# PICTURE GROUPS OF FINITE TYPE AND COHOMOLOGY IN TYPE $A_n$

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Abstract. For every quiver of finite type we define a finitely presented group called a picture group. We construct a finite CW complex which is shown in another paper [10] to be a $K(\pi, 1)$ for this picture group. In [5] another independent proof was given for this fact in the special case of type $A_n$ with straight orientation and we use this CW complex to compute the integral cohomology of picture groups of type $A_n$ with straight orientation. It is free abelian in every degree with ranks given by the "ballot numbers". We also compute the ring structure on the cohomology of these groups.

Contents



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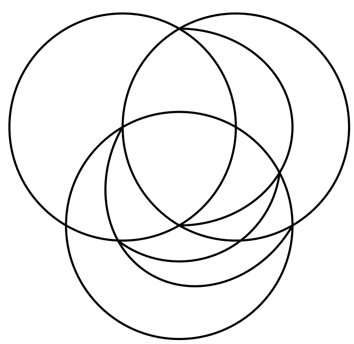
## Introduction

Suppose that $Q$ is a modulated quiver of finite type with $n$ vertices. Then there is an $n-1$ dimensional "picture" $L(Q)$ which is a finite $n-2$ dimensional subcomplex of the sphere $S^{n-1}$. The top dimensional simplices of this complex can be labeled with positive roots of the root system of the quiver which we view as generators of the unipotent Chevalley group $U_Q(\mathbb{Z})$ associated to the root system. This is a nilpotent group with one generator $\epsilon_\beta(1)$, which we denote $x(\beta)$, for every positive roots $\beta$.) On the codimension one simplices, Chevalley relations are displayed. For example, $x(\alpha), x(\beta)$ commute if $\alpha+\beta$ is not a root and is nonzero (See definition in Section 1 below.) Consequently, every region in the complement of $L(Q)$ in $S^{n-1}$ can be labelled with an element of the nilpotent group $N(Q)$ in such a way that the group label in adjacent regions differ by right multiplication by the generator labelling the wall separating the regions.

However, this nilpotent group is not the minimal group supporting the picture $L(Q)$. Although all positive roots occur as labels, only a subset of the Chevalley relations occur. We let $G(Q)$ be the finitely presented group with only those generators and relations which actually occur in the picture $L(Q)$. This group depends on the orientation of the quiver $Q$ but we believe the cohomology of the group depends only on the underlying root system. We call this the *picture group* of $Q$.

As an example, consider the following picture.



This picture has 6 smooth curves without inflection points. These curves meet transversely at 9 vertices. Such a picture determines a "picture group", unique up to isomorphism, with 6 generators and 9 relations as follows.

(1) Label each smooth curve with a different letter. These are the 6 generators, say, $a, b, c$ for the circles and $x, y, z$ for the arcs.
(2) At each vertex we obtain a relation by reading the labels of the arcs coming into the vertex counterclockwise starting at any point. Read the label as $g, g^{-1}$ depending on the curvature of the arc. For example, at the top vertex, we get the relation:
$$r_1 = aba^{-1}x^{-1}b^{-1}$$
since, going counterclockwise around this vertex we: enter circle $a$, enter circle $b$, exit circle $a$, exit the circle $x$ for which there is only an arc in the picture and, finally, we exit circle $b$ to return to the starting point.

Following the construction in [9] of the nilmanifold for a torsion-free nilpotent group, we view an $n-1$ dimensional picture as the attaching map for an $n$-cell in a finite CW complex. The minimal CW complex which supports the attachment of the single $n$-cell given by the spherical semi-invariant picture we call $X(Q)$. In a later paper [10] we prove that this is an

Eilenberg-MacLane space $K(\pi, 1)$ with $\pi_1 = G(Q)$. In the present paper we prove this in the special case when $Q$ is $A_n$ with straight orientation. And we focus on this case.

The group $G(A_n)$ has generators $x_{ij}$ for all $0 \le i < j \le n$ subject to the following relations.

(1) $x_{ij}, x_{k\ell}$ commute if either $j < k$ or $i < k < \ell < j$.
(2) $[x_{ij}, x_{jk}] = x_{ik}$ for all $i < j < k$ where $[x, y] := y^{-1}xyx^{-1}$.

These imply that $G(A_n)$ is generated by the $n$ elements $x_{j-1,j}$ for $j = 1, \cdots, n$. The space $X(A_n) \simeq K(G(A_n), 1)$ is a CW-complex with Narayama number $N(n, k)$ of $k$-cells. The cohomology group $H^k(G(A_n))$ is free abelian of rank given by "ballot numbers" $b(n, n-2k)$. The ring structure is also easy to describe.

These spaces and groups have many convenient properties. For example, the groups $G(A_n)$ form both a directed system and an inverse system since $X(A_n)$ is a retract of $X(A_{n+1})$. Also $X(A_n) \times X(A_m)$ is a retract of $X(A_{n+m+1})$. The filtration of $X(A_n)$ by subcomplexes $X(A_m)$ for $m < n$ has a refinement by subcomplexes which are all $K(\pi, 1)$'s. These properties help to determine the ring structure of the cohomology of $G(A_n)$.

Outline of paper:

In Section 1 we construct the $\boxed{\textit{spherical semi} - \textit{invariant picture}\ \mathrm{L(Q)}}$ for any valued quiver $Q$ of finite type. More precisely, we construct $L(\Lambda)$ for any finite dimensional hereditary algebra and show that it depends only on the underlying valued quiver of $\Lambda$.

In Section 2 we define the $\boxed{\textit{picture group}\ \mathrm{G(Q)}}$ and show that the picture $L(Q)$ is a picture for the group $G(Q)$ with its defining presentation. The picture group is the universal group with this property.

In Section 3 we construct the $\boxed{\textit{picture space}\ \mathrm{X(Q)}}$. This is a CW-complex with one $k$-cell for every set of $k$ hom-orthogonal roots $\alpha_1, \cdots, \alpha_k \in \Phi_+(Q)$.

In Section 4 we compute the integral $\boxed{\textit{cohomology of the space } X(A_n)}$. It is generated by indecomposable elements with square zero. There are $(n - 2k + 2)C_k$ indecomposable elements of degree $k$ for $2k - 1 \le n$ where $C_k$ is the $k$th Catalan number.

In Section 5 we outline a $\boxed{\textit{proof that } X(A_n) \textit{ is a } K(\pi, 1)}$ and that, therefore, the calculation in Section 4 computes the cohomology of these groups. The proof uses a filtration of $X(A_n)$ by subcomplexes $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n$ where $Y_0 = X(A_{n-1})$, $Y_n = X(A_n)$ and each $Y_m$ is a $K(\pi, 1)$. We give some examples to see what these spaces and groups are. Missing details in this proof can be found in another paper [10] which gives a categorical approach to the construction of the picture space $X(Q)$ for any modulated quiver of finite type and extended it to finite convex subsets of the Auslander-Reiten quiver of a hereditary algebra of possibly infinite type. The first author has written another topological approach to the same special case of $A_n$ with straight orientation in [5]. Since these other papers give two independent detailed and rigorous proofs of the main theorem of Section 5, we present only an outline of our original idea which was the inspiration for these other papers. This shows that the cohomology of the space $X(A_n)$ is the same as the cohomology of the group $G(A_n)$.

Finally, in Section 6 we determine the $\boxed{\textit{cup product structure}}$ of the cohomology ring of the picture group $G(A_n)$.

## 1. Spherical semi-invariant picture $L(Q)$

We construct the spherical semi-invariant picture $L(\Lambda)$ for any valued quiver $Q$ of finite type. This is a codimension one subcomplex of the $n-1$ sphere with suitable simplicial decomposition where $n$ is the number of vertices of $Q$. This is defined in terms of the representations of a hereditary algebra $\Lambda$ of finite type. However, it depends only on the underlying valued quiver $Q$ of $\Lambda$. So, we often denote it by $L(Q)$ instead of $L(\Lambda)$.

1.1. **Notation.** Suppose that $\Lambda$ is a finite dimensional hereditary algebra over a field $K$. We assume that $\Lambda$ has finite representation type. Here is a summary of well-known facts and our notation. See [7], [8] for more details. Also [3] is the classical reference for valued quivers.

So, the quiver of $\Lambda$ is a valued quiver which is a disjoint union of Dynkin quivers. Recall that the *quiver* $Q$ is a directed graph with one vertex for every (isomorphism class of) simple module $S_i$, $i = 1, \cdots, n$ with one arrow $i \to j$ if $\mathrm{Ext}(S_i, S_j) \neq 0$. The quiver $Q$ has *valuation* given by $f_i = \dim_K F_i$ where $F_i = \mathrm{End}(S_i)$ at each vertex $i$ and edge valuation $(d_{ij}, d_{ji})$ on any arrow $i \to j$ if $d_{ij} = \dim_{F_j} \mathrm{Ext}(S_i, S_j)$ and $d_{ji} = \dim_{F_i} \mathrm{Ext}(S_i, S_j)$ so that $d_{ij} f_j = d_{ji} f_i$.

Given any $\Lambda$-module $M$, the *dimension vector* $\underline{\dim}\, M$ is the vector in $\mathbb{N}^n$ whose $i$th coordinate is $\dim_{F_i} \mathrm{Hom}_\Lambda(P_i, M)$ where $P_i$ is the projective cover of $S_i$ with endomorphism ring canonically identified with $F_i = \mathrm{End}(S_i)$. A *virtual representation* is a homomorphism between projective modules $p : P \to P'$ (thought of as the presentation of a module $M$) with morphisms given by homotopy classes of chain maps. Up to isomorphism, the indecomposable virtual representations are presentations of indecomposable modules and shifted indecomposable projective modules $P_i[1] := (P_i \to 0)$. The dimension vector of a virtual representation $P \to P'$ is defined to be $\underline{\dim}\, P' - \underline{\dim}\, P$. Then the dimension vector of the presentation of any module is equal to the dimension vector of the module.

The *Euler matrix* $E$ is the $n \times n$ integer matrix with entries

$$E_{ij} = \dim_K \mathrm{Hom}(S_i, S_j) - \dim_K \mathrm{Ext}(S_i, S_j)$$

Then, the *Euler-Ringel form* $\langle \cdot, \cdot \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$, defined by $\langle v, w \rangle = v^t E w$, satisfies

$$\langle \underline{\dim}\, M, \underline{\dim}\, N \rangle = \dim_K \mathrm{Hom}(M, N) - \dim_K \mathrm{Ext}(M, N).$$

Let $\Phi_+(Q)$ be the set of *positive roots* of $Q$. These are the dimension vectors of the indecomposable $\Lambda$-modules. If $\pi_i = \underline{\dim}\, P_i$ then we call $-\pi_i$ the *negative projective roots*. These are the dimension vectors of the virtual representations $P_i[1] = (P_i \to 0)$. We say that $\beta$ is an *almost positive root* if it is either a positive root or a negative projective root. Using the notation $hom(\alpha, \beta) = \dim_K \mathrm{Hom}(M_\alpha, M_\beta)$ and $ext(\alpha, \beta) = \dim_K \mathrm{Ext}(M_\alpha, M_\beta)$, we say that $\alpha, \beta$ are *hom-orthogonal* if $hom(\alpha, \beta) = 0 = hom(\beta, \alpha)$ and *ext-orthogonal* if $ext(\alpha, \beta) = 0 = ext(\beta, \alpha)$. We use the notation $|P[1]| = P$ and $|-\beta| = \beta$. So, $|M_\beta| = M_{|\beta|}$.

The *cluster complex* $\Sigma(\Lambda)$ of $\Lambda$ is defined to be the $n-1$ dimensional simplicial complex whose vertices are the almost projective roots of $Q$. The $k$-simplices of $\Sigma(\Lambda)$ are $k+1$ tuples of pairwise ext-orthogonal almost positive roots. Since $\Lambda$ has finite type, $\Sigma(\Lambda)$ is a finite complex whose geometric realization is the sphere $S^{n-1}$.

One definition of the *picture* $L(\Lambda) \subset S^{n-1}$ is that it is the geometric realization of the $n-2$ skeleton of $\Sigma(\Lambda)$. This is a "picture" for the group $G(\Lambda)$ as defined below. In this definition we use normal orientations on $n-3$ dimensional (codimension 2 in $S^{n-1}$) simplices

$\rho$ of $L$. Such a normal orientation induces a cyclic ordering of the $n-2$ simplices of $L$ which contain $\rho$. We use $k = n-1$ in the definition.

**Definition 1.1.** Suppose that $G$ is a group given by generators and relations: $G = \langle \mathcal{X} | \mathcal{Y} \rangle$ where each $y \in \mathcal{Y}$ is a word in $\mathcal{X} \cup \mathcal{X}^{-1}$ and $k \geq 2$. Then a $k$-dimensional *picture* for $G$ is defined to be a $k-1$ dimensional subcomplex $L$ of a triangulated $k$ sphere $S^k$ together with an orientation of the normal bundle in $S^k$ of every $k-1$ simplex and $k-2$ simplex of $L$ and labels $x(\sigma) \in \mathcal{X}$ for each $k-1$ simplex $\sigma$ in $L$ so that

(1) There exists a locally constant function $g : S^k \backslash L \to G$ so that, for every $k-1$ simplex $\sigma$ of $L$, $g(\tau') = g(\tau)x(\sigma)$ when $\tau'$ is on the positive side of $\sigma$.
(2) For every $k-2$ simplex $\rho$ of $L$, the $k-1$ simplices $\sigma_i$ of $L$ which contain $\rho$, can be numbered in agreement with the cyclic ordering given by the normal orientation of $\rho$ in the sphere, so that

$$\prod x(\sigma_i)^{\varepsilon_i} \in \mathcal{Y} \cup \{xx^{-1} \,|\, x \in \mathcal{X}\}$$

where $\varepsilon_i = +1$ if the positive side of $\sigma_i$ faces $\sigma_{i+1}$ and $\varepsilon_i = -1$ if not. We use the notation $y(\rho) = \prod x(\sigma_i)^{\varepsilon_i}$.

Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the set of all labels $x(\sigma)$ for all $k-1$ simplices $\sigma$ of $L$ and let $\mathcal{Y}_0 \subseteq \mathcal{Y}$ be the set of all elements of $\mathcal{Y}$ which occur as labels $y(\rho)$ for some $k-2$ simplex $\rho$ of $L$. Then we call $G_0 = \langle \mathcal{X}_0 \,|\, \mathcal{Y}_0 \rangle$ the group *determined by* $L$ with its normal orientation and system of labels $x(\sigma)$.

*Remark* 1.2. For some applications ([6],[9]), it is useful to specify the starting point of the word $y(\rho)$ in (2). Such a starting point is given by one of the $k$-dimensional simplices of $S^k$ which contains $\rho$. This is called the *base point direction*. Without this choice, the relation $y(\rho)$ will only be well-defined up to cyclic orientation. Without the orientation of $\rho$, the relation $y(\rho)$ would only be well-defined up to inversion. But, given the labels $x(\sigma)$ and normal orientations for the $k-1$ simplices of $L$, the group $G_0$ is still uniquely determined (thereby justifying the terminology). If $L$ is a picture for $G$ then there is a canonical homomorphism $G_0 \to G$ induced by the inclusion $\mathcal{X}_0 \hookrightarrow \mathcal{X}$.

Let $L$ be a one-dimensional subcomplex of any triangulation of the 2-sphere $S^2$ which, when considered as a graph, contains no leaves. (Every vertex of $L$ is adjacent to at least two edges.) Choose a normal orientation of each edge and vertex in $L$. Let $x : L_1 \twoheadrightarrow \mathcal{X}$ by any surjective mapping of the set of edges of $L$ to any finite set $\mathcal{X}$. For each vertex $v \in L_0$, let $y(v)$ be the product of the labels $x(e_i)^{\varepsilon_i}$ on the edges adjacent to $v$ starting with any edge and going either clockwise or counterclockwise according to the orientation of $v$, with exponent $\varepsilon_i = \pm 1$ according to the orientation of $e_i$. If the words $y(v)$ are cyclically reduced and aperiodic then $L$ is a picture for the group $G_0 = \langle x(e), e \in L_1 \,|\, y(v), v \in L_0 \rangle$ and $G_0$ is the group determined by $L \subset S^2$. Figure 1 gives an example.

More generally we have the following.

**Proposition 1.3.** *If $L$ is any codimension one subcomplex of any triangulated $k$-sphere $S^k$ with normal orientations on $k-1$ and $k-2$ simplices and labels in a set $\mathcal{X}$ on the $k-1$ simplices, there is group $G_0$, unique up to isomorphism, determined by $L$. Furthermore, if $L$ is a picture for another group $G$, there is a unique homomorphism $G_0 \to G$ which takes the generators of $G_0$ to the corresponding generators of $G$.*

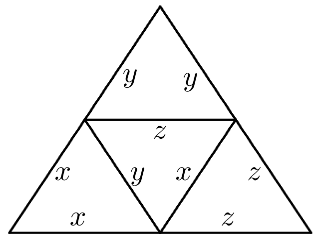



Figure 1. This graph with indicated labels and normal orientation, given by placing the labels on the positive side of each edge and taking positive orientation at each vertex, determines the group

$$G_0 = \left\langle x, y, z \,|\, xyz^{-1}y^{-1}, yzx^{-1}z^{-1}, zxy^{-1}x^{-1} \right\rangle$$

which is the fundamental group of the complement of the trefoil knot.

The proof of Proposition 1.3 is given by Remark 1.2 above.

We will use semi-invariants to provide a system of labels and normal orientations for $L(\Lambda) \subseteq S^{n-1}$. This will simultaneously define a group $G(\Lambda)$ and show that $L(\Lambda)$ is an $n-1$ dimensional picture for this group.

1.2. **Semi-invariants.** For every positive root $\beta \in \Phi_+(Q)$, let $M_\beta$ be the unique indecomposable $\Lambda$-module with dimension vector $\beta$. Then the (integral) *support* of $\beta$ is defined to be the set of all dimension vectors of all virtual representations $V = (p : V_1 \to V_0)$ so that $\mathrm{Hom}(V, M_\beta) = 0 = \mathrm{Ext}(V, M_\beta)$ or, equivalently,

$$\mathrm{Hom}(p, M_\beta) : \mathrm{Hom}(V_0, M_\beta) \cong \mathrm{Hom}(V_1, M_\beta)$$

is an isomorphism. The determinant of this linear map is a semi-invariant of weight $\beta$. (See [8].)

The *real support* of $\beta$, denoted $D(\beta)$, is defined to be the closure in $\mathbb{R}^n$ of the set of all vectors in $\mathbb{Q}^n$ an integer multiple of which lies in the integral support of $\beta$. The *virtual stability theorem* [8] states that

$$D(\beta) = \{v \in \mathbb{R}^n \mid \langle v, \beta \rangle = 0 \text{ and } \left\langle v, \beta' \right\rangle \leq 0 \ \forall \beta' \subseteq \beta\} \tag{1.1}$$

where $\beta' \subseteq \beta$ means that $M_\beta$ contains a submodule isomorphic to $M_{\beta'}$. Formula (1.1) implies in particular that $D(\beta)$ depends only on the valued quiver $Q$.

Note that $D(\beta)$ is the closure of a convex open subset of the hyperplane

$$H(\beta) = \{v \in \mathbb{R}^n \mid \langle v, \beta \rangle = 0\}.$$

This hyperplane has a normal orientation. The positive side is given by

$$H_+(\beta) = \{v \in \mathbb{R}^n \mid \langle v, \beta \rangle \geq 0\}.$$

Thus, each $D(\beta)$ is a normally oriented codimension one subspace of $\mathbb{R}^n$.

**Lemma 1.4.** *For every cluster $T_1, \cdots, T_n$ in the cluster category of $\Lambda$ there are unique roots $\gamma_1, \cdots, \gamma_n \in \Phi(Q)$ so that*

$$\langle \underline{\dim}\, T_i, \gamma_j \rangle = \delta_{ij} \dim_K \mathrm{End}(T_i).$$

*Furthermore, $\underline{\dim}\, T_i$ lies in $D(|\gamma_j|)$ for all $i \neq j$.*

*Proof.* By an observation of Schofield, the modules $|T_i|$ can be ordered to form an exceptional sequence. (Put the shifted projectives last.) For each $i$ there is a unique exceptional module $M_i$ so that $(M_i, |T_1|, |T_2|, \cdots, \widehat{|T_i|}, \cdots, |T_n|)$ is an exceptional sequence. This implies that $\mathrm{Hom}(|T_j|, M_i) = 0 = \mathrm{Ext}(|T_j|, M_i)$ for all $j \neq i$. We claim that

$$\langle \underline{\dim}\, T_i, \underline{\dim}\, M_i \rangle = \pm \dim_K \mathrm{End}(T_i)$$

Then $\gamma_i = \pm \underline{\dim}\, M_i$ with the same sign as above is the solution of the equation.

The claim follows from that fact that $\underline{\dim}\, M_i$ is equal to $\pm \underline{\dim}\, T_i$ modulo the $\mathbb{Z}$-span of $\underline{\dim}\, T_1, \cdots, \underline{\dim}\, T_{i-1}$. This follows easily from the fact that braid mutation of exceptional sequences preserves the $\mathbb{Z}$-span of the dimension vectors of the modules. □

Since $D(|\gamma_i|)$ is convex it contains all nonnegative linear combinations of $\underline{\dim}\, T_j, j \neq i$.

**Theorem 1.5.** *Let $D(\Lambda) \subset \mathbb{R}^n$ be the union*

$$D(\Lambda) = \bigcup_{\beta \in \Phi_+(Q)} D(\beta)$$

*Then $L(\Lambda) = D(\Lambda) \cap S^{n-1}$ where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$.*

*Proof.* By the lemma, $D(\Lambda)$ contains the $k-2$ skeleton of $\Sigma(\Lambda)$ and therefore, $L(\Lambda)$ is a subset of $D(\Lambda) \cap S^{n-1}$. Conversely, suppose that $v \in D(\Lambda) \cap S^{n-1}$ and $v \notin L(\Lambda)$. Since $L(\Lambda)$ is a closed set and every point in $D(\Lambda)$ is a limit of rational points, there is a rational vector $w \in D(\Lambda)$ so that $w/||w||$ is not in $L(\Lambda)$. By definition of $L(\Lambda)$ this implies that some positive scalar multiple of $w$ has the form $mw = \sum a_i \underline{\dim}\, T_i$ for some cluster $T_1, \cdots, T_n$ where $a_i$ are positive integers. But, $\bigoplus T_i^{a_i}$ is the generic module of dimension vector $mw$. So, $\sum a_i \underline{\dim}\, T_i \in D(\beta)$ implies $\underline{\dim}\, T_i \in D(\beta)$ for all $i$. But this is impossible since $\underline{\dim}\, T_i$ are linearly independent and $D(\beta)$ is a subset of a hyperplane through the origin. □

Since the subsets $D(\beta) \cap S^{n-1} \subseteq L(\Lambda)$ are normally oriented and labeled with positive roots $\beta \in \Phi_+(Q)$ and depend only on $Q$, we get the following.

**Corollary 1.6.** *$L(\Lambda) \subset S^{n-1}$ is an $n-1$ dimensional picture for a group $G(\Lambda)$ with generators $x(\beta)$ for $\beta \in \Phi_+(Q)$. Furthermore, $L(\Lambda)$ together with its normal orientation and system of labels depends only on the underlying valued quiver $Q$ of $\Lambda$.*

Because of this we write $L(\Lambda) = L(Q)$.

Summary: Section 1 constructs the spherical semi-invariant picture $L(\Lambda) = D(\Lambda) \cap S^{n-1}$ where $D(\Lambda)$ is the union of domains $D(\beta)$ of virtual semi-invariants of weight $\beta$. These sets are normally oriented and labelled $\beta$. So, they determine a group $G(Q)$ which is universal with the property that $L(\Lambda)$ is a picture for $G(Q)$.

## 2. Picture group $G(Q)$

We define the *picture group* $G(Q)$ to be the group determined by the subcomplex $L(Q) \subseteq S^{n-1}$. By Proposition 1.3, $L(Q)$ is a picture for the group $G(Q)$ with its defining presentation.

The generators of the picture group are, by definition, the labels of the walls in $L(Q)$. Since we sometimes think of $L(Q)$ as an $n-2$ dimensional subcomplex of $S^{n-1}$ and sometimes as an $n-1$ dimensional subcomplex of $\mathbb{R}^n$, we will refer to the codimension instead of the dimension of its pieces. The walls are the codimension one sets. Since these walls are $D(\beta)$ for all positive roots $\beta$ of $Q$, we have a generator $\boxed{x(\beta)}$ for each $\beta \in \Phi_+(Q)$.

We now consider a codimension $p \geq 2$ simplex $\rho$ of $L(Q) = L(\Lambda)$. In this section we are only interested in the case $p = 2$, but the general case is needed for the next section. By definition the vertices of $\rho$ form a partial cluster $T_1, \cdots, T_{n-p}$ in the cluster category of $\Lambda$. By Lemma 1.4, the codimension 1 simplices of $L(Q)$ which contain $\rho$ are contained in $D(\beta_j)$, $j > n-p$, for some extension $T_1, \cdots, T_n$ of the partial cluster to a full cluster where $\beta_j = |\gamma_j| \in \Phi_+(Q)$ in the notation of the lemma. Furthermore, the condition $T_i \in D(\beta_j)$ for $i = 1, 2, \cdots, n-p$ is equivalent to the condition that $M_{\beta_j}$ lies in the right hom-ext perpendicular category $|T|^\perp$ of the underlying module $|T|$ of $T = T_1 \oplus \cdots \oplus T_{n-p}$. Since $T$ has $n-p$ components, $|T|^\perp$ is wide subcategory of *mod*-$\Lambda$ of rank $p$ which is the module category of a hereditary algebra with $p$ nonisomorphic simple objects.

Let $M_{\alpha_1}, \cdots, M_{\alpha_p}$ be the simple objects of the wide subcategory $|T|^\perp$. Since $\Lambda$ has finite type, we can number the roots so that $ext(\alpha_i, \alpha_j) = 0$, or equivalently, $\langle \alpha_i, \alpha_j \rangle = 0$ for $i < j$. All other objects of $|T|^\perp$ have the form $M_\gamma$ where $\gamma = \sum r_j \alpha_j$, $r_j \geq 0$, is a nonnegative integer linear combination of the $\alpha_j$.

**Lemma 2.1.** *For any positive root $\gamma \in \Phi_+(Q)$, the following are equivalent.*

1. *There is an $n-1$ simplex $\sigma$ in $L(Q)$ so that $\rho \subset \sigma \subseteq D(\gamma)$.*
2. *The indecomposable module $M_\gamma$ lies in $|T|^\perp$.*
3. *The modules $|T_i|$, $i = 1, \cdots, n-p$, lie in ${}^\perp M_\gamma$.*
4. *$\gamma$ has the form $\gamma = \sum r_i \alpha_i$ where $r_i \geq 0$.*

We denote by $\boxed{\Phi_+(\alpha_*)}$ the set of all $\gamma \in \Phi_+(Q)$ satisfying these equivalent conditions.

*Proof.* As explained above, Lemma 1.4 implies that (1) and (2) are equivalent. (2) and (3), which are clearly equivalent, imply (4) since $M_{\alpha_i}$ are the unique simple objects in the category $|T|^\perp$. Conversely, suppose that $\gamma = \sum r_j \alpha_j \in \Phi_+(Q)$. Then

$$\langle \underline{\dim}\, T_i, \gamma \rangle = \sum r_j \langle \underline{\dim}\, T_i, \alpha_j \rangle = 0$$

Since $\Lambda$ has finite representation type, this implies that $M_\gamma \in |T|^\perp$. So, all four statement are equivalent. □

**Lemma 2.2.** *The interior of the simplex $\rho$ with vertices $\underline{\dim}\, T_i$ lies in the interior of each $D(\alpha_j)$ but it lies on the boundary of $D(\gamma)$ for any $\gamma = \sum r_j \alpha_j$ which is not one of the $\alpha_j$.*

*Proof.* Take any fixed $v \in int\, \rho$. Then, $v$ lies in $D(\gamma)$ if and only if $\rho \subseteq D(\gamma)$. This happens if and only if $\gamma = \sum r_j \alpha_j$ for some $r_j \geq 0$. It follows that $v$ is not contained in $D(\beta)$ if $\beta$ is a subroot of any $\alpha_j$. By the virtual stability theorem (1.1), this implies that $\langle v, \beta \rangle < 0$. Since this is an open condition, we have $\langle w, \beta \rangle < 0$ for all $w$ in some neighborhood of $v$. Therefore, $v$ lies in the interior of each $D(\alpha_j)$.

