CLUSTER CATEGORIES COMING FROM CYCLIC POSETS

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Abstract. Cyclic posets are generalizations of cyclically ordered sets. In this paper we show that any cyclic poset gives rise to a Frobenius category over any discrete valuation ring R. The stable category of a Frobenius category is always triangulated and has a cluster structure in many cases. The continuous cluster categories of [14], the infinity-gon of [12], the m-cluster category of type A_∞ ($m \geq 3$) [13] are examples of this construction as well as some new examples such as the cluster category of $\mathbb{Z}^2$. An extension of this construction and further examples are given in [16].

Introduction

The goal of this paper is to give a simple uniform procedure for constructing many algebraically triangulated categories having cluster structures in the sense of [3]. We start with an elementary combinatorial construction which we call a "cyclic poset." This has several equivalent descriptions, one of which is a (special kind of) $\mathbb{N}$-category. By a linearization process over a discrete valuation ring with uniformizer t, we construct a $t^{\mathbb{N}}$-category which contains the original cyclic poset: every cyclic poset gives a $t^{\mathbb{N}}$-category and all $t^{\mathbb{N}}$-categories are given in this way (Proposition 1.2.5). From the $t^{\mathbb{N}}$-category we construct a Frobenius category using a variation of the matrix factorization construction.

The stable category of any Frobenius category is a triangulated category [11]. However, it will not always be 2-Calabi-Yau. We define a twisted version of the Frobenius category and show that it gives a 2-CY triangulated category for appropriately twisted cases (Theorem 2.4.5). The twist is given by an automorphism of the cyclic poset. Different choices of automorphisms can give higher Calabi-Yau categories.

This paper combines two concepts which have appeared in recent literature: matrix factorization categories and $\mathbb{N}$-categories. Matrix factorization was introduced by Eisenbud [9] and developed by Buchweitz in an unpublished paper [5] (see also [6]), Orlov [20], [21] and many others [19], [8]. $\mathbb{N}$-categories are a version of the $\mathbb{Z}$-categories considered by Drinfeld [7] who, in turn, attributes it to Besser [2] and Grayson [10]. We take the multiplicative version which we call a $t^{\mathbb{N}}$-category, very similar to a construction which occurs in van Roosmalen [22]. We call the basic underlying structure a "cyclic poset."

In more detail, the construction is as follows.

Given a discrete valuation ring R with uniformizer t, we define a $t^{\mathbb{N}}$-*category* $\mathcal{R}$ over R to be a small R-category with two properties (Definition 1.2.1): (1) $\mathcal{R}(x,y) \cong R$ for any two objects x, y of $\mathcal{R}$, (2) $\mathcal{R}(x,y)$ has a generator f_{xy} with the property that, for any three objects x, y, z,

$$f_{yz} \circ f_{xy} = t^n f_{xz}$$

where $n \in \mathbb{N}$ is a nonnegative integer. We call f_{xy} the "basic morphism" from x to y.

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In the above definition, $n \in \mathbb{N}$ is a function of x, y, z which we denote $n = c(xyz)$. If we denote the set of objects of $\mathcal{R}$ by X, this gives a function $c : X^3 \to \mathbb{N}$ which is necessarily a reduced (2)-cocycle where the cocycle condition, equivalent to associativity of composition in $\mathcal{R}$, is

$$c(xyz) - c(wyz) + c(wxz) - c(wxy) = 0$$

for any $w, x, y, z \in X$ and "reduced" means $c(xxy) = 0 = c(xyy)$ for any $x, y \in X$ (equivalent to f_{xx} being the identity on x). Conversely, any reduced cocycle $c : X^3 \to \mathbb{N}$ on any set X defines a $t^{\mathbb{N}}$-category denoted $\mathcal{R}(X, c)$ or simply $\mathcal{R}(X)$ (Lemma 1.2.2 and Proposition 1.2.5).

We define a cyclic poset to be a pair (X, c) where X is a set and $c : X^3 \to \mathbb{N}$ is a reduced cocycle on X. We construct from this a Frobenius category $\mathcal{MF}(X)$, which will be a special case of the twisted version, below, when ϕ is the identity map on X.

Suppose ϕ is an admissible automorphism of (X, c) (Definition 1.4.1). Then ϕ extends to an R-linear automorphism of $\mathcal{P}(X) = add\,\mathcal{R}(X)$ and there is a natural transformation $\eta_V : V \to \phi(V)$ given on each component of any $V \in \mathcal{P}(X)$ by the basic morphism $x \to \phi(x)$. Let $\mathcal{MF}_\phi(X)$ denote the category of all pairs (V, d) where V is an object of $\mathcal{P}(X)$ and d is an endomorphism of V satisfying the following two properties.

(1) $d^2 = \cdot t$ is multiplication by t.
(2) d factors through $\eta_V : V \to \phi(V)$.

If ϕ is admissible, then we show that $\mathcal{MF}_\phi(X)$ is a Frobenius category. Let $\mathcal{C}_\phi(X)$ be the stable category of $\mathcal{MF}_\phi(X)$. Then $\mathcal{C}_\phi(X)$ is a triangulated category.

