langle v, \beta_{aj}\rangle = \pi_v(j) - \pi_v(a) \le 0$. In the second case, $\langle v, \beta_{bj}\rangle = \pi_v(j) - \pi_v(b) \le 0$. So Claim 1 always holds.

Claim 2: $\pi_v(i) \ge \pi_v(a)$ whenever $\varepsilon_i = -$.

This is similar using the fact that either $a < i < b$ in which case $\beta_{ib} \subseteq \beta_{ab}$ or $b < i < a+n$ in which case $\beta_{i,a+n} \subseteq \beta_{b,a+n}$. Both imply $\pi_v(i) \ge \pi_v(a)$.

Claim 3: $v \in D(\eta)$.

Proof: let β_{ij} be any preprojective root. I.e., $i < j$ and $\varepsilon_i = -, \varepsilon_j = +$. Then $\pi_v(i) \ge \pi_v(a) \ge \pi_v(j)$. So,

$$\langle v, \beta_{ij}\rangle = \pi_v(j) - \pi_v(i) \le 0$$

which implies that $v \in D(\eta)$.

Take c with $\varepsilon_c = +$ so that $\pi_v(c)$ is maximal. Then $v \in D_{ac}(\eta) \cap D_{bc}(\eta)$. So, every $v \in D(\beta_{ab}) \cap D(\beta_{b,a+n})$ lies in $D_{ac}(\eta) \cap D_{bc}(\eta)$ for some c.

Conversely, let v be an element of $D_{ac}(\eta) \cap D_{bc}(\eta)$ for some c for some c with $\varepsilon_c = +$. Then

$$\pi_v(i) \ge \pi_v(c) \ge \pi_v(a) = \pi_v(b) \ge \pi_v(j)$$

for all i with $\varepsilon_i = -$ and all j with $\varepsilon_j = +$. This implies that the subroots $\beta' = \beta_{aj}$ of β_{ab} and $\beta' = \beta_{bj}$ of $\beta_{b,a+n}$ satisfy $\langle v, \beta'\rangle \le 0$. All other subroots of β_{ab} and $\beta_{b,a+n}$ are preprojective, so, we have this condition automatically since $v \in D(\eta)$. So, $v \in D(\beta_{ab}) \cap D(\beta_{b,a+b})$.

Since this set is given by two independent linear equations $\pi_v(a) = \pi_v(b) = \pi_v(a+n)$ and a finite list of linear inequalities and there is a solution set of full dimension (the interior) given by letting π_v take $n-1$ distinct values, it is a convect polytope of codimension 2.

Since a point v in the interior of $D(\beta_{ab}) \cap D(\beta_{b,a+n})$ is characterized by $n-1$ distinct values of π_v, it cannot lie on the boundary of either $D(\beta_{ab})$ or $D(\beta_{b,a+n})$. So, all statements hold. □

Similarly, we have the following.

Lemma 3.22. *The interior of $D_{ab}(\eta)$ meets $D(\beta_{ij})$ if and only if either*

(1) $b < i < j < b + n$ *and* $\varepsilon_i = \varepsilon_j = +$ *or*
(2) $a < i < j < a + n$ *and* $\varepsilon_i = \varepsilon_j = -$.

Proof. In terms of π_v, the interior of $D_{ab}(\eta)$ is given by $\pi_v(i) < \pi_v(b) < \pi_v(a) < \pi_v(k)$ for all $i \neq b$ with $\varepsilon_i = +$ and all $k \neq a$ with $\varepsilon_k = -$. And, for $v \in D(\beta_{ij})$ with $\varepsilon_i = \varepsilon_j = +$ we must also have $\pi_v(i) = \pi_v(j) > \pi_v(c)$ for all $i < c < j$ with $\varepsilon_c = +$. So, we cannot have $i < b < j$. $\square$

Proposition 3.23. *Suppose that L_ℓ is the cluster picture for $\widetilde{A}^\varepsilon_{n-1}$ of level $\ell \geq n$ and ρ is an open $n-3$ simplex in L_ℓ which lies in the interior of $D(\eta)$. Then one of the following holds.*