Any nontrivial linear combination $\gamma = \sum r_j \alpha_j$, will contain some $\alpha_j$ as a subroot. Also, $\alpha_j, \gamma$ will be linearly independent. So, the hyperplanes $H(\alpha_j), H(\gamma)$ intersect transversely along a codimension 2 subspace which contains the simplex $\rho$. Since $\langle v, \alpha_j \rangle \leq 0$ for all $v \in D(\gamma)$, the set $D(\gamma)$ is restricted to the negative side of $H(\alpha_j)$. So, $\rho$ lies on the boundary of $D(\gamma)$ as claimed. □

Finally, we need to characterize which pairs of roots $\alpha, \beta \in \Phi_+(\Lambda)$ arise in the way that we described.

**Lemma 2.3.** *Suppose that $\alpha_1, \cdots, \alpha_p \in \Phi_+(\Lambda)$. Then $M_{\alpha_j}$ are the simple objects of a wide subcategory of mod-$\Lambda$ of rank $p$ if and only if they are pairwise hom-orthogonal.*

*Proof.* We prove only the sufficiency of this condition is clearly necessary. Since $\Lambda$ has finite type, we can number the roots so tha $ext(\alpha_i, \alpha_j) = 0$ for $i < j$. Then, reversing the order gives an exceptional sequence $M = (M_{\alpha_p}, \cdots, M_{\alpha_1})$ making $\mathcal{A} = (^\perp M)^\perp$ into a rank $p$ wide subcategory with complete exceptional sequence $M$. Since the $\alpha_j$ are hom-orthogonal, $M_{\alpha_j}$ are the simple objects of $\mathcal{A}$. □

We will use the notation $\mathcal{A}b(\alpha_*) = (^\perp M)^\perp$ for this wide subcategory. By Lemma 2.1, $\Phi_+(\alpha_*)$ is the set of dimension vectors of indecomposable objects of $\mathcal{A}b(\alpha_*)$. We call $\mathcal{A}b(\alpha_*)$ the *wide subcategory spanned by* $\alpha_*$ since "generated" is not the right word.

**Theorem 2.4.** *If $Q$ is a valued Dynkin quiver, the picture group $G(Q)$ determined by the spherical semi-invariant picture $L(Q)$ has the following presentation.*

(1) *$G(Q)$ has one generator $x(\beta)$ for every positive root $\beta \in \Phi_+(Q)$.*
(2) *For each pair $(\alpha, \beta)$ of hom-orthogonal roots in $\Phi_+(Q)$ so that $ext(\alpha, \beta) = 0$, we have the relation:*

$$x(\alpha)x(\beta) = \prod x(r_i\alpha + s_i\beta_i) \tag{2.1}$$

*where the product is over all positive roots of the form $r_i\alpha + s_i\beta$ in increasing order of the ratio $r_i/s_i$ (going from $0/1$ to $1/0$).*

*Proof.* Each codimension one face simplex of $L(Q)$ lies in $D(\beta)$ for some positive roots $\beta$ and is labeled $x(\beta)$. By Lemma 2.2, the relation which occurs around a codimension two simplex $\rho$ of $L(Q)$ is a word in $x(r\alpha + s\beta)$ in which the letters $x(\alpha), x(\beta)$ occur twice and the other letters occur once. In the semi-simple case where $M_\alpha, M_\beta$ do not extend each other, the only $D(\gamma)$ containing $\rho$ are $D(\alpha), D(\beta)$ which meet transversely with $\rho$ in their intersection. So, the relation around $\rho$ is $x(\alpha)x(\beta) = x(\beta)x(\alpha)$ in this case.

If $\mathrm{Ext}(M_\beta, M_\alpha) \neq 0$ then there are extensions $M_\gamma$ where $\gamma = r\alpha + s\beta$. (Example 2.5 below gives a case by case description.) Figure 2 shows where $D(r\alpha + s\beta)$ occur. They are oriented counterclockwise as shown in the figure and the slope of the positive normal direction is proportional to $r/s$. Therefore, the sets $D(r\alpha + s\beta)$ are in cyclic order according to this slope and we get the relation (2.1) above. □

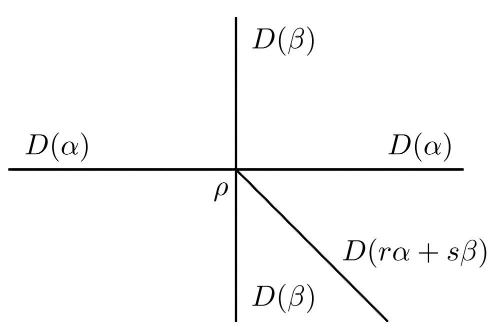



FIGURE 2. Image of $L(Q)$ under the projection $\mathbb{R}^n \to \mathbb{R}^2$ given by $v \mapsto (\langle v, \beta \rangle, \langle v, \alpha \rangle)$. By definition, $D(\alpha)$, $D(\beta)$ map to the $x$ and $y$ axes. In the non-semisimple case, $\langle \alpha, \beta \rangle = 0$ and $\langle \beta, \alpha \rangle < 0$, all sets $D(r\alpha + s\beta)$ for $r, s > 0$ map to the fourth quadrant as shown.

**Example 2.5.** There are only six types of relations (2.1) which occur in the presentation given in the theorem. This is because the wide category $({}^\perp M)^\perp$ is equivalent to the module category of a hereditary algebra of finite type with two vertices. And there are only four possibilities as listed below. (But Cases (3) and (4) have two subcases depending on whether the arrow points towards the short root or the long root. So, the total is six.)

(1) $A_1 \times A_1$. This corresponds to the case when the modules $M_\alpha, M_\beta$ do not extend each other and the wide category that they generate is semi-simple. So, $\Phi_+(\alpha, \beta) = \{\alpha, \beta\}$ and the relation is:

$$x(\alpha)x(\beta) = x(\beta)x(\alpha).$$

(2) $A_2$. Here $\text{Ext}(M_\beta, M_\alpha)$ is one dimensional over both $F_\beta = \text{End}(M_\beta)$ and $F_\alpha = \text{End}(M_\alpha)$. The wide category has 3 indecomposable objects forming an exact sequence $M_\alpha \to M_{\alpha+\beta} \to M_\beta$ and $G(Q)$ has relation:

$$x(\alpha)x(\beta) = x(\beta)x(\alpha+\beta)x(\alpha).$$

(3) $B_2 \cong C_2$. In this case, either $\text{Ext}(M_\beta, M_\alpha)$ is 1-dimensional over $F_\beta$ and 2-dimensional over $F_\alpha$ or vise versa. In the first case, where $\beta$ is the long root, we have $\Phi_+(\alpha, \beta) = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta\}$ and the relation is

$$x(\alpha)x(\beta) = x(\beta)x(\alpha+\beta)x(2\alpha+\beta)x(\alpha).$$

(4) $G_2$. Here $\text{Ext}(M_\beta, M_\alpha)$ is 1-dimensional over $F_\beta$ and 3-dimensional over $F_\alpha$ or vise versa. There are six positive roots giving the relation:

$$x(\alpha)x(\beta) = x(\beta)x(\alpha+\beta)x(3\alpha+2\beta)x(2\alpha+\beta)x(3\alpha+\beta)x(\alpha).$$

In all cases there are irreducible morphism between the corresponding modules in the opposite order than how they appear in the relations. For example, in Case (4) there are irreducible morphisms

$$M_\alpha \to M_{3\alpha+\beta} \to M_{2\alpha+\beta} \to M_{3\alpha+2\beta} \to M_{\alpha+\beta} \to M_\beta.$$

If we compare these relations with the Chevalley relations for the generators of the maximal unipotent subgroup $U_Q$ of the algebraic group of the underlying Dynkin diagram of $Q$, we see that there is an epimorphism $G(Q) \twoheadrightarrow U_Q(\mathbb{Z})$. (Send $x(\beta)$ to $\epsilon_\beta(1)$ in the notation of [11].)

Summary: In Section 2 (Theorem 2.4) we gave a presentation of the picture $G(Q)$ which is determined by the labeled picture $L(Q)$ constructed in Section 1.

## 3. Picture space $X(Q)$

In Section 3 we will construct the picture space $X(\Lambda)$ assuming that $\Lambda$ is a hereditary algebra of finite representation type. This will be a finite CW-complex together with a system of closed codimension-one subsets $J(\beta) \subset X(Q)$ for all $\beta \in \Phi_+(Q)$. Since $X(\Lambda)$ will depend only on the underlying valued quiver $Q$, we will write $X(Q) = X(\Lambda)$.

### 3.1. **Local properties of $D(\beta)$.**

The construction of the space $X(Q)$ depends on the local property of the sets $D(\beta)$ as given in Proposition 3.2 below for $\beta$ in a wide subcategory $\mathcal{A}b(\alpha_*)$ spanned by a pairwise hom-orthogonal set of roots $\alpha_*$. Roughly speaking, it says that the intersection pattern of these sets depends only on the valued quiver of $\mathcal{A}b(\alpha_*)$. This is the quiver with one vertex for each $\alpha_j$ with valuation $f_j = hom(\alpha_j, \alpha_j)$ and an arrow $i \to j$ whenever $ext(\alpha_i, \alpha_j) \neq 0$ with valuation $(d_{ij}, d_{ji})$ so that $d_{ij} f_j = d_{ji} f_i = ext(\alpha_i, \alpha_j)$.

Let $Q(\alpha_*)$ denote this valued quiver. Then $Q(\alpha_*)$ depends only on the numbers $\langle \alpha_i, \alpha_j \rangle$ since this is equal to $f_j$ when $i = j$ and $-ext(\alpha_i, \alpha_j)$ when $i \neq j$.

Let $\alpha_* = \{\alpha_1, \cdots, \alpha_p\} \subseteq \Phi_+(Q)$ be any set of hom-orthogonal roots and let $T = \{T_1, \cdots, T_{n-p}\}$ be any partial cluster in the cluster category of $\Lambda$ so that $\mathcal{A}b(\alpha_*) = |T|^\perp$. For example, take the components of any tilting object of ${}^\perp\mathcal{A}b(\alpha_*)$. Let $\mathbb{R}\alpha_*$ denote the $p$-plane spanned by the vectors $\alpha_1, \cdots, \alpha_p$. Then, it follows from Lemma 1.4 that the span of the roots $\underline{\dim}\, T_i$ is equal to

$$ {}^\perp\mathbb{R}\alpha_* = \{w \in \mathbb{R}^n \mid \langle w, \alpha_j \rangle = 0 \text{ for all } j\}. $$

Let $\pi_{\alpha_*} : \mathbb{R}^n \to \mathbb{R}\alpha_*$ be the projection along ${}^\perp\mathbb{R}\alpha_*$. Then, for every $x \in \mathbb{R}^n$, $\pi_{\alpha_*}(x) \in \mathbb{R}\alpha_*$ is uniquely determined by the condition $\langle x, \alpha_j \rangle = \langle \pi_{\alpha_*}(x), \alpha_j \rangle$ for all $j$.

**Lemma 3.1.** *For $\beta_*$ a hom-orthogonal set of roots in $\Phi_+(\alpha_*)$ we have $\pi_{\beta_*} x = \pi_{\beta_*}\pi_{\alpha_*}(x)$. I.e., $\pi_{\beta_*} : \mathbb{R}^n \to \mathbb{R}\beta_*$ factors uniquely through $\pi_{\alpha_*} : \mathbb{R}^n \to \mathbb{R}\alpha_*$.*

*Proof.* Since $\mathbb{R}\beta_* \subseteq \mathbb{R}\alpha_*$, ${}^\perp\mathbb{R}\alpha_* = \ker \pi_{\alpha_*} \subseteq {}^\perp\mathbb{R}\beta_* = \ker \pi_{\beta_*}$. The lemma follows. □

Let $\rho$ be the $n-p-1$ simplex in the cluster complex $\Sigma(\Lambda)$ spanned by the almost positive roots $\underline{\dim}\, T_i$. Then $\rho \subset {}^\perp\mathbb{R}\alpha_*$. Let $v$ be any point in the interior of $\rho$. (I.e., $v = \sum v_i \underline{\dim}\, T_i$ where $v_i > 0$ for all $i$.)

**Proposition 3.2.** *For each $\beta \in \Phi_+(Q)$ let $D(\beta, \rho)$ be the set of all vectors $x \in \mathbb{R}^n$ so that, for all sufficiently small $\epsilon > 0$, $v + \epsilon x \in D(\beta)$. Then $D(\beta, \rho)$ is nonempty if and only if $\beta \in \Phi_+(\alpha_*)$ and $D(\beta, \rho)$ is the set of all $x \in \mathbb{R}^n$ satisfying the condition:*

(3.1) $\qquad \langle x, \beta \rangle = 0$ *and* $\langle x, \beta' \rangle \le 0$ *for all subroots $\beta'$ of $\beta$ which lie in $\Phi_+(\alpha_*)$.*

*Remark* 3.3. Let $D_{\alpha_*}(\beta) \subset \mathbb{R}\alpha_*$ be the set of all $x \in \mathbb{R}\alpha_*$ satisfying (3.1). This proposition says that $D(\beta, \rho)$ is the inverse image of $D_{\alpha_*}(\beta)$ under $\pi_{\alpha_*} : \mathbb{R}^n \to \mathbb{R}\alpha_*$.