The category $\mathcal{MF}_\phi(X)$ is a matrix factorization category, not by definition but by Theorem 2.1.7 which says that any object (V, d) in $\mathcal{MF}_\phi(X)$ decomposes in $\mathcal{P}(X)$ as a direct sum of two objects $V = V_0 \oplus V_1$ so that $d : V \to V$ has the form

$$d = \begin{bmatrix} 0 & \beta \\ \alpha & 0 \end{bmatrix} : \qquad V_0 \underset{\alpha}{\overset{\beta}{\rightleftarrows}} V_1 \, .$$

In other words, $\alpha : V_0 \to V_1$, $\beta : V_1 \to V_0$ are morphisms so that $\alpha\beta = \cdot t$ and $\beta\alpha = \cdot t$. In the analogous case when V_0, V_1 are replaced by free modules over a commutative ring R, such morphisms are given by square matrices A, B so that $AB = BA = tI_n$, i.e., (A, B) is a matrix factorization of t in the traditional sense [9]. In a matrix factorization category, one usually assumes that objects are $\mathbb{Z}/2$-graded. For us, the grading is undefined and there is no difference between even and odd morphisms.

To summarize: we start with any cyclic poset (X, c) given by a reduced cocycle c. This is also given by a covering poset (Definition 1.1.1) and there is an associated $\mathbb{N}$-category (Definition 1.1.15). By a linearization procedure we form a $t^{\mathbb{N}}$-category $\mathcal{R}(X)$ where R is a DVR with uniformizer t. Let $\mathcal{P}(X) = add\,\mathcal{R}(X)$ be the additive category generated by $\mathcal{R}(X)$. From $\mathcal{P}(X)$ we construct a matrix factorization category $\mathcal{MF}(X)$. This is a Frobenius category. The stable category is a triangulated category $\mathcal{C}(X)$ which depends only on X, R and t. Given an admissible automorphism ϕ of the cyclic poset X, we construct a twisted version of this to give another Frobenius category $\mathcal{MF}_\phi(X)$ whose stable category $\mathcal{C}_\phi(X)$ will be 2-Calabi-Yau with suitable choice of X and ϕ. The untwisted version is the case $\phi = id_X$.

In the remainder of the paper we explore examples and special cases of Frobenius categories of the form $\mathcal{MF}_\phi(X)$ and their stable categories $\mathcal{C}_\phi(X)$. These include:

(1) Continuous Frobenius categories over R
(2) Cluster categories of type A_n over the field $\mathbf{k} = R/(t)$ where $1 \le n \le \infty$
(3) m-cluster categories of type A_∞ over $\mathbf{k}$ for $m \ge 3$

(4) New examples of categories with cluster structure such as $\mathcal{C}_\phi(\mathbb{Z} * \mathbb{Z})$.

There is a chart at the end of the paper listing the main examples explained in this paper.

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1. General theory

We will go through the general theory of how a cyclic poset gives a Frobenius category.

1.1. **Cyclic posets.**

The purpose of this subsection is to give three equivalent descriptions of cyclic posets and set up basic terminology for use in the rest of the paper. A cyclic poset can be given as the orbit space of a special kind of poset having a free action of the additive group $\mathbb{Z}$. A cyclic poset structure on a set X is also given by a function $c : X^3 \to \mathbb{N}$ called a (reduced) cocycle. We give examples and show (Corollary 1.1.11) that these two notions are equivalent. Finally, we show (Proposition 1.1.16) that cyclic posets are a special case of $\mathbb{N}$-categories which we define. The basic property is that $\mathbb{N}$ acts on the set of morphism from x to y so that all morphisms have the form $n \cdot f_{xy}$ where $n \in \mathbb{N}$ for some basic morphism $f_{xy} : x \to y$.

1.1.1. *Covering posets.* We recall that a *poset* is a set X with a transitive, reflexive relation $\leq$. Two elements of X are *equivalent* and we write $x \approx y$ if $x \leq y$ and $y \leq x$. We write $x < y$ if $x \leq y$ and $x \not\approx y$. A morphism of posets $f : X \to Y$ is a mapping which preserves the relation: $x \leq y \Rightarrow f(x) \leq f(y)$.

Definition 1.1.1. A *covering poset* of set X is a poset $\tilde{X}$ together with a surjective mapping $\pi : \tilde{X} \to X$ satisfying the following.

(1) The inverse image $\pi^{-1}(x) \subseteq \tilde{X}$ of any $x \in X$ is isomorphic as a poset to the set of integers with the usual ordering.
(2) There exists an automorphism σ of the poset $\tilde{X}$ so that, for any $\tilde{x} \in \tilde{X}$, $\sigma\tilde{x}$ is the smallest element of $\pi^{-1}(\pi(\tilde{x}))$ which is greater than $\tilde{x}$.
(3) For every pair $\tilde{x}, \tilde{y} \in \tilde{X}$ there exists an integer $m \geq 0$ so that $\tilde{x} \leq \sigma^m\tilde{y} \leq \sigma^{2m}\tilde{x}$.

An *equivalence* of covering posets $f : \tilde{X} \to \tilde{X}'$ is a σ-equivariant poset isomorphism over X, i.e., $\pi' f = \pi$ and $\sigma f = f\sigma$.

Remark 1.1.2. In the above definition, given (1), the automorphism σ of the covering poset $\tilde{X}$ is uniquely determined.

Example 1.1.3. The simplest example of this is a set with n elements $Z_n = \{1, 2, \cdots, n\}$. One covering poset of Z_n is given by the set of integers $\tilde{Z}_n = \mathbb{Z}$ with the usual ordering and the automorphism $\sigma x = x + n$. Therefore, $\cdots < 1 < 2 < \cdots < n < \sigma 1 < \sigma 2 < \cdots$.