(0) *(trivial case) ρ lies in the interior of some $D_{ab}(\eta)$ and ρ is not contained in any other support set. In particular, the word given by the labels of the $n-2$ simplices around ρ is $\eta_{ab}\eta_{ab}^{-1}$.*
(1) *ρ lies in the interior of $D_{ab}(\eta)$ and the interior of $D(\beta_{ij})$ for a, b, i, j distinct modulo n and ρ is not contained in any other support set. In this case, the words at ρ is a commutator of η_{ab} and $x(\beta_{ij})$.*
(2) *ρ lies in $D_{ki}(\eta) \cap D_{kj}(\eta) \subseteq D(\beta_{ij}) \cap D(\beta_{j,i+n})$ where $\varepsilon_i = \varepsilon_j = +$, $\varepsilon_k = -$ and the word around ρ is*
$$x(\beta_{ij})\eta_{kj}x(\beta_{j,i+n})x(\beta_{ij})x^{-1}\eta_{ki}^{-1}x(\beta_{j,i+n})^{-1}$$
(3) *ρ lies in $D_{ik}(\eta) \cap D_{jk}(\eta) \subseteq D(\beta_{ij}) \cap D(\beta_{j,i+n})$ where $\varepsilon_i = \varepsilon_j = -$, $\varepsilon_k = +$ and the word around ρ is*
$$x(\beta_{ij})\eta_{ik}x(\beta_{j,i+n})x(\beta_{ij})x^{-1}\eta_{jk}^{-1}x(\beta_{j,i+n})^{-1}$$

In all nontrivial cases, the word around ρ is one of the given relations for $\hat{G}(\widetilde{A}^\varepsilon_{n-1})$.

Proof. Since ρ has codimension 2 it can only satisfy two of its defining linear equalities and consequences of these. So, ρ does not meet any other support sets except for the ones listed. Also, in Case (1), ρ lies in the interior of $D(\beta_{ij})$ since being in the boundary would increase its codimension. So, the word around ρ is a commutator as stated.

In Case (2), we will recover the exact word order by using the linear projection map $f : \mathbb{R}^n \to \mathbb{R}^2$ given by
$$f(v) = (\langle v, \beta_{ij}\rangle, \langle v, \eta\rangle) = (\pi_v(j) - \pi_v(i), \pi_v(i+n) - \pi_v(i))$$
Then $\rho \subseteq D(\beta_{ij}) \cap D(\eta) \subseteq \ker f$. The horizon H maps to the x-axis and $D_{ki}(\eta), D_{kj}(\eta)$ map to the negative and positive x-axis. The set $D(\beta_{ij})$ maps to the y-axis and $D(\beta_{j,i+n})$ maps to the diagonal since its points satisfy $\pi_v(j) = \pi_v(i+n)$. These four sets are oriented as show in Figure 11 since the positive direction faces the projective P which maps to
$$f(P) = (j - i, n)$$
which is in the first quadrant above the diagonal since $0 < j - i < n$. So, the relation is as given. In Case (3), we take the linear projection f with the same equation as in Case (2), however the positive x-axis, where $\pi_v(j) > \pi_v(i)$, is now $D_{ik}(\eta)$ since $\varepsilon_i = -$. And the negative x-axis is $D_{jk}(\eta)$. So, we get the same relation as in Case (2) with η_{ki} replaced with η_{jk} and η_{kj} replaced with η_{ik}. $\square$

Finally, we come to the case where the open codimension 2 simplex ρ is disjoint from $D(\eta)$ but still unstable. Then the closure of ρ meets $D(\eta)$. So, ρ can only be contained

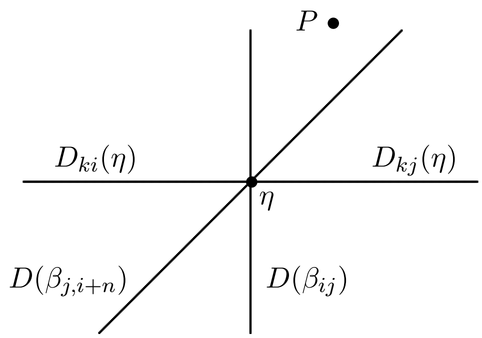


FIGURE 11. Case (2): Image in $\mathbb{R}^2$ of $D(\beta_{ij}), D(\beta_{j,i+n})$, $D_{ki}(\eta), D_{kj}(\eta)$. All supports are oriented towards P since P is projective.

in $D(\beta)$ for regular β which have length $< n$. When we increase the level ℓ, new support sets $D(\beta')$ of larger length will be added to the picture, subdividing ρ into smaller simplices of the same dimension. Each point of ρ will eventually become stable. However, the word on the labels around ρ will remain unchanged since the new $D(\beta')$ will be transverse to ρ. So, the stable relation that we get will be the same word as the word around ρ which is in question. This proves the following.

Proposition 3.24. *If ρ is a codimension 2 simplex in L_ℓ with $\ell \geq n$ so that ρ is not contained in $D(\eta)$. If the closure of ρ meets $D(\eta)$ then the supports $D(\beta)$ which contain ρ are regular and form one of the finite relations of $\hat{G}(\widetilde{A}^{\varepsilon}_{n-1})$.* $\square$

This completes the proof of the propicture theorem.