Suppose that $\tau$ is a simplex containing $\rho$ and let $M_{\beta_j}$ be the simple objects in the right perpendicular category of $|T'|$ where $T'$ is the direct sum of all vertices $T_i$ of $\tau$. Then $\Phi_+(\beta_*) \subseteq \Phi_+(\alpha_*)$. If $\beta \in \Phi_+(\beta_*)$ then $D(\beta, \tau)$ is a nonempty subset of $D(\beta, \rho)$ and $D(\beta, \tau)$ is the inverse image under $\pi_{\beta_*} : \mathbb{R}^n \to \mathbb{R}\beta_*$ of the subset $D_{\beta_*}(\beta) \subseteq \mathbb{R}\beta_*$. By Lemma 3.1, $\pi_{\beta_*}$ factors through $\mathbb{R}\alpha_*$. So, $D_{\alpha_*}(\beta) \subseteq \mathbb{R}_{\alpha_*}$ contains the inverse image of $D_{\beta_*}(\beta)$ under the induced map $\mathbb{R}_{\alpha_*} \to \mathbb{R}_{\beta_*}$.

*Proof.* Since $D(\beta)$ is a closed set, the condition $v + \epsilon x \in D(\beta)$ for small $\epsilon$ implies that $v \in D(\beta)$. This implies that $\rho \subset D(\beta)$ which holds if and only if $\beta \in \Phi_+(\alpha_*)$ by Lemma 2.1. So, $D(\beta, \rho)$ is nonempty only in this case.

We show that the condition (3.1) is necessary. If $v + \epsilon x \in D(\beta)$ then $\langle v + \epsilon x, \beta \rangle = 0$ and $\langle v + \epsilon x, \beta' \rangle \le 0$ for all $\beta' \subseteq \beta$. Since $\langle v, \beta \rangle = 0$ and $\epsilon > 0$, these conditions imply (3.1). Conversely, suppose that (3.1) holds. Then $\langle v + \epsilon x, \beta \rangle = \langle v, \beta \rangle + \epsilon \langle x, \beta \rangle = 0$. Let $\gamma \subset \beta$ be any subroot. If $\gamma \in \Phi_+(\alpha_*)$ then (3.1) implies that $\langle v + \epsilon x, \gamma \rangle \le 0$ for all $\epsilon > 0$. If $\gamma \notin \Phi_+(\alpha_*)$ then we know that $v \notin D(\gamma)$. Therefore $\langle v, \gamma \rangle < 0$. (Otherwise, $\langle v, \gamma \rangle = 0$ and $\langle v, \gamma' \rangle \le 0$ for all $\gamma' \subseteq \gamma \subset \beta$ making $v \in D(\gamma)$.) So, $\langle v + \epsilon x, \gamma \rangle < 0$ for sufficiently small $\epsilon$ which implies that $v + \epsilon x$ lies in $D(\beta)$ for sufficiently small positive $\epsilon$ as claimed. □

Recall that the cluster complex $\Sigma(Q)$ is a simplicial complex whose geometric realization is $|\Sigma(Q)| \cong S^{n-1}$. Since $S^{n-1}$ is a manifold, the dual cell decomposition is an $n-1$ dimensional CW complex. We attach a single $n$ cell to this dual cell complex to get an $n$ dimensional CW complex which we denote by $E(Q)$. Then $|E(Q)| \cong D^n$. Proposition 3.2

gives us an equivalence between certain $p$-cells in this cell decomposition. By identifying these equivalent cells we will obtain the picture space $X(Q)$.

We recall that the *dual cell* $E(\rho)$ to the $n-p-1$ simplex $\rho$ in $|\Sigma(Q)|$ is the $p$-cell which is the union of all simplices $\tau$ in the first barycentric subdivision of $|\Sigma(Q)|$ so that $\tau \cap \rho$ is the barycenter of $\rho$. This implies that the other vertices of $\tau$ are barycenters of simplices $\sigma$ which contain $\rho$. For every $\beta \in \Phi_+(Q)$ let $J(\beta,\rho)$ be the subcomplex of $E(\rho)$ consisting of all simplices $\tau$ whose vertices are barycenters of simplicies $\sigma_i$ which are contained in $D(\beta)$. $J(\beta,\rho)$ is nonempty if and only if $\beta \in \Phi_+(\alpha_*)$.

We use the general fact that $E(\rho)$ is simplicially isomorphic to the cone on the first barycentric subdivision of the *link* of $\rho$ in $\Sigma(Q)$. This is the simplicial complex whose vertices are all almost positive roots $\gamma$ which are ext-orthogonal to $\underline{\dim}\, T_1, \cdots, \underline{\dim}\, T_{n-p}$. The vertices $\gamma_i$ span a simplex in $Lk(\rho)$ if and only if they are ext-orthogonal.

**Corollary 3.4.** *There is a simplicial isomorphism $\varphi_\rho : Lk(\rho) \cong \Sigma(Q(\alpha_*))$ which is uniquely determined by the property that, for every vertex $\gamma$ of $Lk(\rho)$, $\varphi_\rho(\gamma)$ is a positive scalar multiple of $\pi_{\alpha_*}(\gamma) \in \mathbb{R}\alpha_*$.*

*Remark* 3.5. (a) By Proposition 3.2 this implies that a vertex $\gamma$ of $Lk(\rho)$ lies in $D(\beta)$ for some $\beta \in \Phi_+(\alpha_*)$ if and only if $\varphi_\rho(\gamma)$ lies in $D_{\alpha_*}(\beta)$.

(b) We also obtain as a consequence the following naturality condition on $\varphi_\rho$. Suppose that $\tau$ is a simplex in $Lk(\rho)$ and $\sigma = \rho * \tau$ is the smallest simplex in $\Sigma(Q)$ containing $\rho$ and $\tau$. Then $Lk(\sigma) \subseteq Lk(\rho)$. Take $\varphi_\sigma : Lk(\sigma) \cong \Sigma(Q(\beta_*))$ where $\beta_*$ gives the simple objects in $|\sigma|^\perp$. Let $\tau' = \varphi_\rho(\tau)$ and let $Lk'(\tau')$ be the link of $\tau'$ in $\Sigma(Q(\alpha_*))$. Then, $\beta_*$ also gives the simple objects in the right perpendicular category of $|\tau'|$ in $\mathcal{A}b(\alpha_*)$. So, we get two isomorphisms $Lk(\sigma) \cong \Sigma(Q(\beta_*))$. We need to know that they agree. Equivalently, the following diagram commutes.

$$\begin{array}{ccccc} & & Lk(\sigma) & \xrightarrow{\subset} & Lk(\rho) \\ & \swarrow^{\varphi_\tau} & \downarrow{\scriptstyle \varphi_\rho|Lk(\tau)} & & \downarrow \\ \Sigma(Q(\beta_*)) & \xleftarrow{\varphi_{\tau'}} & Lk'(\sigma) & \xrightarrow{\subset} & \Sigma(Q(\alpha_*)) \end{array}$$

It suffices to show that the triangle commutes. But this follows from Corollary 3.4 above since each vertex $\gamma$ of $Lk(\sigma)$ maps by $\varphi_\tau$ to the unique vertex of $\Sigma(Q(\beta_*))$ which is proportional to $\pi_{\beta_*}(\gamma)$ which comes from $\pi_{\alpha_*}(\gamma) \propto \varphi_\rho(\gamma)$ by Remark 3.3.

*Proof.* By Lemma 1.4, each vertex $\gamma$ in $Lk(\rho)$ is contained in $D(\beta)$ for $p-1$ linearly independent vectors $\beta \in \Phi_+(\alpha_*)$. This implies that $v + \epsilon(\gamma - v)$ lies in each of these $D(\beta)$ for all small $\epsilon > 0$ where $v \in int\rho$. So, $\gamma - v \in D(\beta,\rho) = \pi_{\alpha_*}^{-1} D_{\alpha_*}(\beta)$. Since $\pi_{\alpha_*}(v) = 0$, this implies that $\pi_{\alpha_*}(\gamma)$ lies in each $D_{\alpha_*}(\beta)$. So, it is a scalar multiple of an almost positive root in $\Phi(\alpha_*)$ which we define to be $\varphi_\rho(\gamma)$.

To see that $\varphi_\rho$ takes simplices to simplices, take a maximal simplex in $Lk(\rho)$ spanned by $p$ vertices $\gamma_1, \cdots, \gamma_p$. Then, for sufficiently small $\epsilon_j > 0$, we have that $v + \sum \epsilon_j(\gamma_j - v)$ does not lie in $D(\beta)$ for any $\beta$ since it lies in the interior of a top dimensional simplex of the cluster complex. This condition characterizes which sets of vertices in $Lk(\rho)$ form a simplex.

By the proposition, this implies that, when $r_j > 0$, $\sum r_j \varphi_\rho(\gamma_j)$ does not lie in $D_{\alpha_*}(\beta)$ for any $\beta \in \Phi_+(\alpha_*)$. This is equivalent to the condition that $\varphi_\rho(\gamma_j)$ are ext-orthogonal and therefore form a simplex in $\Sigma(Q(\alpha_*))$. So, $\varphi_\rho$ is a simplicial isomorphism. ∎

**Corollary 3.6.** *Let $\rho, \rho'$ be two $n-p-1$ simplices spanned by the dimension vectors of two partial clusters $T = \{T_1, \cdots, T_{n-p}\}$ and $T' = \{T'_1, \cdots, T'_{n-p}\}$ in the cluster category of $\Lambda$ so that $\mathcal{A}b(\alpha_*) = |T|^\perp = |T'|^\perp$. Then there is a simplicial isomorphism $\psi_\rho : E(\rho) \cong E(\rho')$ which sends $J(\beta, \rho)$ onto $J(\beta, \rho')$. Furthermore, if $\rho \subseteq \sigma = \rho * \tau$ so that $E(\sigma) \subset E(\rho)$, then the isomorphism $\psi_\rho$ restricts to the isomorphism $\psi_\sigma : E(\sigma) \cong E(\sigma')$ where $\sigma' = \rho' * \tau$. We also note that $J(\beta, \sigma) = J(\beta, \rho) \cap E(\sigma)$.*

*Proof.* We use the general fact that $E(\rho)$ is the cone on the first barycentric subdivision of the *link* $Lk(\rho)$ of $\rho$ in $\Sigma(Q)$. By Corollary 3.4, $Lk(\rho), Lk(\rho')$ are both isomorphic to $\Sigma(Q(\alpha_*))$. So, $Lk(\rho) \cong Lk(\rho')$ and, therefore, $E(\rho) \cong E(\rho')$. We refer to the elements of $E(\rho)$ corresponding to vertices of $Lk(\rho)$ as the *corners* of the cell $E(\rho)$. (The vertex $\gamma$ of $Lk(\rho)$ corresponds to the barycenter of the $n-p$ simplex $\rho * \gamma$.)

By Remark 3.5(a), the set $J(\beta, \rho) \subseteq E(\rho)$ is the cone on the inverse image of the subsets $D_{\alpha_*}(\beta) \subseteq \Sigma(Q(\alpha_*))$ under the isomorphism $\partial E(\rho) \cong Lk(\rho) \cong \Sigma(Q(\alpha_*))$. So, $\psi_\rho : E(\rho) \to E(\rho')$ must send $J(\beta, \rho)$ to $J(\beta, \rho')$.

By Remark 3.5(b), the subset $Lk(\rho * \tau) \subseteq Lk(\rho)$ maps to $Lk(\rho' * \tau)$ under $\varphi_{\rho'}^{-1}\varphi_\rho$ and the induced map is equal to $\varphi_{\rho'*\tau}^{-1}\varphi_{\rho*\tau}$. The two maps agree on where they send each vertex of $Lk(\rho * \tau)$. Therefore they agree on where they send each corner of $E(\rho * \tau)$ under the map $\psi_{\rho*\tau}$. So, $\psi_{\rho*\tau}$ agrees with $\psi_\rho$. □

### 3.2. **Construction of the picture space.**

**Definition 3.7.** The *picture space* $X(Q)$ is defined to be CW complex obtained from $E(Q)$ by identifying $p$ cells $E(\rho) \cong E(\rho')$ using the simplicial isomorphisms given by the corollary above. The compatibility of the map $\psi_\rho$ with $\psi_\sigma$ that we just proved implies that the identifications on the $p$-cells agrees with the identifications on lower cells. So, $X(Q)$ is a well defined CW complex constructed one cell at a time by induction on dimension of cells.

**Theorem 3.8.** *$X(Q)$ is an $n$-dimensional CW complex with one cell of dimension $k$ for every set $\alpha_* = \{\alpha_1, \cdots, \alpha_k\}$ of pairwise hom-orthogonal roots in $\Phi_+(Q)$. Denote it $e^k_{\alpha_*}$. Then the closure of the cell $e^k_{\alpha_*}$ is a subcomplex of $X(Q)$ isomorphic to $X(Q(\alpha_*))$. In particular, $X(Q)$ is the closure of the single cell $e^n_{\epsilon_1,\cdots,\epsilon_n}$ where $\epsilon_i$ are the simple root in $\Phi_+(Q)$. The cell $e^p_{\beta_*}$ is in the closure of $e^k_{\alpha_*}$ if and only if $\beta_* \subseteq \Phi_+(\alpha_*)$.*

*Proof.* The cell $e^k_{\alpha_*}$ is the one obtained by identifying all $E(\rho)$ where $\mathcal{A}b(\alpha_*) = |\rho|^\perp$, equivalently $\rho$ is a cluster inside the cluster category of ${}^\perp\mathcal{A}b(\alpha_*)$. So, the cells of $X(Q)$ are indexed by all such sets $\alpha_*$ which are the spanning sets of wide subcategories $\mathcal{A}b(\alpha_*)$. When $\rho \subset \sigma$ then $\sigma = \rho * \tau$ and $E(\sigma) \subset E(\rho)$. So, $|\sigma|^\perp = \mathcal{A}b(\beta_*) \subseteq \mathcal{A}b(\alpha_*)$. And conversely ($|\sigma|^\perp \subseteq |\rho|^\perp$ iff $\rho \subseteq \sigma$). But $E(\rho)$ is the cone on $\partial E(\rho) \cong sd\, Lk(\rho) \cong sd\, \Sigma(Q(\alpha_*))$ by Corollary 3.4. And, $X(Q(\alpha_*))$ is obtained from $sd\, \Sigma(Q(\alpha_*))$ by the same recipe as $X(Q)$:

$$X(Q(\alpha_*)) = \coprod_\tau E_{\alpha_*}(\tau)/\sim$$

where $E_{\alpha_*}(\tau)$ is the cell in $sd\, \Sigma(Q(\alpha_*))$ dual to $\tau$. The subscript indicates that we are working in the quiver $Q(\alpha_*)$. In the larger quiver, we have $E_{\alpha_*}(\tau) = E(\rho * \tau)$. Thus the cells of $X(Q(\alpha_*))$ correspond to those cells of $X(Q)$ which are identified with $E(\rho * \tau)$ for various $\tau$. These are exactly the simplices which contain $\rho$ and the cells are identified in the same way in both cell complexes by Remark 3.5(b). □

We will examine in more detail the structure of the space $X(Q)$ and $J(\beta) \subset X(Q)$ one cell at a time by induction on dimension. We use the notation $X(Q)^k$ for the $k$-skeleton of $X(Q)$ and $J(\beta)^{k-1} = J(\beta) \cap X(Q)^k$.