1.1.2. *Cocycles.* We will show that covering posets over X are classified by reduced cocycles $c : X^3 \to \mathbb{N}$ as defined below.

Definition 1.1.4. A *cocycle* on X is a function $c : X^3 \to \mathbb{N}$, written $c(xyz)$, so that

$$\delta c(wxyz) := c(xyz) - c(wyz) + c(wxz) - c(wxy) = 0$$

for all $w, x, y, z \in X$. We say that c is *reduced* if $c(xxy) = c(xyy) = 0$ for all $x, y \in X$.

Example 1.1.5. Central group extensions $\mathbb{Z} \to E \to G$ are classified by elements of $H^2(G;\mathbb{Z})$ and represented by factor sets $f : G^2 \to \mathbb{Z}$ which may be taken to be reduced in the sense that $f(g,h) = 0$ if $g = 1$ or $h = 1$. If this reduced factor set happens to take only nonnegative values, it gives a reduced cocycle c by the formula $c(xyz) = f(x^{-1}y, y^{-1}z)$. Every such cocycle gives a distinct cyclic poset structure on the underlying set of G.

Remark 1.1.6. It can be shown that the usual notion of a *cyclic ordering* on a set X is equivalent to a reduced cocycle on X taking only values $0, 1$ where $c(xyz) = 0$ if $x, y, z \in X$ are in cyclic order and $c(xyz) = 1$ if not.

Definition 1.1.7. Given a covering poset $\pi : \tilde{X} \to X$ and a section $\lambda : X \to \tilde{X}$, we define the corresponding *distance function* $b_\lambda : X^2 \to \mathbb{Z}$ by letting $b_\lambda(xy) = m$ be the smallest integer so that $\lambda(x) \le \sigma^m \lambda(y)$.

We note that b_λ is *reduced* in the sense that $b_\lambda(xx) = 0$ for all $x \in X$.

Lemma 1.1.8. *Let $\pi : \tilde{X} \to X$ be a covering poset and λ a section of π. Then*

$$c(xyz) := \delta b_\lambda(xyz) = b_\lambda(xy) - b_\lambda(xz) + b_\lambda(yz)$$

is a reduced cocycle $c : X^3 \to \mathbb{N}$ and c is independent of the choice of sections $\lambda : X \to \tilde{X}$.

Proof. Since $\delta^2 = 0$, it follows that c is a cocycle. Also, c is easily seen to be reduced. To prove uniqueness, suppose that λ' is another section of π. Then $\lambda'(x) = \sigma^{a(x)}\lambda(x)$ for some function $a : X \to \mathbb{Z}$. Then the two distance function are related by $b_{\lambda'}(xy) = b_\lambda(xy) - a(y) + a(x)$ or $b_{\lambda'} = b_\lambda - \delta a$. So, $\delta b_{\lambda'} = \delta b_\lambda - \delta^2 a = \delta b_\lambda$ as claimed. □

Remark 1.1.9. Since c is uniquely determined for any covering poset $\pi : \tilde{X} \to X$, we refer to it as the *structure cocycle* of the covering poset.

We say that $x, y \in X$ are *equivalent* and we write $x \approx y$ if $c(xyx) = c(yxy) = 0$. It is easy to see that this is an equivalence relation and that $x \approx y$ if and only if $\tilde{x} \approx \tilde{y}$, i.e., $\tilde{x} \le \tilde{y} \le \tilde{x}$, for some liftings $\tilde{x}, \tilde{y}$ of x, y to the covering poset $\tilde{X}$.

Theorem 1.1.10. *Given any reduced cocycle $c : X^3 \to \mathbb{N}$ on any set X, there exists a covering poset $\tilde{X} \to X$ whose structure cocycle is c. Furthermore, $\tilde{X}$ is unique up to poset isomorphism over X.*

Proof. Choose a base point $x_0 \in X$. Then we can express c as $c = \delta b$ where $b : X^2 \to \mathbb{Z}$ is given by $b(xy) = c(x_0xy)$. A covering poset $\tilde{X}$ can now be given as follows. As a set, let $\tilde{X} = X \times \mathbb{Z}$ with σ action given by $\sigma(x,i) = (x, i+1)$. Then the partial ordering of $\tilde{X}$ is given by: $(x,j) \le (y,k)$ if $k \ge j + b(xy)$. In other words, $b(xy)$ is the distance function for the lifting $\tilde{x} = (x, 0)$. Therefore the structure cocycle of $\tilde{X}$ is $\delta b = c$.

If $\tilde{X}' \to X$ is another covering poset with structure cocycle c then a σ-equivariant poset isomorphism $f : X \times \mathbb{Z} \to \tilde{X}'$ is given as follows. Choose a fixed lifting $\tilde{x}_0 \in \tilde{X}'$. Then for each $x \in X$ let $\tilde{x} \in \tilde{X}'$ be the unique lifting of x so that $\tilde{x}_0 \le \tilde{x}$ but $\sigma\tilde{x}_0 \not\le \tilde{x}$. Then, for any $x, y \in X$, the smallest integer j so that $\tilde{x} \le \sigma^j\tilde{y}$ is $j = c(x_0xy) = b(xy)$. Therefore, the covering posets $\tilde{X}'$ and $X \times \mathbb{Z}$ have the same distance function which implies that they are isomorphic with isomorphism $f : X \times \mathbb{Z} \to \tilde{X}'$ given by $f(x,j) = \sigma^j\tilde{x}$. □

Corollary 1.1.11. *For any set X, there is a 1-1 correspondence between equivalence classes of covering posets over X and reduced cocycles $c : X^3 \to \mathbb{N}$.* □

Intuitively, a cyclic poset structure on a set X is an equivalence class of covering posets $\tilde{X} \to X$. However, because of the above theorem, we use the following simpler definition.