3.7. **Presentation of pro-groups of type $\widetilde{A}_{n-1}$.** Now we write the pro-finite presentation of any quiver of type $\widetilde{A}_{n-1}$ in general. Since we know the relations and generators for the pro-picture group $G(A^{\epsilon}_{n-1})$, we apply the mutation formula from [BHIT], to obtain a complete set of generators and relations for the pro-picture group $G(Q)$, where Q is a general quiver of type $\widetilde{A}_{n-1}$.

Suppose $\epsilon : [n] \to \{+,-\}$ is a surjective function and define $\widetilde{A}^{\epsilon}_{n-1}$ to be the unoriented n-cycle with vertex set $\{1, 2, \ldots, n\}$ and arrows $i \to i+1$ if $\epsilon_i = -$ and $i \leftarrow i+1$ otherwise. Label the arrow $i \leftrightarrow i+1$ by a_i.

Given a quiver Q of type $\widetilde{A}_n$, Q is mutation equivalent to $\widetilde{A}^{\epsilon}_{n-1}$. By [IK], Q has a unique minimal unoriented cycle η. Following the notation in [DWZ], it is clear that the arrows of Q will be labeled by products of the form $[a^{\pm}_r a^{\pm}_{r+1} \cdots a^{\pm}_s]$ for some $s > r \geq 0$, where $a^+ = a$ and $a^- = a^*$. The surjective function ϵ extends to the arrows in Q in the following sense:

(1) $\epsilon_{[a^{\pm}_r a^{\pm}_{r+1} \cdots a^{\pm}_s]} = \epsilon_{a^{\pm}_r}$ (here note that we could have chosen any integer between r and s).
(2) $\epsilon_{a^-} = -\epsilon_a$.

In [IKTW], generators for A^{ϵ}_{n-1} break into two categories:

(1) Schur roots $x(\beta_{ij})$, where β_{ij} is an edge vector.
(2) Null generators η_{ab} for all $1 \leq a, b \leq n$ such that $\epsilon_a = +$ and $\epsilon_b = -$.

Since our quiver Q is mutation equivalent to $\widetilde{A}^{\epsilon}_{n-1}$ for some ϵ, we expect to have Schur roots and null generators as above. From tilting theory (?), we know that indecomposable string modules are invariant under mutation. Thus, Schur roots are mapped to Schur roots and null generators are mapped to null generators via mutation. The generators in the pro-finite presentation of the pro-group $\hat{G}(Q)$ are:

(1) $x(\beta)$ where β is a Schur root of degree equal to the dimension of the string module $M(\beta)$.
(2) η_{ab}, where η is the minimal unoriented cycle of Q and $a = [a_{i_1}a_{i_2}\cdots a_{i_k}]$ or $b = [a_{j_1}a_{j_2}\cdots a_{j_k}]$ with $\epsilon_a = +$, $\epsilon_b = -$.

Schur roots correspond to indecomposable string modules in kQ-mod while null generators for Q correspond precisely to the null generators of $\widetilde{A}_k$ for some $k \leq n$.

(Here draw diagram showing how η_{ab} and η_{ac} collapse into $\eta_{[ab]c}$ after mutating at vertex i, where $a = i-1, b = i$, $\epsilon_a = \epsilon_b = +$ ad $\epsilon_c = 0$).

Given a quiver Q, we have its dual periodic tree T_Q. By [IK], indecomposable modules correspond to segments supported on the tree T_Q. Now we give the following definition:

Definition: Let α and β be two Schur roots of type Q and u and v their corresponding strings. Then, we say that α and β are noncrossing if the segments s_u and s_v satisfy the following:

(1) s_u and s_v do not share endpoints
(2) s_u and s_v admit simple curves γ and δ of the same color in some Garver-McConville diagram supported on T_Q.

(Please refer to [IK] for terminology).

3.8. **Relations.** Throughout the list below we will use the notation u_β, where β is a root, to refer to a string (as in [5]) with dimension vector β in kQ/I for some quiver Q and admissible ideal I. Moreover, we denote $M(\alpha) = M(u_\alpha)$ (refer to [5] for details).