The cell complex $X(Q)$ has a single 0-cell (vertex) $e^0$. It have a 1-cell $e^1_\beta$ for every positive root $\beta \in \Phi_+(Q)$. The endpoints of each 1-cell are attached to the unique 0-cell. This gives a 1-dimensional CW complex $X(Q)^1$ whose fundamental group is the free group with generators $x(\beta)$ where $\beta \in \Phi_+(Q)$. The generator $x(\beta)$ is represented by the cell $e^1_\beta$.

Before attaching more cells to $X(Q)^1$, we give a recursive description of the sets $J(\beta)^k$ in terms of the attaching maps of the cells.

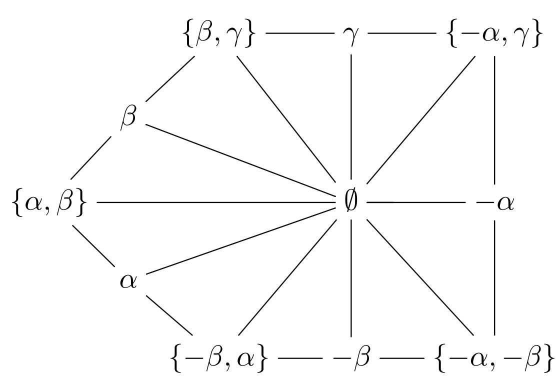
**Proposition 3.9.** *The 0-dimensional subset $J(\beta)^0 \subset X(Q)^1$ consists of the center point of the cell $e^1_\beta$. Given $J(\beta)^{k-1} \subset X(\beta)^k$ for $k \geq 1$ and attaching maps $\eta_i : S^k = \partial D^{k+1} \to X(Q)^k$, the set $J(\beta)^k$ is the union of $J(\beta)^{k-1}$ and certain subsets of each cell $e^{k+1}_i$ as follows. For each $k+1$ cells, $J(\beta)^k \cap e^{k+1}_i$ is the cone of the inverse image of $J(\beta)^{k-1}$ under the attaching map $\eta_i : S^k \to X(Q)^k$ assuming the image of $\eta_i$ meets $J(\beta)^{k-1}$. Otherwise $J(\beta)^k \cap e^{k+1}_i$ is empty.*

Continuing with the construction of $X(Q)$, we take one 2-cell $e^2_{\alpha,\beta}$ for every unordered pair of hom-orthogonal roots $\alpha, \beta$. This 2-cell is attached to the 1-skeleton $X(Q)^1$ using the relation (2.1) corresponding to the pair $\{\alpha, \beta\}$. In Example 2.5, Case (1), this gives a torus $S^1 \times S^1$ with $J(\alpha) = S^1 \times *$ and $J(\beta) = * \times S^1$. In Case (2) we get a torus with one boundary component given by the 1-cell $e^1_{\alpha+\beta}$. Cases (3) and (4) also give closed subsets of tori. To be more precise, we define each 2-cell to be a convex polygon with $m+2$ sides where $m$ is the number of elements of $\Phi_+(\alpha, \beta)$ ($m = 2, 3, 4, 6$ in Cases (1), (2), (3), (4), respectively). The attaching map sends these $m+2$ sides to the 1-cells corresponding to the letters in the relation (2.1).

For all $k$, the cell complex $X(Q)$ will have one $k$-cell $e^k_{\alpha_*}$ for every unordered set of $k$ pairwise hom-orthogonal roots $\alpha_* = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$.

**Example 3.10.** Consider the quiver $A_2 : 1 \leftarrow 2$.

(1) $\Sigma(A_2)$ is a pentagon with five vertices and 5 edges. These vertices are the almost positive roots $\alpha, \beta, \gamma, -\alpha, -\beta$ connected in a cycle. The simple roots are $\alpha, \gamma$.
(2) The following figure represents the cone on the first barycentric subdivision of $\Sigma(A_2)$. Except for the cone point $\emptyset$, each point $b_\rho$ is the barycenter of a simplex $\rho$ in $\Sigma(A_2)$. The point $b_\rho$ is labeled by the set of vertices of $\rho$.



(3) The entire object is $E(\emptyset)$ and the sides are $E(\alpha), E(\beta), E(\gamma), E(-\alpha), E(-\beta)$. The vertices are $E(\rho_i)$ where $\rho_i$ are the 2-simplices of $\Sigma(A_2)$.
(4) Since $|\rho_i|^\perp = 0$, these vertices are all identified to one point $e^0$ in $X(A_2)$. The 1-cells $E(\alpha), E(-\alpha)$ are identified since $|\alpha|^\perp = |-\alpha|^\perp = \mathcal{A}b(\gamma)$ and $E(\beta), E(-\beta)$ are also identified since $\beta^\perp = \mathcal{A}b(\alpha)$. Finally, $\gamma^\perp = \mathcal{A}b(\beta)$.
(5) $J(\alpha)$ is the cone on the two points $\beta, -\beta$, $J(\beta)$ is the cone on the point $\gamma$ and $J(\gamma)$ is the cone on the points $\alpha, -\alpha$.

Summary: In Section 3 we constructed the picture space $X(Q)$. It is a quotient space of the cone on the first barycentric subdivision of the cluster complex $\Sigma(Q)$ under certain identifications. The space $C(sd\,\Sigma(Q))$ is decomposed as a union of cells $E(\rho)$ for all simplices $\rho$ in $\Sigma(Q)$ (plus one big cell $E(\emptyset) := C(sd\,\Sigma(Q)))$ and $E(\rho)$ is identified with $E(\rho')$ if and only if the right perpendicular categories of $|\rho|, |\rho'|$ agree.

## 4. Homology of $X(A_n)$

In this section we compute the homology of $X(A_n)$ with straight orientation. The first steps in the computation work in general in the sense that the cellular chain complex of $X(Q)$ always has a weight filtration. In the case $Q = A_n$ with straight orientation, the homology of the associated graded complex is easy to compute and equal to the homology of the actual complex by an easy argument. We will see that the integral homology of $X(A_n)$ has no torsion. So the cohomology is also free abelian with the same rank. The cup product structure will be determined at the end.

Recall that the $k$-cells of $X(Q)$ are indexed by all sets of $k$ pairwise hom-orthogonal positive roots $\beta_i$ of $Q$. Let $\langle\beta_1, \cdots, \beta_k\rangle \in C_k(Q, \mathbb{Z})$ denote the corresponding free generator of the cellular chain complex $C_*(Q, \mathbb{Z})$ of $X(Q)$. We understand the order of the $\beta_i$ to be given up to even permutation. Under an odd permutation, the sign changes. Thus:

$$\left\langle\beta_{\sigma(1)}, \cdots, \beta_{\sigma(k)}\right\rangle = \operatorname{sgn}(\sigma)\left\langle\beta_1, \cdots, \beta_k\right\rangle$$

This generator has *degree* $k$. We often call this generator (with either sign) a *cell* and denote it by $\beta_*$. We define the *weight* of the cell $\langle\beta_1, \cdots, \beta_k\rangle$ to be the vector $\sum \beta_i \in \mathbb{N}^n$. Given two weights $w, w'$ we say that $w \le w'$ if $w_i \le w_i'$ for $i = 1, \cdots, n$ and $w < w_i$ if $w \le w'$ and $w \ne w'$. Note that if $w < w'$ then $w$ comes before $w'$ in lexicographic order.

Given $\beta_* = \langle\beta_1, \cdots, \beta_k\rangle$, recall that $\mathcal{A}b(\beta_*)$ is the abelian category spanned by the roots $\beta_i$ and $\Phi_+(\beta_*)$ is the set of all positive roots which can be written as nonnegative integer linear combinations of the roots $\beta_i$.

For each $\alpha \in \Phi_+(\beta_*)$, recall that $D_{\beta_*}(\alpha)$ is the set of all $\sum v_i\beta_i \in \mathbb{R}\beta_*$ satisfying the stability conditions (3.1). Then, by Theorem 1.5, the union of the $D_{\beta_*}(\alpha)$ intersected with the unit sphere $S^{k-1}$ in $\mathbb{R}\beta_*$ is the spherical semi-invariant picture for the hereditary abelian category $\mathcal{A}b(\beta_*)$.

**Lemma 4.1.** *Let $\beta_* = \langle\beta_1, \cdots, \beta_k\rangle$ be a set of hom-orthogonal positive roots. Then there is a 1-1 correspondence between positive roots $\gamma \in \Phi_+(\beta_*)$ and hom-orthogonal subsets $\alpha_* = \langle\alpha_1, \cdots, \alpha_{k-1}\rangle$ of $\Phi_+(\beta_*)$. The correspondence is given by $\gamma^\perp = span(\alpha_*)$. Furthermore, $\gamma$ is in the interior of $D_{\beta_*}(\alpha_i)$ for all $i$. Finally, $wt(\alpha_*) \ge wt(\beta_*)$ if and only if $M_\gamma$ is not a projective object in the abelian category $\mathcal{A}b(\beta_*)$.*

*Proof.* The formula $\gamma^\perp = span(\alpha_*)$ gives the 1-1 correspondence. Assume for simplicity of notation that $k = n$ and $\beta_i$ are simple roots. Then $wt(\beta_*) = (1, 1, \cdots, 1)$ and $wt(\alpha_*) \ge$

$wt(\beta_*)$ iff and only is $\sum \alpha_j$ is sincere, i.e., there is no index $i$ so that the $i$th coordinate of each $\alpha_j$ is zero. But, if this happens then the $i$-th projective root $\pi_i$ is left perpendicular to all $\alpha_j$ which implies $\gamma = \pi_i$. So, the last statement holds. The statement that $\gamma$ lies in the interior of each $D_{\beta_*}(\alpha_i)$ was already shown in Lemma 2.2. □

### 4.1. Weight filtration of $C_*(Q;\mathbb{Z})$ in general.

**Proposition 4.2.** *The boundary of $\beta_* = \langle \beta_1, \cdots, \beta_k\rangle$ in the cellular chain complex $C_*(Q;\mathbb{Z})$ is the sum of terms $\alpha_* = \langle \alpha_1, \cdots, \alpha_{k-1}\rangle$ listed below, with coefficient $\epsilon(\beta_*, \alpha_*) = \pm 1$.*

1. *One of the roots $\alpha_i$ is equal to the sum of two of the roots $\beta_j$ and the remaining $\alpha$'s are equal to the remaining $\beta$'s. In this case, $\alpha_*, \beta_*$ have the same weight.*
2. *$\alpha_i$ are hom-orthogonal roots in $\Phi_+(\beta_*)$ and $\sum \alpha_i > \sum \beta_j$.*

*Furthermore, $\epsilon(\beta_*, \alpha_*)$ is the sign of the change of basis matrix from the basis $\langle \beta_1, \cdots, \beta_k\rangle$ to the basis $\langle \alpha_1, \cdots, \alpha_{k-1}, \gamma\rangle$, each ordered up to even permutation, where $\gamma$ is the unique positive root so the $\alpha_* = \gamma^\perp$.*

*Proof.* (1) is the only way that the $k-1$ positive roots can add up to $\sum \beta_i$. (2) is the only way that $\sum \alpha_i > \sum \beta_i$. The only remaining cases are $\left\langle \beta_1, \cdots, \widehat{\beta_i}, \cdots, \beta_k \right\rangle$ which are right perpendicular to projective roots $\pi_i$. But these terms occur twice as summands of $d\beta_*$ with opposite sign corresponding to the vertices $\pi_i$ and $-\pi_i$ in the spherical semi-invariant picture for $\beta_*$. So, they cancel. The signs comes from the definition of induced orientation on the boundary of a disk. The plane $\mathbb{R}\alpha_*$ plays the role of the tangent plane to the unit sphere in $\mathbb{R}\beta_*$ at the vector $\gamma$. The induced orientation is $\epsilon(\beta_*, \alpha_*)$. The two cancelling terms have signs given by the bases $\left\langle \beta_1, \cdots, \widehat{\beta_i}, \cdots, \beta_k, \pi_i \right\rangle$ and $\left\langle \beta_1, \cdots, \widehat{\beta_i}, \cdots, \beta_k, -\pi_i \right\rangle$. Which are opposite. □

**Corollary 4.3.** *$d\beta_* = 0$ if and only if $\mathcal{A}b(\beta_*)$ is semi-simple. Equivalently, the sum of any two of the roots $\beta_i$ is not a root.*

We call $\beta_*$ a *semi-simple cell* or *semi-simple set* if these conditions hold.