Definition 1.1.12. A *cyclic poset* is defined to be a pair (X, c) where X is a set and $c : X^3 \to \mathbb{N}$ is a reduced cocycle on X.

Example 1.1.13. The *product* of cyclic posets $X_1 \times X_2$ has reduced cocycle $c(xyz) := c_1(x_1y_1z_1) + c_2(x_2y_2z_2)$ where c_1, c_2 are the cocycles of X_1, X_2 and $x = (x_1, x_2)$, etc.

Example 1.1.14. Suppose that X is a cyclic poset with covering poset $\tilde{X}$ and P is another poset. Then another cyclic poset $X * P$ with underlying set $X \times P$ can be constructed as follows. The covering poset $\widetilde{X * P}$ of $X * P$ is the Cartesian product $\tilde{X} \times P$ with lexicographic order. So, $(\tilde{x}, p) \leq (\tilde{y}, q)$ if either $\tilde{x} < \tilde{y}$ or $\tilde{x} \approx \tilde{y}$ and $p \leq q$. The σ action is given only on the first coordinate: $\sigma(\tilde{x}, p) = (\sigma\tilde{x}, p)$. An important example is $P = \mathbb{Z}$. If X is cyclically ordered, then so is $X * \mathbb{Z}$.

1.1.3. $\mathbb{N}$-*categories.* Following, van Roosmalen [22], p.10 and Drinfeld [7], p.5, (and [2], [10]), we note that a cyclic poset structure on a set X is equivalent to a special case of an $\mathbb{N}$-category structure on (the object set) X.

Definition 1.1.15. An $\mathbb{N}$-*category* is a category $\mathcal{Z}$ with the property that the additive monoid $\mathbb{N}$ acts freely on every Hom set

$$\mathbb{N} \times \mathcal{Z}(x, y) \to \mathcal{Z}(x, y)$$

so that composition satisfies:

$$nf \circ mg = (n + m)fg : x \to z$$

Acting freely means Hom sets are disjoint unions of copies of $\mathbb{N}$: $\mathcal{Z}(x, y) = \coprod \mathbb{N}f_i$.

Proposition 1.1.16. *A cyclic poset structure on a set X is the same as an $\mathbb{N}$-category $\mathcal{Z}$ with object set X so that every Hom set $\mathcal{Z}(x, y)$ is freely generated by one morphism f_{xy}.*

Proof. Given three objects, $x, y, z \in X$, we have

$$f_{yz}f_{xy} = nf_{xz} \tag{1.1}$$

for some $n = n(x, y, z) \in \mathbb{N}$. It is not hard to see that (1.1) is a valid composition rule for a category if and only if $n(x, y, z)$ is a reduced cocycle. (See the proof of Lemma 1.2.2 below.) □

We need the following construction and notation to go directly from a covering poset $\tilde{X} \to X$ to an $\mathbb{N}$-category structure on X. For any $\tilde{x}, \tilde{y}$ in $\tilde{X}$, let $[\tilde{x}, \tilde{y}]$ be the orbit of the pair $(\tilde{x}, \tilde{y})$ under the action of σ given by $\sigma(\tilde{x}, \tilde{y}) = (\sigma\tilde{x}, \sigma\tilde{y})$.

Definition 1.1.17. If $\tilde{X}$ is a covering poset of X, let $\mathbb{N}\tilde{X}$ be the $\mathbb{N}$-category with object set X and morphism sets

$$\mathbb{N}\tilde{X}(x, y) = \{[\tilde{x}, \tilde{y}] \,:\, \tilde{x} \leq \tilde{y} \text{ where } \tilde{x}, \tilde{y} \in \tilde{X} \text{ cover } x, y\}.$$

The action of $\mathbb{N}$ is given by $n \cdot [\tilde{x}, \tilde{y}] = [\tilde{x}, \sigma^n\tilde{y}]$ and composition is given by $[\tilde{y}, \sigma^n\tilde{z}] \circ [\tilde{x}, \sigma^m\tilde{y}] = [\tilde{x}, \sigma^{n+m}\tilde{z}]$.

Proposition 1.1.18. *The $\mathbb{N}$-category $\mathbb{N}\tilde{X}$ is isomorphic to the $\mathbb{N}$-category associated to the cyclic poset X corresponding to $\tilde{X}$.*

Proof. The basic morphism f_{xy} is equal to $[\tilde{x}, \sigma^n\tilde{y}]$ where $n \in \mathbb{Z}$ is minimal so that $\tilde{x} \leq \sigma^n\tilde{y}$. □

1.2. **Linearization of cyclic posets.** Given any category $\mathcal{C}$ and any commutative ring R, the *R-linearization* of $\mathcal{C}$ is a category $R\mathcal{C}$ with the same objects as $\mathcal{C}$ but with morphism sets $R\mathcal{C}(x,y)$ defined to be the free R-module generated by the morphism set $\mathcal{C}(x,y)$ for all objects x,y in $\mathcal{C}$ and composition given by "matrix multiplication". In this subsection we give a modified version of this construction which we call the $t^{\mathbb{N}}$-category of a cyclic poset. We take R to be a discrete valuation ring with uniformizer t. We show (Proposition 1.2.6) that a $t^{\mathbb{N}}$-category over $R=\mathbf{k}[[t]]$ is the same as the completed $\mathbf{k}$-linearization of the covering poset $\tilde{X}$ considered as a category (similar to a construction in [22]). We start with the abstract definition of a $t^{\mathbb{N}}$-category, then show that every $t^{\mathbb{N}}$-category comes from a cyclic poset.