(1) Let α and β be noncrossing Schur roots. Then, $x(\alpha)$ and $x(\beta)$ commute.
(2) Suppose $\mathrm{Ext}^1(M(\alpha), M(\beta))$ is 1-dimensional and $\mathrm{Ext}^1(M(\beta), M(\alpha)) = 0$. Then, we have the relation $x(\alpha)x(\beta) = x(\beta)x(\gamma)x(\alpha)$, where $u_\gamma = u_\beta \xrightarrow{a} u_\alpha$, where a gives the extension of $M(\alpha)$ by $M(\beta)$.
(3) Consider α and η_{ab} such that $a = [a_i^{\pm}a_{i+1}^{\pm}\cdots a_{i+r}^{\pm}]$ and $b = [a_j^{\pm}a_{j+1}^{\pm}\cdots a_{j+s}^{\pm}]$, where $\epsilon_a = -$ and $\epsilon_b = +$. Denote by β the root corresponding to the string starting at vertex $j+s+1 (\mathrm{mod}\ n)$ and ending at vertex i in the minimal cycle η. Then, $x(\alpha)$ and η_{ab} commute if α and $\beta + k\eta$ are noncrossing for all $k \in \mathbb{N}$.
(4) Let θ be a Schur root such that $\mathrm{Supp}M(\theta) \bigcap \mathrm{Supp}M(\eta) = \emptyset$, $\dim_{\mathbb{K}}\mathrm{Ext}^1(M(\eta), M(\theta)) = \dim_{\mathbb{K}}\mathrm{Ext}^1(M(\theta), M(\eta)) = 1$. Then, we have the relation: $x(\theta)x(\gamma_1)\eta_{[ab]k} = \eta_{[ab]k}x(\gamma_2)x(\theta)$, where $u_{\gamma_1} = u_\theta \xleftarrow{a^*} u_\eta$ and $u_{\gamma_2} = u_\theta \xrightarrow{b^*} u_\eta$ and $\epsilon_{[ab]} = -$ and $\epsilon_k = +$. Similarly, we have $x(\theta)x(\gamma_1)\eta_{k[cd]} = \eta_{k[cd]}x(\gamma_2)x(\theta)$, where $u_{\gamma_1} = u_\theta \xrightarrow{c^*} u_\eta$ and $u_{\gamma_2} = u_\theta \xleftarrow{d^*} u_\eta$ and $\epsilon_{[cd]} = +$ and $\epsilon_k = -$.
(5) Let α and β be Schur roots such that $\alpha + \beta = \eta$ and $\dim_{\mathbb{K}}\mathrm{Ext}^1(M(\alpha), M(\beta)) = \dim_{\mathbb{K}}\mathrm{Ext}^1(M(\beta), M(\alpha)) = 1$ with corresponding arrows $u_\beta \xleftarrow{a} u_\alpha$ and $u_\alpha \xleftarrow{b} u_\beta$, respectively. Moreover, assume $\epsilon_a = \epsilon_b = -$. Let c be an arrow satisfying $\epsilon_c = +$. Then, we have the relation: $x(\beta)\eta_{ac}x(\alpha) = x(\alpha)\eta_{bc}x(\beta)$. Similarly, if $u_\beta \xrightarrow{c} u_\alpha$ and

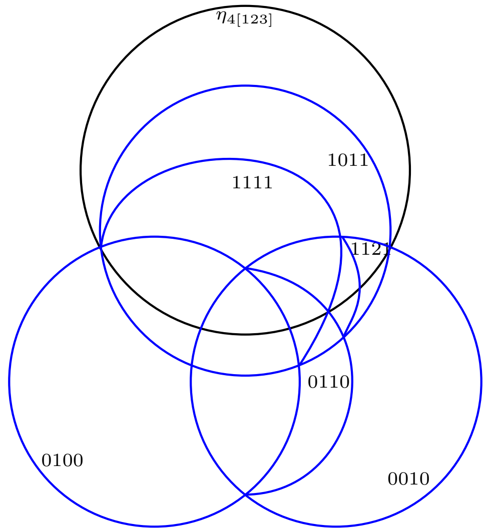


FIGURE 12. Picture for $\widetilde{A}_3$ with minimal cycle $\eta = 1 \leftarrow 4 \rightarrow 1$.

$u_\alpha \xrightarrow{d} u_\beta$, where $\epsilon_c = \epsilon_d = +$ and some arrow a satisfies $\epsilon_a = -$, then we have the relation: $x(\alpha)\eta_{ac}x(\beta) = x(\beta)\eta_{ad}x(\alpha)$.

(6) Suppose α and β are Schur roots such that $\alpha+\beta = \eta$ and $\dim_{\mathbb{K}}\mathrm{Ext}^1(M(\alpha), M(\beta)) = 2$. Label the corresponding arrows by a and b such that $\epsilon_a = -$ and $\epsilon_b = +$. Then, we have the following relation: $x(\alpha)x(\beta) = x(\beta)x(\beta+\eta)x(\beta+2\eta)\cdots\eta_{ab}\cdots x(\alpha+2\eta)x(\alpha+\eta)x(\alpha)$.