*Proof.* Under this condition, there are no $\alpha_*$ as described in the Proposition. □

**Corollary 4.4.** *The chain complex $C_*(Q;\mathbb{Z})$ is filtered by weight in the sense that the additive subgroup generated by the cells of weight $\geq w$ form a subcomplex $C_*(Q;\mathbb{Z})_w$.* □

**Definition 4.5.** We will say that $\beta_* = \langle \beta_1, \cdots, \beta_k\rangle$ is a *resolution* of $\alpha_* = \langle \alpha_1, \cdots, \alpha_\ell\rangle$ and the $\alpha_*$ is a *degeneration* of $\beta_*$ if $k > \ell$ and there is an epimorphism $p : \{1, \cdots, k\} \twoheadrightarrow \{1, \cdots, \ell\}$ so that each $\alpha_j$ is the sum of all $\beta_i$ for $i \in p^{-1}(j)$. If $\beta_*$ has no degenerations we call it semi-simple. (This is equivalent to $\mathcal{A}b(\beta_*)$ being semi-simple.) If $\beta_*$ has no resolutions, we call it *primitive.*

**Example 4.6.** In type $A_5$ with straight orientation, $\langle \beta_{04}, \beta_{45}, \beta_{12}, \beta_{23}\rangle$ is a resolution of $\langle \beta_{05}, \beta_{13}\rangle$ since $\beta_{05} = \beta_{04} + \beta_{45}$ and $\beta_{13} = \beta_{12} + \beta_{23}$. The later is semi-simple and the former is primitive. We use the notation

$$\beta_{ij} = \epsilon_{i+1} + \epsilon_{i+2} + \cdots + \epsilon_j$$

where $\epsilon_j$ is the $j$th simple root of the root system $A_n$.

Note that, for any $\beta_*$ which is not semi-simple, the degeneration of $\beta_*$ with minimal degree is necessarily semi-simple.

4.2. **Semi-simple categories in type $A$.** Suppose now that $Q = Q(A_n)$ with straight orientation:

$$1 \leftarrow 2 \leftarrow \cdots \leftarrow n$$

The positive roots are labeled $\beta_{ij}$ where $0 \le i < j \le n$ as described in the example above.

**Lemma 4.7.** *Two roots $\beta_{ij}, \beta_{k\ell}$ are hom-orthogonal if and only if one of the following two conditions holds.*

(1) *$i < k < \ell < j$ or $k < i < j < \ell$.*
(2) *$j \le k$ or $\ell \le i$.*

*Furthermore, $\beta_{ij}, \beta_{k\ell}$ are hom-ext-orthogonal if and only if, in addition, equality cannot hold in (2).* □

We say that the closed intervals $[i, j], [k, \ell]$ are *noncrossing* if they are either disjoint or one is contained in the interior of the other. The lemma says that $\beta_{ij}, \beta_{k\ell}$ are hom-ext-orthogonal if and only if the intervals $[i, j], [k, \ell]$ are noncrossing

Let $\alpha_* = \langle\alpha_1, \cdots, \alpha_k\rangle$ be a semi-simple cell. Then each root $\alpha_s$ is equal to $\beta_{ij}$ for some $0 \le i < j \le n$. And the closed intervals $[i, j]$ are pairwise noncrossing. We define the *blocks* of $\alpha_*$ to be the maximal intervals $B_s = \beta_{i_s j_s}, s = 1, \cdots, m$ among these $\alpha$'s. We arrange these so that

$$0 \le i_1 < j_1 < i_2 < j_2 < \cdots < i_m < j_m \le n\,.$$

Then the weight $w$ of $\beta_*$ has the property that $w_t \ne 0$ if and only if $t$ lies in one of the intervals $[i_s + 1, j_s]$. Furthermore, $w_t = 1$ if $t = i_s + 1$ or $t = j_s$ for some $s$. Thus the blocks $B_s$ are completely determined by the weight $w$ of the semi-simple cell $\alpha_*$.

We say that a weight $w \in \mathbb{N}^n$ is *admissible* if it is the weight of some hom-orthogonal set $\beta_*$. This includes $w = 0$ which is the weight of the empty hom-orthogonal set.

**Lemma 4.8.** *A weight $w \in \mathbb{N}^n$ is admissible if and only if $|w_i - w_{i+1}| \le 1$ for all $0 \le i \le n$ with the convention that $w_0 = 0 = w_{n+1}$.*

*Proof.* The condition is clearly necessary. For example if $w_{i+1} \ge w_i + 2$ then any $\beta_*$ with weight $w$ will have two objects $\beta_{ij}$ and $\beta_{ik}$ one of which is a subroot of the other and are therefore not hom-orthogonal.

To see that the condition is sufficient, express $w$ as the sum of the smallest possible number of roots. This is the number of indices $i$ with the property that $w_i < w_{i+1}$. This is also the number of indices $j$ so that $w_j > w_{j+1}$. If we view each $i$ as an open parenthesis "(" and each $j$ as a closed parenthesis ")" then we see that they are paired. Then the collection of roots $\beta_{ij}$ for such pairs $i, j$ forms a hom-orthogonal set of roots with total weight $\sum \beta_{ij} = w$. So, $w$ is admissible. □

The pairing between $i$ and $j$ used in the above proof can be described as follows. Given $j$, the corresponding $i$ is the largest integer $i < j$ so that $w_i < w_j$. Since the degree of this cell $\langle\beta_{ij}\rangle$ is minimal, it is semi-simple. The following lemma shows that there are no other semi-simple objects with the same weight.

**Lemma 4.9.** *For every admissible weight $w \in \mathbb{N}^n$ there is a unique semi-simple set $\alpha_*$ with weight $w$.*

*Proof.* As observed above, the blocks of $\alpha_*$ are uniquely determined by $w = \sum \alpha_i$. They are $B = \beta_{ij}$ where $[i+1, j]$ is a maximal connected component of the support of $w$. Each of these blocks is a element of the cell $\alpha_*$. When we remove these blocks, we are left with a semi-simple cell which is uniquely determined by its weight $w'$ by induction on $w$. But $w'$ is uniquely determined by $w$. So, $\alpha_*$ is uniquely determined by $w$. □

An admissible weight $w$ will be called *primitive* if the unique semi-simple set with weight $w$ is primitive. This is equivalent to the statement that there exists a unique hom-orthogonal set with weight $w$. We will show that the primitive semi-simple sets freely generate the homology of $X(A_n)$. Then we will enumerate all such hom-orthogonal sets and obtain "ballot numbers."

**Proposition 4.10.** *An admissible weight $w$ is primitive if and only if $w_i \neq w_{i+1}$ for all $i$ except in the case $w_i = 0 = w_{i+1}$.*

*Proof.* Suppose that two consecutive coordinates of $w$ are equal and positive: $w_a = w_{a+1} > 0$. Then we will show that $w$ is not primitive. Let $\beta_*$ be the unique semi-simple cell of weight $w$. Then we first note that, for any $\beta_{ij}$ in $\beta_*$, $i, j$ are not equal to $a$ by the description of the set of $i, j$ given in the proof of Lemma 4.8.

Consider all objects $\beta_{ij}$ in $\beta_*$ so that $i < a < j$ and let $\beta_{pq}$ be the one with minimal length. Then a resolution of $\beta_*$ is given by replacing $\beta_{pq}$ with $\beta_{pa}$ and $\beta_{aq}$. To see that this works, note that every other object of $\beta_*$ either contains $\beta_{pq}$ in its interior (and therefore contains $\beta_{pa}$ and $\beta_{aq}$ in its interior) or lies in the interior of $\beta_{pq}$, i.e., has the form $\beta_{ij}$ where $p < i < j < q$. Minimality of $[p, q]$ implies that either $j < a$ in which case $\beta_{ij}$ lies in the interior of $\beta_{pa}$ or $a < i$ in which case $\beta_{ij}$ lies in the interior of $\beta_{aq}$.

Conversely, suppose that $w$ is not primitive. Let $\beta_*$ be the minimal hom-orthogonal set with weight $w$ with, say, $k$ elements. Let $\alpha_*$ be a resolution of $\beta_*$ with $k+1$ elements. Take the object of $\beta_*$ which resolves into an extension of two objects of $\alpha_*$, say $\beta_{ij}$ is in $\beta_*$ and $\beta_{ia}, \beta_{aj}$ are in $\alpha_*$. Then the index $a$ cannot occur as a first or second index of any other objects of $\beta_*$. For example if $\beta_{pa}$ is another object in $\beta_*$ then $\beta_{pa}, \beta_{ia}$ are not hom-orthogonal. Therefore $w_a = w_{a+1} \geq 1$. □

**Corollary 4.11.** *Suppose that $w$ is a primitive block and $\beta_* = \langle \beta_1, \cdots, \beta_k \rangle$ is the unique semi-simple set with weight $w$. Let $\beta_{ij}$ be the longest root in $\beta_*$, so that $w_p \neq 0$ only for $i < p \leq j$. Then $j - i = 2k - 1$.*

*Proof.* The number of roots $k$ in a semi-simple set with weight $w$ is equal to the sum $\sum |w_{i+1} - w_i|$ since each root contributes 2 to this sum. When these numbers are equal to 1 then the sum is equal to the number of terms which is $j - i + 1$. □

4.3. **Non-primitive weights.** Suppose that $w$ is a non-primitive weight. Then we will show that the weight subquotient complex of $C_*(A_n)$ given by

$$C_*(A_n)_{(w)} := C_*(A_n)_w / \sum_{w' > w} C_*(A_n)_{w'}$$

has zero homology. We do this using the *resolution set* of $w$ which we define to be the set of all integers $i$ so that $w_i = w_{i+1}$. By Proposition 4.10, this set is nonempty.

Suppose that $\beta_*$ is the unique semi-simple set with weight $w$. Then the resolutions of $\beta_*$ are in bijection with nonempty subsets $S$ of the resolution set. The resolution $\beta(S)$ corresponding to $S$ is given recursively as follows. If $S$ is empty, then $\beta(S) = \beta_*$. Otherwise, let $s_0$ be the largest element of $S$ and let $S_0 = S \backslash s_0$. Then $\beta(S)$ is given by replacing the object $\beta_{ij}$ of $\beta(S_0)$ of smallest length so that $i < a < j$ and replacing it with the two objects $\beta_{is_0}, \beta_{s_0 j}$. In the reverse direction, a resolution $\alpha_*$ of $\beta_*$ determines resolution set $S$ of all indices $s$ which occur as both a first and last index of an object of $\alpha_*$.

By Proposition 4.2, the boundary of $\beta(s_1, \cdots, s_r)$ is equal to the sum

$$d(\beta(s_1, \cdots, s_r)) = \sum_{i=1}^{r} \pm\beta(s_1, \cdots, \widehat{s_i}, \cdots, s_r)$$

or $r$ terms, each with coefficient plus or minus 1.

**Lemma 4.12.** *For any non-primitive weight $w$, the subquotient complex $C_*(A_n)_{(w)}$ is contractible.*

*Proof.* Since $w$ is non-primitive, the resolution set $S$ of $w$ is nonempty. Choose one element $s_0 \in S$. Then a chain contraction $h$ of $C_*(A_n)_{(w)}$ is given by

$$h(\beta(T)) = \begin{cases} 0 & \text{if } s_0 \in T \\ \pm\beta(T \cup \{s_0\}) & \text{otherwise} \end{cases}$$

where the sign is front of $\beta(T \cup \{s_0\})$ is the same as the sign of $\beta(T)$ as a term in $d\beta(T \cup \{s_0\})$. This will insure that $\beta(T)$ occurs with coefficient 1 in $(dh + hd)(\beta(T))$. The other terms cancel as a result of the fact that $d^2\beta(T \cup \{s_0\}) = 0$. □

This proves the following theorem.

**Theorem 4.13.** *The homology of the associated graded complex $\bigoplus_w C_*(A_n)_{(w)}$ is freely generated by the primitive semi-simple sets $\beta_*$.* □

**Corollary 4.14.** *The homology of the space $X(A_n)$ is freely generated by the primitive semi-simple sets $\beta_*$.*

*Proof.* Semi-simple (ss) sets are cycles in $C_*(A_n)$. By the theorem, the primitive ss cells $\beta_*$ generate the homology of the chain complex. It remains to show that no integer linear combination of such cycles is a boundary.

Suppose not. Let $z$ be an integer linear combination of primitive ss cells of degree $k$ which is the boundary of a $k + 1$ chain $z = dc$. Let $w$ be a weight which is minimal in lexicographic order so that $c_w \neq 0$ where $c_w$ is the component of $c$ of weight $w$. Choose $c$ so that this minimal weight $w$ is maximal in lexicographic order. Then $w$ is non-primitive since, otherwise, $dc_w = 0$ and $c$ can be replaced with $c - c_w$ contradicting the maximality of the minimal weight $w$. This implies that $z_w = 0$. So, the image of $c_w$ in $C_*(A_n)_{(w)}$ is a cycle and therefore a boundary. Say, $c_w = dx$ in $C_*(A_n)_{(w)}$. In the chain complex $C_*(A_n)$, the boundary of $x$ may have higher weight terms. So, $c - dx$ has no terms of weight $w$ but has new higher weight terms. This contradicts the maximality of $w$ in all cases. So, we conclude that $z$ is not a boundary and no linear combination of primitive semi-simple cells is a boundary.

Equivalently, the homology of $C_*(A_n)$ is isomorphic to the homology of the associated graded chain complex. □

It remains to determine the list of all primitive weights.

4.4. **Primitive weights.** By Corollary 4.14, $H_k(X(A_n); \mathbb{Z})$ is free abelian for every $n, k$. Let $r(n, k)$ denote its rank. Then $r(n, k)$ is the number of primitive semi-simple weights $w \in \mathbb{N}^n$ of degree $k$. We show that these numbers are equal to the "ballot numbers" by showing that they satisfy the same recursion.

By Lemma 4.8 and Proposition 4.10, the primitive weights $w \in \mathbb{N}^n$ of degree $k$ are all sequences of nonnegative integers $w = (w_1, \cdots, w_n)$ satisfying the following conditions where $w_0 = 0 = w_{n+1}$ by convention.

(1) $w_{i+1} = w_i + 1$ or $w_{i+1} = \max(w_i - 1, 0)$ for all $i = 0, \cdots, n$.

(2) There are exactly $k$ values of $i$ for which $w_{i+1} = w_i + 1$.

We recall that a *block* of this weight $w$ is a maximal sequence of consecutive nonnegative coordinates. By condition (1) each block has odd length. For example

$$w = (12123210012101)$$

has three blocks $B_{07}, B_{9,12}, B_{13,14}$ of lengths $7, 3, 1$ and degrees $4, 2, 1$ respectively. There is only one possible block of length 1 and of length 3 which are as given in the example. However, there are two possible blocks of length 5: 12321 and 12121. And there are 5 possible blocks of weight 7:

$$1234321, 1232321, 1212321, 1232121, 1212121$$

Also, a block of length $2j + 1$ has degree $j + 1$.