1.2.1. *$t^{\mathbb{N}}$-categories.*

Definition 1.2.1. A $t^{\mathbb{N}}$-*category* over R is defined to be a small category $\mathcal{R}$ with the following two properties:

(1) $\mathcal{R}(x,y)$ is a free R-module with one generator f_{xy} for all x,y in the object set X of $\mathcal{R}$. Thus $\mathcal{R}(x,y)=\{rf_{xy}\,|\,r\in R\}$.
(2) For any objects $x,y,z\in X$, there is a nonnegative integer n so that for all $r,s\in R$,

$$rf_{yz}\circ sf_{xy}=rst^nf_{xz}.$$

Note that this is an *R-category*: Hom sets are R-modules and composition is R-bilinear. Note that the integer n in the above equation is uniquely determined since $t^n=t^m$ in R if and only if $n=m$. So, n is a well defined element of $\mathbb{N}$ which depends only on x,y,z.

Lemma 1.2.2. *Given a $t^{\mathbb{N}}$-category $\mathcal{R}$ over R, there is a unique reduced cocycle $c:X^3\to\mathbb{N}$ on its set of objects X so that $f_{yz}f_{xy}=t^{c(xyz)}f_{xz}$ for all $x,y,z\in X$.*

Proof. We have already remarked that n gives a well-defined function $X^3\to\mathbb{N}$ which we will denote by c. It remains to show that c is a reduced cocycle.

First note that, if rf_{xx} is the identity map on $x\in X$ then, for any $y\in X$, we must have $f_{xy}=f_{xy}(rf_{xx})=rt^{c(xxy)}f_{xy}$. Therefore, we must have $r=1$ and $c(xxy)=0$. Similarly, $c(xyy)=0$. So, c is reduced. Associativity of composition gives

$$(f_{yz}f_{xy})f_{wx}=t^{c(xyz)}f_{xz}f_{wx}=t^{c(xyz)+c(wxz)}f_{wz}$$
$$=f_{yz}(f_{xy}f_{wx})=t^{c(wxy)}f_{yz}f_{wy}=t^{c(wxy)+c(wyz)}f_{wz}.$$

Therefore, $\delta c=c(xyz)+c(wxz)-c(wxy)-c(wyz)=0$. Thus c is a reduced cocycle. □

Definition 1.2.3. We call (X,c) the *underlying cyclic poset* of the $t^{\mathbb{N}}$-category $\mathcal{R}$.

Example 1.2.4. Suppose that $b:X^2\to\mathbb{N}$ is a distance function for $c:X^3\to\mathbb{N}$ in the sense that $c=\delta b$. Let $\mathcal{R}$ denote the R-category with object set X and morphism sets $\mathcal{R}(x,y)=(t^{b(xy)})$, the ideal in R generated by $t^{b(xy)}$, with composition given by multiplication. Then composition of any morphisms $x\to y\to z$ will be divisible by $t^{b(xy)+b(yz)}=t^{c(xyz)+b(xz)}$ and therefore will lie in the ideal $(t^{b(xz)})$ as required. Then $\mathcal{R}$ is a $t^{\mathbb{N}}$-category with underlying cyclic poset (X,c).

By an *isomorphism* of cyclic posets $(X,c)\cong(X',c')$ we mean a bijection $x\leftrightarrow x'$ so that $c(xyz)=c'(x'y'z')$ for all $x,y,z\in X$.

Proposition 1.2.5. *(a) Given any cyclic poset (X,c), there is a $t^{\mathbb{N}}$-category $\mathcal{R}(X)$ with underlying cyclic poset (X,c).*

(b) A $t^{\mathbb{N}}$-category $\mathcal{R}$ is isomorphic to $\mathcal{R}(X)$ if and only if its underlying cyclic poset is isomorphic to (X,c).

Proof. (a) The $t^{\mathbb{N}}$-category $\mathcal{R}(X)$ is given as follows. We take X to be the object set of $\mathcal{R}(X)$ but we denote by P_x the object in $\mathcal{R}(X)$ corresponding to $x \in X$. For any $x, y \in X$ we take $\mathcal{R}(X)(P_x, P_y) \cong R$ to be the free R module generated by the single element f_{xy}. Composition of morphisms $P_x \to P_y \to P_z$ is given by $(rf_{yz}) \circ (sf_{xy}) = rst^{c(xyz)} f_{xz}$. As in the proof of the lemma above, composition is associative since c is a cocycle and f_{xx} is the identity map on x since c is reduced.