Theorem 3.25. *The set of relations above is invariant under mutation.*

Proof. First, we start with relation (1). Now suppose we have two roots α and β that are noncrossing. By definition, the corresponding string modules $M(\alpha)$ and $M(\beta)$ are *hom*-orthogonal and ext orthogonal. Suppose γ is some simple root. There are two cases to consider: i. $\mathrm{Supp}M(\alpha)\bigcap\mathrm{Supp}M(\beta) = \emptyset$ and ii. $\mathrm{Supp}M(\alpha)\bigcap\mathrm{Supp}M(\beta) \neq \emptyset$. For the first case, suppose $\mathrm{Supp}M(\gamma) \subset \mathrm{Supp}M(\alpha)$ or $\mathrm{Supp}M(\gamma) \subset \mathrm{Supp}M(\beta)$. Without loss of generality, suppose $\mathrm{Supp}M(\gamma) \subset \mathrm{Supp}M(\beta)$. Then, it is clear that $\alpha := \mu_\gamma(\alpha)$ and $\beta' := \mu_\gamma(\beta)$ are noncrossing. Now suppose $\beta_1 \leftrightarrow \cdots \leftrightarrow \beta_r \xleftarrow{a} \gamma \xleftarrow{b} \alpha_1 \leftrightarrow \cdots \leftrightarrow \alpha_s$ is a string, where $u_\beta = \beta_1 \leftrightarrow \cdots \leftrightarrow \beta_r$ and $u_\alpha = \alpha_1 \leftrightarrow \cdots \leftrightarrow \alpha_s$. Mutation at γ creates a nontrivial extension of $M(\beta)$ by $M(\alpha)$. $x(\alpha)$ and $x(\beta)$ are no longer commuting but appear in the relation $x(\beta)x(\alpha) = x(\alpha)x(\alpha+\beta)x(\beta)$ (extension is given by $u_\alpha \xrightarrow{[ab]} u_\beta$), which is relation (2). For the second case, suppose $\{i, j, k\}$ is the common support of string modules $M(\alpha)$ and $M(\beta)$. Then, α and β are noncrossing implies that u_α and u_β are of the form $u_\alpha = s_1 \leftarrow i \leftrightarrow j \leftrightarrow k \leftarrow s_2$ and $u_\beta = t_1 \rightarrow i \leftrightarrow j \leftrightarrow k \rightarrow t_2$. The only interesting mutation happens at either vertex i or vertex k. Without loss of generality, we show what happens when we mutate at vertex i with orientation $i \rightarrow j$. Since $\dim_{\mathbb{K}}\mathrm{Hom}(M(\alpha), S_i) \neq 0$, we apply the mutation X_i^+. Thus, $u_{\mu_i(\alpha)} = s_1 \rightarrow i \rightarrow j \leftrightarrow k \leftarrow s_2$ and $u_{\mu_i(\beta)} = t_1 \rightarrow j \leftrightarrow k \rightarrow t_2$, which are still noncrossing. Suppose $s_1 = i_1 \leftrightarrow \cdots \leftrightarrow i_l$.

Mutation at i_l is similar.
(Draw figure showing an explicit example)

Now let's look at relation (2). Again, assume β is a simple root and apply the mutation formula for β to the following relation:

$$x(\alpha)x(\beta) = x(\beta)x(\gamma)x(\alpha) \quad \text{where } u_\gamma = u_\beta \xrightarrow{a} u_\alpha$$

to obtain:

$$x(\alpha)^{-1}x(\beta) = x(\gamma)x(\beta)x(\alpha)^{-1} \quad \text{where } u_\gamma = u_\alpha \xrightarrow{a^*} u_\beta$$

which can be rewritten as:

$$x(\beta)x(\alpha) = x(\alpha)x(\gamma)x(\beta) \quad \text{where } u_\gamma = u_\alpha \xrightarrow{a^*} u_\beta$$

which is precisely the relation we expect since now we extend $M(\beta)$ by $M(\alpha)$. Now suppose β is not simple and ι is some simple root contained in either the support of $M(\alpha)$ or the support of $M(\beta)$.

$$u_\gamma := \alpha_1 \leftrightarrow \alpha_2 \leftrightarrow \cdots \leftrightarrow \alpha_r \to \beta_1 \leftrightarrow \beta_2 \leftrightarrow \cdots \leftrightarrow \beta_s$$

Without loss of generality, suppose $\mathrm{Supp}(M(\iota)) \subset \mathrm{Supp}(M(\beta))$. Observe that $\iota = \beta_k$, where $1 < k < s$, will not change relation (2). So suppose $\iota = \beta_1$. If we have the orientation $\beta_1 \to \beta_2$, then $\mu_\iota(\alpha) = X_\iota^+(\alpha) = \alpha$, $u_{\mu_\iota(\beta)} = u_{X_\iota^+(\beta)} = \beta_2 \leftrightarrow \beta_3 \leftrightarrow \cdots \leftrightarrow \beta_s$, and $u_{\mu_\iota(\gamma)} = u_{X_\iota^+(\gamma)} = \alpha_1 \leftrightarrow \alpha_2 \leftrightarrow \cdots \leftrightarrow \alpha_r \to \beta_2 \leftrightarrow \cdots \leftrightarrow \beta_s$. So type (2) relations are stable under mutation. Thus, the set of relations (1) and (2) are invariant under mutation.