**Lemma 4.15.** *The number of possible blocks of length* $2j+1$ *is given by the Catalan number*

$$C_j = \frac{1}{j+1}\binom{2j}{j}$$

The proof is left to the reader.

**Lemma 4.16.** *The ranks* $r(n, k)$ *are uniquely determined by the following recursion:* $r(n, 0) = 1$ *for all* $n \geq 0$ *and for* $k > 0$ *we have:*

$$r(n,k) = \begin{cases} 0 & \text{if } n \leq 2k-2 \\ r(n-1,k) + \sum_{1 \leq j \leq k} r(n-2j, k-j)C_{j-1} & \text{otherwise} \end{cases}$$

*where, for convenience of notation, we use the convention that* $r(-1, 0) = 1$.

*Proof.* Since $X(A_n)$ is connected, we have $r(n, 0) = 1$ for $n \geq 0$. The convention $r(-1, 0) = 1$ is used to define ther term $r(n - 2j, k - j)$ when $n = 2k - 1$ and $j = k$. To get from $w_0 = 0$ to $w_{n+1} = 0$ with $k$ steps up and $k$ steps down we must have at least $n + 1 \geq 2k$. So, $r(n, k) = 0$ when $n + 1 < 2k$.

Now consider all primitive semi-simple weight $w$ with $n, k \geq 1$. There are two cases.

Case 1: $w_n = 0$. In that case $(w_1, \cdots, w_{n-1})$ is a primitive semi-simple weight of degree $k$. So, there are $r(n - 1, k)$ weights in this case.

Case 2: $w_n = 1$. Let $2j - 1$ be the length of the last block of $w$. Then $w_{n-2j} = 0$ and $w' = (w_1, \cdots, w_{n-2j-1})$ is a primitive semi-simple weight of degree $k - j$. Since there are $C_{j-1}$ possibilities for the last block of $w$ and there are $r(n - 2j, k - j)$ possibilities for $w'$ we have $C_{j-1}r(n - 2j, k - j)$ possibilities for $w$ in this case. This proves the recursion. □

**Definition 4.17.** The *ballot number* $b(k, j)$ is defined to be the number of ways in which $k$ "yes" votes and $j$ "no" votes can be cast in an ordered sequence in such a way that the number of "yes" votes is always greater than or equal to the number of "no" votes. In particular $b(k, j) = 0$ if $j > k$.

Since the count starts at $(0, 0)$ and votes are cast one at a time by assumption, we have the following recussion: $b(k, j) = 0$ unless $k \geq j \geq 0$, $b(0, 0) = 1$ and

$$b(k, j) = b(k - 1, j) + b(k, j - 1)$$

for $k \geq 1$. It is a well-known property of Catalan numbers that $b(k,k) = C_k$. An extension of this observation is the following recursion.

**Lemma 4.18.** *For $m \geq k \geq 1$ we have*

$$b(m,k) = b(m-1,k) + \sum_{j=1}^{k} b(m-j,k-j)C_{j-1}$$

*Proof.* There are two cases.

Case 1: The last vote cast was "yes". There are $b(m-1,k)$ ways this could happen.

Case 2: The last vote cast was "no". Consider the difference $m-k \geq 0$ between the number of "yes" votes and the number of "no" votes. In case two this number was $m-k+1 \geq 1$ before the last vote. Since this difference starts and ends at a smaller number, this difference must have been equal to $m-k$ at some earlier point. Let $j > 0$ be minimal so that the last $2j$ votes were tied $j$ in favor and $j$ against. There are $C_{j-1}$ ways these last $2j$ votes could have been cast since the last vote was "no" and the first must have been "yes". So, there are $C_{j-1}b(m-j,k-j)$ ways that this could happen.

Adding up all possible cases, we get the stated recursion. □

Comparing the two lemmas, we get:

$$r(n,k) = b(n-k+1,k)$$

which proves the following theorem.

**Theorem 4.19.** *For $n,k \geq 0$ the integral homology group $H_k(X(A_n);\mathbb{Z})$ of the CW-complex $X(A_n)$ for the quiver of type $A_n$ with straight orientation is free abelian with rank equal to the ballot number $b(n-k+1,k)$.* □

Since $b(m,k) = 0$ for $k > m$, $b(k,k) = C_{k-1}$ and $b(n,k) \neq 0$ for $k \leq m$ we get the following.

**Corollary 4.20.** *$H_k(X(A_n);\mathbb{Z}) = 0$ for $k > \frac{n+1}{2}$ and is nonzero for $0 \leq k \leq \frac{n+1}{2}$. Furthermore, $H_k(X(A_{2k-1})) = C_k$.* □

Note that, for any Dynkin quiver $Q$ with $n$ vertices, $X(Q)$ is an $n$-dimensional CW complex. So, we always have: $H_k(X(Q)) = 0$ for $k > n$.

Summary: In Section 4, we show that the homology of the space $X(A_n)$ is freely generated by "primitive semi-simple weights". These are disjoint unions of "blocks". Blocks are enumerated using Catalan numbers and the primitive weights are enumerated by ballot numbers.

## 5. Proof that $X(A_n)$ is a $K(\pi,1)$ for $\pi = G(A_n)$

Finally, in Section 5 we outline a proof that $X(A_n)$ is a $K(\pi,1)$ and that, therefore, the calculation in Section 4 computes the cohomology of these groups. The proof is by induction on $n$ and uses a filtration of $X(A_n)$ by subcomplexes $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n$ where $Y_0 = X(A_{n-1})$, $Y_n = X(A_n)$ and each $Y_m$ is a $K(\pi,1)$. We give some examples to see what these spaces and groups are. Missing details in this proof can be found in another paper [10] where this theorem is extended to the case of a convex subset of the preprojective component of the Auslander-Reiten quiver of a hereditary algebra of not necessarily finite type. The special case of $A_n$ with straight orientation, as considered here, is also redone combinatorially in [5] where the result is slightly strengthened: The space $X(A_n)$ is locally CAT(0).

We will first go over the base cases $n \leq 2$. Then, for $n \geq 3$, we use that fact that $G(A_n)$ is an iterated HNN extension of $G(A_{n-1})$ to construct a sequence of spaces $Y_m$ going from $Y_0 = X(A_{n-1})$ to $Y_n = X(A_n)$ and show by induction on $m$ that each $Y_m$ is a $K(\pi, 1)$.

5.1. **Filtration of $X(A_n)$.** For $n = 0$, the space $X(A_0)$ is a single point which is a $K(\pi, 1)$ with trivial group $\pi$.

For $n = 1$, $X(A_1)$ has $C_2 = 2$ cells: one 0-cell and one 1-cell. So, it is the circle $S^1 = K(\mathbb{Z}, 1)$.

For $n = 2$, $X(A_2)$ has $C_3 = 5$ cells: one 0-cell, three 1-cells giving generators $a, b, x$ for the fundamental group and one 2-cell $e(a, b)$ whose boundary gives the relation

$$\partial e(a, b) = aba^{-1}x^{-1}b^{-1}$$

In other words, the 2-cell $e(a, b)$ is a pentagon (the associahedron) with one side identified with the 1-cell $x$ and the other four sides pasted twice to $a$ and $b$. So, $X(A_2)$ is homeomorphic to a torus with an open 2-disk removed. This is homotopy equivalent to $S^1 \vee S^1$ which is a $K(\pi, 1)$ with $\pi$ being the free group with two generators $F_2$.

Suppose by induction that $n \geq 3$ and $X(A_k)$ is a $K(\pi, 1)$ for $k < n$. Then we consider another filtration of $X(A_n)$ with subcomplexes $Y_m$ for $0 \leq m \leq n$ give as follows.

First, note that $X(A_{n-1})$ is the union of all cells in $X(A_n)$ whose last block is $B_{ij}$ where $i < j < n$. Let $Y_m$ be the union of $X(A_{n-1})$ and all cells $\beta_*$ so that, for any object of the hom-orthogonal set $\beta_*$ of the form $\beta_{kn}$, we have $k < m$. Then

$$Y_0 = X(A_{n-1}) \subseteq Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_n = X(A_n)$$

So, $Y_0$ is a $K(\pi, 1)$ and we need to show that $Y_n$ is a $K(\pi, 1)$. We may assume by induction on $1 \leq m \leq n$ that $Y_{m-1}$ is a $K(\pi, 1)$. To show that $Y_m$ is a $K(\pi, 1)$ we first observe that $\pi_1(Y_m)$ is an HNN extension of $\pi_1(Y_{m-1})$. Then we will show that $Y_m$ is a "graph of groups" [4], p.91, and therefore a $K(\pi, 1)$. We will give examples, definitions and proofs.

**Example 5.1.** Let $n = 3$ and $m = 0$. Then $Y_0 = X(A_2)$ and

$$Y_1 = Y_0 \cup e(\beta_{03}) \cup e(\beta_{03}, \beta_{12})$$

$Y_0 \simeq S^1 \vee S^1$ and we are attaching one more 1-cell giving a new generator $x(\beta_{03})$ for the fundamental group and one new 2-cell giving the commutation relation $[x(\beta_{12}), x(\beta_{03}] = 1$. So, $Y_1 \simeq S^1 \vee (S^1 \times S^1)$ which is a $K(\pi, 1)$ with $\pi$ the free product of $\mathbb{Z}$ with $\mathbb{Z} \times \mathbb{Z}$. For $n = 3, m = 1$ we have:

$$Y_2 = Y_1 \cup e(\beta_{13}) \cup e(\beta_{01}, \beta_{13})$$

As in the case of $Y_1$, $Y_2$ is obtained by attaching a new 1-cell $e(\beta_{mn}) = e(\beta_{13})$ giving the new generator $x(\beta_{13})$ in $\pi_1 Y_2$ and a new 2-cell giving the relation $[x(\beta_{01}), x(\beta_{13})] = x(\beta_{03})$. In other words:

$$x(\beta_{13})^{-1}x(\beta_{01})x(\beta_{13}) = x(\beta_{03})x(\beta_{01})$$

For $n = 5, m = 2$, $Y_3$ is obtained from $Y_2$ by attaching 10 new cells:

$$Y_3 = Y_2 \cup e(\beta_{25}) \cup e(\beta_{01}, \beta_{25}) \cup e(\beta_{12}, \beta_{25}) \cup e(\beta_{02}, \beta_{25}) \cup e(\beta_{01}, \beta_{12}, \beta_{25}) \tag{5.1}$$

$$\cup e(\beta_{25}, \beta_{34}) \cup e(\beta_{01}, \beta_{25}, \beta_{34}) \cup e(\beta_{12}, \beta_{25}, \beta_{34}) \cup e(\beta_{02}, \beta_{25}, \beta_{34}) \cup e(\beta_{01}, \beta_{12}, \beta_{25}, \beta_{34})$$

We will use this to explain the general construction.

5.2. **HNN extensions.**

**Definition 5.2.** Suppose that $H, G$ are groups and $\varphi, \psi : H \to G$ are injective homomorphisms. Then *HNN-extension* $N(H, G, \varphi, \psi)$ is defined to be the quotient of the free product $G * \mathbb{Z}$ with new free generator $t$ modulo the relation

$$t\varphi(h)t^{-1} = \psi(h)$$

for all $h \in H$.

Let $G_{n,m} = \pi_1(Y_m)$ where $Y_m$ is defined above. Since $Y_m$ is a subcomplex of $X(A_n)$, the generators and relations for $G_{n,m}$ are the subsets of the generators and relations for $G(A_n)$:

**Lemma 5.3.** *$G_{n,m} = \pi_1(Y_m)$ has generators and relations given by the 1-cells and 2-cells of $Y_m$ as follows.*

1. *Generators: $x(\beta_{ij})$ where $0 \le i < j \le n$ excluding $x(\beta_{in})$ for $i \ge m$.*
2. *Relations among these generators are:*
   - (a) *$x(\beta_{ij}), x(\beta_{k\ell})$ commute if $i, j, k, \ell$ are distinct and noncrossing and*
   - (b) *$[x(\beta_{ij}), x(\beta_{jk})] = x(\beta_{ik})$ if $0 \le i < j < k \le n$ and where $j < m$ if $k = n$.*

Examination of this list shows that $G_{n,m+1} = \pi_1(Y_{m+1})$ has only one new generator not in $G_{n,m}$, namely $x(\beta_{mn})$ and the only new relations are commutation relations with this new generators which can be expressed as follows.

- (a) $x(\beta_{ij}), x(\beta_{mn})$ commute if either $0 \le i < j < m$ or $m < i < j < n$.
- (b) $x(\beta_{mn})^{-1}x(\beta_{im})x(\beta_{mn}) = x(\beta_{in})x(\beta_{im})$

Equivalently, we have:

**Proposition 5.4.** *$G_{n,m+1}$ is an HNN-extension of $G_{n,m}$ with*

1. $t = x(\beta_{mn})^{-1}$
2. $H = H_m = G(A_m) \times G(A_{n-m-2})$
3. *The monomorphism $\varphi : H_m \to G_{n,m}$ is given on generators by*

$$\varphi(x(\beta_{ij}), x(\beta_{k\ell})) = x(\beta_{ij})x(\beta_{k+m+1,\ell+m+1}) \in G(A_{n-1}) \subseteq G_{n,m}$$

4. *The monomorphism $\psi : H_m \to G_{n,m}$ is given on generators by $\psi(x(\beta_{ij}), x(\beta_{k\ell})) = x(\beta_{ij})x(\beta_{k+m+1,\ell+m+1})$ (same as $\varphi$) if $j < m$ and*

$$\psi(x(\beta_{im}), 1) = x(\beta_{in})x(\beta_{im})$$

*which lies in $G_{n,m}$ since $i < m$.*

*Proof.* We will check that $\psi$ is a group homomorphism by showing that it preserves relations. We check only one relation: $[x(\beta_{ij}), x(\beta_{jm})] = x(\beta_{im})$:

$$[\psi x(\beta_{ij}), \psi x(\beta_{jm})] = [x(\beta_{ij}), x(\beta_{jn})x(\beta_{jm})]$$
$$= x(\beta_{jm})^{-1}x(\beta_{jn})^{-1}x(\beta_{ij})x(\beta_{jn})x(\beta_{jm})x(\beta_{ij})^{-1}$$
$$= x(\beta_{jm})^{-1}x(\beta_{in})x(\beta_{ij})x(\beta_{jm})x(\beta_{ij})^{-1}$$
$$= x(\beta_{in})[x(\beta_{ij}), x(\beta_{jm})] = x(\beta_{in})x(\beta_{im})$$

The other relations are easy to check. □

We use the following well-known theorem. (See [4] for an elementary exposition and proof.)