(b) Suppose we have a $t^{\mathbb{N}}$-category $\mathcal{R}$ with underlying cyclic poset (X', c') and an isomorphism $\Phi : \mathcal{R}(X) \to \mathcal{R}$. Then, Φ gives a bijection between the object set X of $\mathcal{R}(X)$ and the object set X' of $\mathcal{R}$ which we denote $P_x \mapsto x'$. The basic morphism $f_{xy} : P_x \to P_y$ maps to a generator $r_{xy} f_{x'y'} \in \mathcal{R}(x', y')$. So, r_{xy} must be a unit in R. Since Φ sends the identity f_{xx} on P_x to the identity on x', we must have $r_{xx} = 1$. Finally, the relation $f_{yz} f_{xy} = t^{c(xyz)} f_{xz}$ gives the relation

$$r_{yz} r_{xy} f_{y'z'} f_{x'y'} = r_{xz} t^{c(xyz)} f_{x'z'}$$

which implies that

$$r_{yz} r_{xy} t^{c'(x'y'z')} = r_{xz} t^{c(xyz)}$$

Since the r's are units, we conclude that $c'(x'y'z') = c(xyz)$ for all $x, y, z \in X$.

Conversely, if the underlying cyclic poset (X', c') of $\mathcal{R}$ is isomorphic to (X, c) then we get a bijection $x' \leftrightarrow P_x$ on the set of objects and this extends to the morphism sets by mapping $f_{x'y'}$ to f_{xy}. The cyclic poset isomorphism implies that the composition laws correspond. Therefore, we have an isomorphism of categories $\mathcal{R} \cong \mathcal{R}(X)$. □

In the special case when $R = \mathbf{k}[[t]]$, the $t^{\mathbb{N}}$-category $\mathcal{R}(X)$ has another description. Consider the $\mathbf{k}$-linearization $\mathbf{k}\mathcal{Z}$ of the $\mathbb{N}$-category $\mathcal{Z} = \mathcal{Z}(X, c)$ associated to the cyclic poset (X, c). By definition, the object set of $\mathbf{k}\mathcal{Z}$ is X and a morphism $g : x \to y$ is a finite $\mathbf{k}$-linear combination of the morphisms $n \cdot f_{xy} : x \to y, n \in \mathbb{N}$ in $\mathcal{Z}(X, c)$. Thus

$$g = \sum_{n \geq 0} g_n \left(n \cdot f_{xy}\right), \ g_n \in \mathbf{k}, \tag{1.2}$$

where $g_n = 0$ for all but a finite number of indices n. The *completion* $\widehat{\mathbf{k}\mathcal{Z}}$ of this category is given by allowing infinitely many terms to be nonzero in the sum (1.2).

Proposition 1.2.6. *When $R = \mathbf{k}[[t]]$, there is a natural isomorphism of categories $\mathcal{R}(X) \cong \widehat{\mathbf{k}\mathcal{Z}}(X, c)$ given by sending $\sum a_n t^n f_{xy} : x \to y$ to $\sum a_n (n \cdot f_{xy})$.*

Proof. This follows directly from the definitions. □

We will use the notation $\widehat{\mathbf{k}X}$ for $\widehat{\mathbf{k}\mathcal{Z}}(X, c)$.

1.3. Frobenius category.

Starting with any $t^{\mathbb{N}}$-category we construct a Frobenius category. Recall that R is a DVR with uniformizer t. So far we have constructed a $t^{\mathbb{N}}$-category $\mathcal{R}(X)$ for any cyclic poset X. Let $\mathcal{P}(X) = add\,\mathcal{R}(X)$ denote the additive R-category generated by $\mathcal{R}(X)$.

Definition 1.3.1. Let $\mathcal{MF}(X)$ denote the category of all pairs (P, d) where $P \in \mathcal{P}(X)$ and $d : P \to P$ is a morphism so that $d^2 = \cdot t$ (multiplication by t). Morphisms $f : (P, d) \to (Q, d)$ in $\mathcal{MF}(X)$ are maps $f : P \to Q$ in $\mathcal{P}(X)$ so that $df = fd$.

The key property is the *adjunction lemma*:

Lemma 1.3.2. *The functor $G : \mathcal{P}(X) \to \mathcal{MF}(X)$ given by*

$$GP := \left(P \oplus P, \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix} \right) : \qquad P \underset{id}{\overset{\cdot t}{\leftrightarrows}} P$$

is both left and right adjoint to the forgetful functor $F : \mathcal{MF}(X) \to \mathcal{P}(X)$.

For the proof, see the twisted version (Lemma 1.4.4) below.

Theorem 1.3.3. *For any cyclic poset X, $\mathcal{MF}(X)$ is a Frobenius category where a sequence*

$$(A, d) \rightarrowtail (B, d) \twoheadrightarrow (C, d)$$

is defined to be exact in $\mathcal{MF}(X)$ if $A \rightarrowtail B \twoheadrightarrow C$ is (split) exact in $\mathcal{P}(X)$. Objects isomorphic to GP for some P in $\mathcal{P}$ are the projective-injective objects of $\mathcal{MF}(X)$.

The twisted version of this theorem (Theorem 1.4.7) is proved in the next section. The fact that GP is projective and injective follows from the adjunction lemma.