Relation (3) is exactly the same as relation (1), except that we consider infinitely many roots $\beta + k\eta$ for positive k. It is clear that α and $\beta + \eta$ noncrossing implies α and $\beta + k\eta$ are noncrossing for all $k > 1$. Thus, (3) is invariant under mutation.

For relation (4), start with root θ being a simple root. It is clear that mutation at θ takes us to relation (5). Now suppose $u_\theta = j_1 \leftrightarrow j_2 \leftrightarrow \cdots \leftrightarrow j_l$ is not simple such that $u_\eta = i_1 \leftrightarrow \cdots \leftrightarrow i_k$ and $i_1 \leftarrow u_\theta \leftarrow i_k$. Let β be some simple root. The case where $\mathrm{Supp}M(\beta) \subset \mathrm{Supp}M(\theta)$ or $\beta = i_j$ $(j \neq 1, k)$ is quite straightforward. So suppose $\beta = i_1$ without loss of generality. We have the 3-cycle $i_1 \xleftarrow{b^*} j_1 \xleftarrow{a^*} i_k \xleftarrow{ab} i_1$. Suppose $i_2 \xrightarrow{c} i_1$. Then, mutation at i_1 gives us $u_{\theta'} = u_{\mu_{i_1}(\theta)} = i_1 \xrightarrow{b} u_\theta$, $u_{\gamma_2'} = u_{\mu_\beta(\gamma_2)} = i_k \leftrightarrow \cdots \leftrightarrow i_2 \xleftarrow{c^*} i_1 \to u_\theta$, $u_{\gamma_1'} = u_{\mu_\beta(\gamma_1)} = u_\theta \leftarrow i_1 \xleftarrow{[ab]^*} i_k \leftrightarrow i_{k-1} \leftrightarrow \cdots \leftrightarrow i_2$, and $x(\theta')x(\gamma_1')\eta_{[cab]k} = \eta_{[cab]k}x(\gamma_2')x(\theta')$.

Now, let's look at relation (5). Suppose η has dimension vector $\sum_{j=1}^k e_{i_j}$. Let i_r be a vertex in η that is neither a source nor a sink. Then, $\mu_{i_r}(\eta)$ has dimension vector $\sum_{j\neq r}^k e_{i_j}$. Moreover, suppose there exist pairs of roots (θ_1, θ_2) and (κ_1, κ_2) satisfying $\theta_1 + \theta_2 = \eta = \kappa_1 + \kappa_2$, where $\theta_i \neq \kappa_j$ for any i, j and none of them are simple. Also suppose θ_1 and θ_2 satisfy $\mathrm{Hom}(M(i_r), M(\theta_1)) \neq \{0\}$ and $\mathrm{Ext}^1(M(\theta_i), M(\theta_j)) \neq \{0\}$ for $i \neq j$. Similarly, $\mathrm{Hom}(M(\kappa_2), M(i_r)) \neq \{0\}$ and $\mathrm{Ext}^1(M(\kappa_i), M(\kappa_j)) \neq 0$ for $i \neq j$. By construction, $\mu_{i_r}(\theta_2) = X_{i_r}^-(\theta_2) = \theta_2$ and $\mu_{i_r}(\kappa_1) = X_{i_r}^+(\kappa_1) = \kappa_1$. Therefore, $\mu_{i_r}(\theta_1) = \kappa_1$ and $\mu_{i_r}(\kappa_2) = \theta_2$. After mutation, the relations $x(\theta_1)\eta_{ad}x(\theta_2) = x(\theta_2)\eta_{cd}x(\theta_1)$

and $x(\kappa_1)\eta_{bd}x(\kappa_2) = x(\kappa_2)\eta_{cd}x(\kappa_1)$, where $\epsilon_a = \epsilon_b = \epsilon_c = -$ and $\epsilon_d = +$, become one relation: $x(\kappa_1)\eta_{[ab]d}x(\theta_2) = x(\theta_2)\eta_{cd}x(\kappa_1)$. The case $x(\theta_1)\eta_{ab}x(\theta_2) = x(\theta_2)\eta_{ad}x(\theta_1)$ and $x(\kappa_1)\eta_{ac}x(\kappa_2) = x(\kappa_2)\eta_{ad}x(\kappa_1)$ is similar. Now assume β is a simple root satisfying $\alpha + \beta = \eta$, $\mathrm{Ext}^1(M(\alpha), M(\beta)) \neq 0 \neq \mathrm{Ext}^1(M(\beta), M(\alpha))$. Then, mutating at β gives us precisely relation (4). Thus, the set of relations (4) and (5) is closed under mutation.