**Theorem 5.5.** *Suppose that $X, Y$ are base pointed spaces so that $X$ is a $K(H,1)$ and $Y$ is a $K(G,1)$. Let $f, g : X \to Y$ be a pointed continuous mapping which induces the maps $\varphi, \psi : \pi_1(X) = H \to G = \pi_1(Y)$ respectively. Let $Z = X \times I \coprod Y/\sim$ be the quotient space of $X \times I \coprod Y$ modulo the identifications $(x, 0) \sim f(x) \in Y$ and $(x, 1) \sim g(x) \in Y$ for all $x \in X$. Then $Z$ is a $K(\pi, 1)$ with $\pi = N(H, G, \varphi, \psi)$.*

We claim that, using $X = X(A_m) \times X(A_{n-m-2})$, $Y = Y_m$ with $f, g : X \to Y$ appropriately chosen maps, we get $Z$ homeomorphic to $Y_{m+1}$ proving that $Y_{m+1}$ is a $K(\pi, 1)$. See [10] for a detailed proof of a more general theorem proved using the same outline.

**Theorem 5.6.** *The picture space $X(A_n)$ is a $K(\pi, 1)$ where $\pi$ is the picture group $G(A_n)$ with generators $x_{ij}$ for all $0 \le i < j \le n$ and relations where we use the notation $[x, y] := y^{-1}xyx^{-1}$.*

1. $[x_{ij}, x_{jk}] = x_{ik}$ *for all* $i < j < k$
2. $[x_{ij}, x_{k\ell}] = 1$ *if the closed intervals* $[i, j], [k, \ell]$ *are either disjoint or one is contained in the interior of the other.*

We call $[i, j]$ the *extended support* of $x_{ij}$. The actual support is $[i + 1, j]$. We say that two closed intervals are *noncrossing* if they are either disjoint or one is in the interior of the other. So (2) says that generators commute if their extended supports are noncrossing.

*Proof.* The presentation follows from Lemma 4.7 and (1) and (2) in Example 2.5. We have switched to the simplified notation

$$x_{ij} = x(\beta_{ij}).$$

□

Summary: Section 5 outlines the proof that $X(A_n)$ is a $K(\pi, 1)$ for $\pi = G(A_n)$ and, therefore, the homology of $X(A_n)$ is the homology of the picture group $G(A_n)$.

## 6. Cup product structure

We now determine the ring structure on the cohomology $H^*(X(A_n); \mathbb{Z})$. We use the fact that $X(A_n)$ is a $K(\pi, 1)$ for the group $G(A_n)$ proved in detail in [10], [5] and outlined in the last section. So, we deal with the cohomology of the group $G(A_n)$ instead of the space $X(A_n)$. Since the homology is freely generated by the set of primitive weights $w$, the cohomology is also freely generated as an additive group by the dual elements $w^*$. We will show that, as a ring, the cohomology is generated by the duals $w^*$ to weights $w$ having only one block. We call such generators *dual blocks*. First, we need an intrinsic (instead of computational) description of the dual blocks with "full support."

6.1. **Dual blocks.** Each block is supported on a subspace $G(Q_{ij}) \subseteq G(A_n)$ which we will show is a retract. For every $0 \le i < j \le n$ let $Q_{ij}$ be the full subquiver of the quiver $Q(A_n)$ with vertices $i + 1, \cdots, j$:

$$Q_{ij} : \qquad i + 1 \leftarrow i + 2 \leftarrow \cdots \leftarrow j$$

This is the smallest subquiver which supports the root $\beta_{ij}$. Let $G(Q_{ij})$ be the corresponding subgroup of $G(A_n)$ which is generated by $x_{pq} = x(\beta_{pq})$ where $i \le p < q \le j$.

The CW complex $X(A_n)$ contains a subcomplex $X(Q_{ij})$ which is the union of all cells $e^k_{\alpha_*}$ where each element of $\alpha_*$ is a root $\beta_{pq}$ for $i \le p < q \le j$. The inductive proof that $X(A_n)$ is a $K(\pi, 1)$ shows that $X(Q_{ij})$ is a $K(\pi, 1)$ with $\pi = G(Q_{ij})$ and the inclusion $X(Q_{ij}) \subseteq X(A_n)$ induces the inclusion $G(Q_{ij}) \subseteq G(A_n)$.

**Lemma 6.1.** *Let $r_{ij} : G(A_n) \to G(A_{j-i})$ be the homomorphism given on generators by*

$$r_{ij}(x_{pq}) = \begin{cases} x_{p-i,q-i} & \text{if } i \le p < q \le j \\ 1 & \text{otherwise} \end{cases}$$

*Then the composition $G(Q_{ij}) \to G(A_n) \to G(A_{j-i})$ is an isomorphism.*

*Proof.* The map $r_{ij}$ is a homomorphism since it respects the relations of $G(A_n)$. For example, the relation $[x_{ij}, x_{jk}] = x_{ik}$ in $G(A_n)$ becomes the relation $x_{ij}x_{ij}^{-1} = 1$ in $G(A_{j-i})$. On $G(Q_{ij})$, the mapping $r_{ij}$ sends relations among $x_{pq}$ to corresponding relations among $x_{p-i,q-i}$. So, it induces an isomorphism $G(Q_{ij}) \cong G(A_{j-i})$. □

**Proposition 6.2.** *If $0 \le p_1 < q_1 < \cdots < p_k < q_k \le n$ then the subgroups $G(Q_{p_i,q_i})$ of $G(A_n)$ commute with each other. So, we have inclusion and projection morphisms*

$$\prod G(Q_{p_i,q_i}) \hookrightarrow G(A_n) \xrightarrow{\prod r_{p_i,q_i}} \prod G(A_{q_i-p_i})$$

*Whose composition is an isomorphism.*

*Proof.* Generators of $G(Q_{i_s,j_s})$ have extended supports in $[i_s, j_s]$ which are disjoint and therefore commute. □

**Proposition 6.3.** *Suppose that $m = 2k+1$ is odd. Then the cokernel of the homomorphism in integral homology induced by inclusion:*

$$H_p(G(Q_{1,m})) \oplus \bigoplus_{j=2}^{m-1} H_p(G(Q_{0,j-1}) \times G(Q_{j,m})) \oplus H_p(G(Q_{0,m-1})) \to H_p(G(A_m))$$

*is free of rank $C_k = \frac{1}{k+1}\binom{2k}{k}$ if $p = k+1$ and is 0 otherwise.*

*Proof.* The image of this homomorphism contains all primitive semisimple weights which do not have full support. So, the cokernel is freely generated by primitive blocks $w$ with full support, i.e., so that $w_i \neq 0$ for $1 \le i \le n$. These exist only when $m$ is odd. They have degree $k+1$ and there are a Catalan number $C_k$ of these by Lemma 4.15. □

Let $K(G(A_m)) \subseteq H^{k+1}(G(A_m); \mathbb{Z})$ denote the kernel of the homomorphism

$$H^{k+1}(G(A_m)) \to H^{k+1}(G(Q_{1,m})) \oplus \bigoplus_{j=2}^{m-1} H^{k+1}(G(Q_{0,j-1}) \times G(Q_{j,m})) \oplus H^{k+1}(G(Q_{0,m-1}))$$

induced by the inclusions of these subgroups into $G(A_m)$ where $m = 2k+1$. By the proposition above, $K(G(A_m))$ is free abelian of rank $C_k$. (Whereas $H^{k+1}(G(A_m))$ is free abelian of rank $C_{k+1}$.)

**Definition 6.4.** Choose a fixed basis for $K(G(A_m))$. We define the *dual blocks with full support* in $H^{k+1}(G(A_m); \mathbb{Z})$ to be any element of this basis. For $n \ge m$, the images of these $C_k$ generators under the split monomorphisms

$$H^{k+1}(G(A_m)) \xrightarrow{r_{i,i+m}^*} H^{k+1}(G(A_n))$$

induced by the retractions $r_{i,i+m} : G(A_n) \twoheadrightarrow G(A_m)$ for $0 \le i \le n-m$ will be called the *dual blocks* in $H^{k+1}(G(A_n))$ with *extended support* $[i, i+m]$. The image of $K(G(A_m))$ under $r_{i,i+m}^*$ will be denoted $K(Q_{i,i+m})$.

**Proposition 6.5.** *The cohomology of $G(A_n)$ has a direct summand which is freely generated by the dual blocks. I.e., $\bigoplus_i K(Q_{i,i+m})$ is a direct summand of $H^{k+1}(G(A_n))$ for $m = 2k+1$.*

*Proof.* Restricting to one degree, say $k+1$, we are talking about dual blocks with extended support $[p_i, q_i]$ where $q_i - p_i = 2k+1$ for each $i$. Given any additive linear combination of such dual blocks, consider the projection homomorphism

$$H^{k+1}(G(A_n)) \twoheadrightarrow H^{k+1}(G(Q_{p_i,q_i}))$$

induced by the inclusion $G(Q_{p_i,q_i}) \hookrightarrow G(A_n)$. By definition of dual blocks, this homomorphism is zero on dual blocks of degree $k+1$ with extended support not equal to $[p_i, q_i]$ since such extended supports will not be contained in $[p_i, q_i]$. Also, by definition of dual blocks, the dual blocks with extended support $[p_i, q_i]$ map to distinct free generators of a split summand of $H^{k+1}(G(Q_{p_i,q_i}))$. The Proposition follows. □

6.2. **Multiplication of dual blocks.**

**Lemma 6.6.** *If $w_1^*, w_2^* \in H^*(G(A_n))$ are dual blocks whose extended supports intersect then their cup product is zero: $w^1 \cup w^2 = 0$.*

*Proof.* This is just dimension counting. Suppose that $w_i^*$ has extended support $[p_i, q_i]$ with $q_i - p_i = 2k_i + 1$. Then, by definition, $w_i^* \in H^{k_i+1}(G(A_n))$ is the pull back of a dual block of full support in $H^{k_i+1}(G(A_{2k_i+1}))$ under the homomorphism

$$r_{p_i,q_i} : G(A_n) \to G(A_{2k_i+1})$$

Therefore, their cup product is the pull back of the cross product of these cohomology classes under the homomorphism

$$(r_{p_1,q_1}, r_{p_2,q_2}) : G(A_n) \to G(A_{2k_1+1}) \times G(A_{2k_2+1})$$

Let $a = \min(p_1, p_2)$ and $b = \max(q_1, q_2)$. Then the homomorphism $(r_{p_1,q_1}, r_{p_2,q_2})$ above factors through $G(A_{b-a})$ which has cohomological dimension $\lfloor \frac{b-a+1}{2} \rfloor$ by Corollary 4.20. If the extended supports of $w_1^*, w_2^*$ meet then we will have $b - a \leq 2k_1 + 2k_2 + 2$ making $\lfloor \frac{b-a+1}{2} \rfloor \leq k_1 + k_2 + 1$ which is less than the degree of $w_1^* \cup w_2^*$. So, this cup product must be zero. □

**Lemma 6.7.** *If $w_1^*, \cdots, w_k^* \in H^*(G(A_n))$ are dual blocks with disjoint extended supports then their cup product is nonzero.*

*Proof.* This follows immediately from Proposition 6.2. □

**Theorem 6.8.** *The integral cohomology of the picture group $G(A_n)$ is generated, as a ring, by the set of dual blocks $w_i^*$. The cup product of these dual blocks is nonzero if and only if their extended supports are disjoint. Furthermore, as an additive group, $H^*(G(A_n); \mathbb{Z})$ is freely generated by the nonzero cup products of the dual blocks.*

*Proof.* A simple counting argument shows that the number of nonzero cup products of dual blocks is equal to the total rank of the cohomology of $G(A_n)$. Therefore, it suffices to show that these cup products are linearly independent and generate a direct summand of $H^*(G(A_n))$.

We work in a fixed degree, say $d$. Consider an arbitrary integer linear combination of degree $d$ cup products of dual blocks. Take one of these expressions: $w_1^* \cup w_2^* \cup \cdots \cup w_k^*$ where

$w_i^*$ is a dual block with extended support $[p_i, q_i]$ so that $k$ is maximal. Then $\sum(q_i - p_i) = 2d - k$ is minimal. This implies that, the restriction homomorphism

$$H^d(G(A_n)) \to H^d\left(\prod_{i=1}^k G(Q_{p_i,q_i})\right)$$

is zero on all cup products in the given linear combination which have different extended support. The reason is that any other extended support will contain a point not in the set $\bigcup[p_i, q_i]$ and this implies that that term will vanish under the restriction map and therefore the cup product will also go to zero. The set of all cup products of dual blocks with the same extended support as the given one freely generate the subgroup

$$K(Q_{p_1,q_1}) \otimes K(Q_{p_2,q_2}) \otimes \cdots \otimes K(Q_{p_k,q_k}) \tag{6.1}$$

which is a direct summand of $H^d\left(\prod_{i=1}^k G(Q_{p_i,q_i})\right)$ and thus of $H^p(G(A_n))$ since each $K(Q_{p_i,q_i})$ is a direct summand of $H^*(G(Q_{p_i,q_i}))$. Proceeding by downward induction on $k$, we obtain a complete direct sum decomposition of $H^d(G(A_n))$ as claimed. □

*Remark* 6.9. We observe that this proof shows that $H^*(G(A_n))$ is a direct sum of subgroups of the form (6.1).

When $Q$ is a quiver of type $A_n$, the group $G(Q)$ depends on the orientation of $Q$. This is a computer calculation using GAP which was carried out by D.Ruberman. Although the group depends on the orientation, we believe that the homology of the groups is independent of the orientation. According to He Wang, the difference lies in the Massey product structure of the cohomology. These are questions for further research.

Summary: Section 6 determines the cup product structure of the integral cohomology of $G(A_n)$.

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