Example 1.3.4. The main example of this construction is the *continuous Frobenius category* $\mathcal{MF}(S^1)$ which comes from the cyclically ordered set S^1. The stable category is the *continuous cluster category* $\underline{\mathcal{MF}}(S^1) = \mathcal{C}_\pi$. These are topological categories. The topology is used to define the cluster structure on $\mathcal{C}_\pi$, namely, two objects X, Y are *compatible* if the ordered pair (X, Y) lies in the closure of the set of all (X', Y') satisfying $\mathrm{Ext}^1(X', Y') = 0 = \mathrm{Ext}^1(Y', X')$ and a *cluster* is defined to be a maximal set of pairwise compatible indecomposable objects which is also a discrete set (every element lies in an open set containing no other objects of the cluster). See [14], [15] for details.

To obtain other kinds of examples, we need to take a twisted version $\mathcal{MF}_\phi(X)$ of the Frobenius category.

1.4. **Twisted version.** Given an admissible automorphism ϕ (Definition 1.4.1) of a cyclic poset (X, c) we define a twisted construction (Definition 1.4.2) and show that the result is a Frobenius category (Theorem 1.4.7).

Definition 1.4.1. An automorphism ϕ of (X, c) will be called *admissible* if it is covered by a σ-equivariant order preserving bijection $\tilde{\phi}$ of the covering poset $\tilde{X}$ to itself satisfying the property:

$$x \le \tilde{\phi} x \le \tilde{\phi}^2 x < \sigma x$$

for all $x \in \tilde{X}$. There is a corresponding additive R-linear automorphism of $\mathcal{P}(X)$, which we also call ϕ, defined on indecomposable objects by $\phi P_x = P_{\phi x}$ and on basic morphisms by $\phi f_{xy} = f_{\phi x, \phi y}$.

In $\mathcal{P}(X)$ the condition above gives morphisms

$$P_x \xrightarrow{\eta_x} \phi P_x = P_{\phi x} \xrightarrow{\xi_x} P_x$$

giving natural transformations

$$P \xrightarrow{\eta_P} \phi P \xrightarrow{\xi_P} P$$

whose composition $\xi_P \circ \eta_P : P \to P$ is multiplication by t. Since t is not a zero divisor, neither are ξ_P nor η_P. Therefore, the reverse composition $\eta_P \circ \xi_P : \phi P \to \phi P$ is also multiplication by t since $\xi \eta \xi = t \xi = \xi t$ making $\xi(\eta \xi - t) = 0$.

Definition 1.4.2. Let $\mathcal{MF}_\phi(X)$ be the full subcategory of $\mathcal{MF}(X)$ of all (P,d) where $d: P \to P$ factors through $\eta_P : P \to \phi P$.

An example of an object of $\mathcal{MF}_\phi(X)$ is given by

$$G_\phi P := \left(P \oplus \phi P, \begin{bmatrix} 0 & \xi_P \\ \eta_P & 0 \end{bmatrix} \right) : \qquad P \underset{\eta_P}{\overset{\xi_P}{\rightleftarrows}} \phi P$$

We will show that $\mathcal{MF}_\phi(X)$ is a Frobenius category with projective-injective objects given by $G_\phi P$. We use the key observation that η_P, ξ_P are not zero divisors. This implies that, for any object (P,d) in $\mathcal{MF}_\phi(X)$, the factorization

$$P \xrightarrow{\eta_P} \phi P \xrightarrow{\theta_P} P, \qquad d = \theta_P \eta_P$$

is unique. Similarly, $d\theta_P = \xi_P : \phi P \to P$ since $d\theta_P \eta_P = d^2 = \cdot t = \xi_P \eta_P$.

Furthermore, for all morphisms $f : (V,d) \to (W,d)$, the following commutes.

$$\begin{array}{ccccccc} \phi V & \xrightarrow{\xi_V} & V & \xrightarrow{\eta_V} & \phi V & \xrightarrow{\theta_V} & V \\ \downarrow{\scriptstyle \phi f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \phi f} & & \downarrow{\scriptstyle f} \\ \phi W & \xrightarrow{\xi_W} & W & \xrightarrow{\eta_W} & \phi W & \xrightarrow{\theta_W} & W \end{array} \tag{1.3}$$

This is because η_V is not a zero divisor: $(f\theta_V - \theta_W \phi f)\eta_V = fd - \theta_W \eta_W f = fd - df = 0$ implies $f\theta_V = \theta_W \phi f$. Note that the left two squares in the diagram commute for any morphism $f : V \to W$ in $\mathcal{P}(X) = add\,\mathcal{R}(X)$ since ξ, η are natural transformations.

Lemma 1.4.3. *$\mathcal{MF}_\phi(X)$ is an exact category where a sequence*

$$(A,d) \rightarrowtail (B,d) \twoheadrightarrow (C,d)$$

in $\mathcal{MF}_\phi(X)$ is defined to be exact if the underlying sequence $A \to B \to C$ is split exact in $\mathcal{P}(X)$.

The proof is straighrforward and identical to the case of $X = S^1$ detailed in [14].

Lemma 1.4.4. *The functor $G_\phi : \mathcal{P}(X) \to \mathcal{MF}_\phi(X)$ is left adjoint to the forgetful functor $\mathcal{MF}_\phi(X) \to \mathcal{P}(X)$ and the functor $G_\phi \circ \phi^{-1}$ is right adjoint to the forgetful functor. In other words, we have natural isomorphisms:*

$$\mathcal{MF}_\phi(G_\phi V, (W,d)) \cong \mathcal{P}(V,W) \cong \mathcal{MF}_\phi((V,d), G_\phi \phi^{-1} W)$$

Proof. A morphism $f : V \to W$ corresponds to the morphisms $(f, \theta_W \phi f) : G_\phi V \to (W,d)$ and $(\phi^{-1}(f\theta_V), f) : (V,d) \to G_\phi \phi^{-1} W$. This is verified by a calculation in which we should remember that $\theta_W \phi f \neq f\theta_V$. However, since d commutes with itself, we have $\theta_V \phi d = d\theta_V$.