Lastly, we treat the relation in (6). Let β be a simple root and α a Schur root such that $\alpha + \beta = \eta$ and $\dim_{\mathbb{K}}\mathrm{Ext}^1(M(\alpha), M(\beta)) = 2$. Here note that β is a source. We apply the mutation formula for simple root β to the relation:

$$x(\alpha)x(\beta) = x(\beta)x(\beta+\eta)x(\beta+2\eta)\cdots\eta_{ab}\cdots x(\alpha+2\eta)x(\alpha+\eta)x(\alpha)$$

we obtain:

$$x(\alpha)x(\beta)^{-1} = x(\beta)^{-1}x(\alpha+\eta)x(\alpha+2\eta)\cdots\eta_{b^*a^*}\cdots x(\beta+2\eta)x(\beta+\eta)$$

which is what we expect since now β is a sink. Also note that we have the null generator $\eta_{b^*a^*}$ since $\epsilon_{a^*} = -\epsilon_a$ and $\epsilon_{b^*} = -\epsilon_b$. Thus, the resulting relation is the same type, namely (6). Now assume that neither α nor β is a simple root. Without loss of generality, let γ be a simple root such that $\mathrm{Supp}M(\gamma) \subset \mathrm{Supp}M(\beta)$ and suppose $u_\beta = \beta_1 \leftrightarrow \beta_2 \leftrightarrow \cdots \leftrightarrow \beta_r$ and $u_\alpha = \alpha_1 \leftrightarrow \alpha_2 \leftrightarrow \cdots \leftrightarrow \alpha_s$. Now suppose we have the arrows a and b such that :

$$\cdots \xleftarrow{a} \beta_1 \leftrightarrow \cdots \leftrightarrow \beta_r \xrightarrow{b} \cdots$$

$$\xrightarrow{a} \alpha_1 \leftrightarrow \cdots \leftrightarrow \alpha_s \xleftarrow{b}$$

If $u_\gamma \neq \beta_1$ or β_r, then (6) remains intact except for the root β which is replaced with $\beta' = \mu_\gamma(\beta)$. Otherwise, consider $u_\gamma = \beta_1$ ($u_\gamma = \beta_r$ is similar). Suppose we have the orientation $\cdots \xleftarrow{a} \beta_1 \xrightarrow{c} \beta_2 \leftrightarrow \cdots$. Then, mutation at γ will produce $u_{\beta'} := u_{\mu_\gamma(\beta)} = \beta_2 \leftrightarrow \beta_3 \leftrightarrow \cdots \leftrightarrow \beta_r$, $u_{\alpha'} = u_{\mu_\gamma(\alpha)} = \beta_1 \xleftarrow{a^*} \alpha_1 \leftrightarrow \alpha_2 \leftrightarrow \cdots \leftrightarrow \alpha_s$, and the relation:

$$x(\alpha')x(\beta') = x(\beta')x(\beta'+\eta)x(\beta'+2\eta)\cdots\eta_{c^*b}\cdots x(\alpha'+2\eta)x(\alpha'+\eta)x(\alpha')$$

Now, suppose we have the orientation: $\xleftarrow{a} \beta_1 \xleftarrow{c} \beta_2 \leftrightarrow \cdots$. Then, mutating at γ, we have $\mu_\gamma(\alpha) = \alpha$ and $u_{\beta'} := u_{\mu_\gamma(\beta)} = \beta_2 \leftrightarrow \beta_3 \leftrightarrow \cdots \leftrightarrow \beta_r$. So our mutated relation becomes:

$$x(\alpha)x(\beta') = x(\beta')x(\beta'+\eta)x(\beta'+2\eta)\cdots\eta_{[ac]b}\cdots x(\alpha+2\eta)x(\alpha+\eta)x(\alpha)$$

Thus, relation of type (6) remains the same under mutation.

□

Theorem 3.26. *Let Q be mutation equivalent to A^ϵ_{n-1} for some surjective ϵ. Then, the set of rigid string modules in A^ϵ_{n-1} correspond to the set of rigid string modules in Q.*

Proof. It is enough to show that indecomposable string modules remain indecomposable and that rigid modules remain rigid even after mutation. The former is clear since mutation of a string will produce another string. The latter we refer the reader to the categorical equivalence found in 1.3 of [1]. □

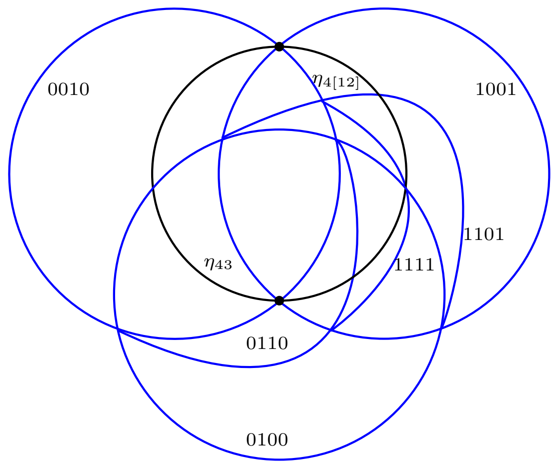


FIGURE 13. Picture for $\widetilde{A}_3$, where the minimal cycle $\eta = 1 \leftarrow 3 \leftarrow 4 \rightarrow 1$.

3.9. **Null pictures.** Figures 6 and 7 showed two examples of null pictures of $\widetilde{A}_3$. In this subsection we give the general definition and construction of the null picture for $\widetilde{A}_{n-1}^{\varepsilon}$.

We define the *null group* N_ε to be the finitely presented group with generators $x(\beta)$ for regular real Schur roots β and all null generators η_{ab} of $\hat{G}_\varepsilon$ modulo those relations of $\hat{G}_\varepsilon$ involving only these generators. Since the generators and relations for N_ε are subsets of those for $\hat{G}_\varepsilon$, any picture for N_ε is also a special case of a picture for $\hat{G}_\varepsilon$.

Lemma 3.27. *Given a d-dimensional picture $L \subseteq S^d$ for any group G and any vertex x of L, the intersection of L with a sufficiently small sphere around x will be another picture $L_x \subseteq S^{d-1}$.*

We call L_x the *link* of x in L. We define the *null picture* for $\widetilde{A}_{n-1}^{\varepsilon}$ to be the link of η in the cluster picture L_n of level n for $\widetilde{A}_{n-1}^{\varepsilon}$. This is the same as the link of η in L_m for any $m \geq n$ since $\eta \in D(\beta)$ if and only if β is regular. So, the support of any root of length $> n$ avoid some neighborhood of η.

For any fixed $m \geq n$, the null picture for $\widetilde{A}_{n-1}^{\varepsilon}$ is an $n-2$ dimensional picture for $\hat{G}_\varepsilon$. Since it involves none of the preprojective or preinjective generators and is the same for any $m \geq n$, the null picture is a picture for the null group N_ε.

We will now construct the null picture. For any regular root β_{ij}, define $D^0(\beta_{ij})$ to be the set of all $v \in \mathbb{R}^n$ so that $v + t\eta \in D(\beta_{ij})$ (equivalently, $\eta + \frac{1}{t}v \in D(\beta_{ij})$) for all sufficiently large real numbers t. This is equivalent to the following conditions on π_v:

(1) $\pi_v(i) = \pi_v(j)$
(2) $\varepsilon_i \pi_v(i) \geq \varepsilon_i \pi_v(k)$ for all $i < k < j$ so that $\varepsilon_k = \varepsilon_i$.

(So, $D^0(\beta_{ij})$ contains $D(\beta_{ij})$.)

Similarly, let $D^0_{ab}(\eta)$ be the set of all $v \in \mathbb{R}^n$ so that $v + t\eta \in D_{ab}(\eta)$ (equivalently, $\eta + \frac{1}{t}v \in D_{ab}(\eta)$) for sufficiently large t. This is equivalent to the following conditions on π_v:

(1) $\pi_v(a) = \pi_v(a+n)$ (i.e., π_v has slope 0)
(2) $\pi_v(i) \geq \pi_v(b)$ whenever $\varepsilon_i = -$

(3) $\pi_v(j) \leq \pi_v(a)$ whenever $\varepsilon_j = +$

Note that if $v \in D^0(\beta_{ij})$ then $v + t\eta \in D^0(\beta_{ij})$ for all real t and similarly for $D^0_{ab}(\eta)$.

Theorem 3.28. *Let S^{n-2} be the unit sphere in the hyperplane N perpendicular to η with respect to the usual Euclidean metric on $\mathbb{R}^n$ (not the one given by the Euler form). Then the null picture is the union of the intersections $S^{n-2} \cap D^0(\beta)$ for all regular generators $x(\beta)$ of N_ε and $S^{n-2} \cap D^0_{ab}(\eta)$ for all null generators η_{ab} of N_ε. These codimension 1 subsets are oriented towards the orthogonal projection of P in N.*

Proof. The construction is the general method of constructing the link of any vertex in any linear simplical complex. The support sets are oriented towards P since $\langle P, \beta \rangle$ is equal to the length of β which is positive. $\square$

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Department of Mathematics, Brandeis University, Waltham, MA 02454
E-mail address: `igusa@brandeis.edu`

Department of Mathematics, Brandeis University, Waltham, MA 02454
E-mail address: `moseskim@brandeis.edu`

Department of Mathematics, Northeastern University, Boston, MA 02115
E-mail address: `g.todorov@neu.edu`