For example, to prove right adjunction, we need to verify that $\phi^{-1}(f\theta_V)d = (\xi_{\phi^{-1}W})f$ and $\eta_{\phi^{-1}W}\phi^{-1}(f\theta_V) = fd$. This follows from:

$$f\theta_V \phi d = fd\theta_V = f\xi_V = \xi_W \phi f$$

Applying ϕ^{-1} to both sides gives $\phi^{-1}(f\theta_V)d = \phi^{-1}(\xi_W)f = \xi_{\phi^{-1}W} f$ since ξ is natural. Also

$$(\eta_W f \theta_V)\eta_V = \eta_W f d = \phi f \eta_V d = (\phi f \phi d)\eta_V$$

Cancel η_V from both sides and apply ϕ^{-1} to get $\eta_{\phi^{-1}W}\phi^{-1}(f\theta_V) = fd$.

Verification of left adjunction is similar but easier. □

Proposition 1.4.5. *If $P = P_x$ is indecomposable in $\mathcal{P}(X)$ then the endomorphism ring of $G_\phi P_x$ is isomorphic to the local ring $R[u]$ where $u^2 = t$.*

Proof. Morphisms $G_\phi V \to (W, d)$ are given by $(f, \theta_W \phi f)$ where $f : V \to W$ is any morphism in $\mathcal{P}(X)$. If $V = P_x$ and $(W, d) = G_\phi P_x$ then $W = P_x \oplus \phi P_x$ and $f : P_x \to P_x \oplus \phi P_x$ is a sum of two morphisms $f_0 = r_0 f_{xx} : P_x \to P_x$ and $f_1 = r_1 f_{x\phi x} : P_x \to \phi P_x$ where $r_0, r_1 \in R$. Since $\theta_W = id_x \oplus \xi'_x$ we get the following formula for a general endomorphism of $G_\phi P_x$:

$$(f, \theta_W \phi f) = \begin{bmatrix} f_0 & \xi'_x \phi f_1 \\ f_1 & \phi f_0 \end{bmatrix} = \begin{bmatrix} r_0 f_{xx} & t r_1 f_{\phi x x} \\ r_1 f_{x\phi x} & r_0 f_{\phi x \phi x} \end{bmatrix}$$

Since the second column is determined by the first, we can write this as (r_0, r_1). Then composition is given by $(r_0, r_1)(s_0, s_1) = (r_0 s_0 + t r_1 s_1, r_1 s_0 + r_0 s_1)$ which is exactly the multiplication rule for $(r_0, r_1) = r_0 + r_1 u$ in $R[u]$ where $u^2 = t$. □

Corollary 1.4.6. *Every object $G_\phi V$ can be expressed uniquely up to isomorphism as a direct sum of indecomposable objects $G_\phi P_{x_i}$ where P_{x_i} are the components of V.*

Proof. If $V = \bigoplus P_{x_i}$ then $G_\phi V = \bigoplus G_\phi P_{x_i}$ where each $G_\phi P_{x_i}$ is indecomposable. □

Theorem 1.4.7. *$\mathcal{MF}_\phi(X)$ is a Frobenius category with projective-injective objects given by $G_\phi V$ for all V in $\mathcal{P}(X)$.*

Proof. By the lemma above, $G_\phi V$ is projective for all V and $G_\phi \phi^{-1} W$ is injective for all W. To see this note that $\mathcal{MF}_\phi(G_\phi V, -) = \mathcal{P}(V, F(-))$ and $\mathcal{MF}_\phi(-, G_\phi \phi^{-1} W) = \mathcal{P}(F(-), W)$ are exact since the forgetful functor $F : \mathcal{MF}_\phi(X) \to \mathcal{P}(X)$ takes exact sequences to split exact sequences. Letting $W = \phi V$ we see that $G_\phi V$ is projective and injective for all V.

For every (V, d) there is a quotient map $G_\phi V \to (V, d)$ adjoint to the identity map on V. Similarly there is a cofibration $(V, d) \to G_\phi \phi^{-1} V$. Thus, there are enough projective-injective objects. If (V, d) is projective or injective these morphisms must split, making (V, d) a summand of either $G_\phi V$ or $G_\phi \phi^{-1} V$. By the corollary, this implies (V, d) is a direct sum of $G_\phi P_{x_i}$. Since $\mathcal{MF}_\phi(X)$ is an exact category with enough projectives all of which are also injective, it is a Frobenius category by definition. □

2. Cluster categories

In Section 2 we consider the stable categories of the Frobenius categories constructed in Section 1. Recall that the *stable category* $\underline{\mathcal{F}}$ of a Frobenius category $\mathcal{F}$ has the same set of objects as $\mathcal{F}$ with morphism sets:

$$\underline{\mathcal{F}}(A, B) = \frac{\mathcal{F}(A, B)}{\mathcal{P}(A, B)}$$

where $\mathcal{P}(A, B)$ is the set of all morphisms $A \to B$ which factor through some projective-injective object of $\mathcal{F}$. By a theorem of Happel [11], these are all triangulated categories.

Theorem 2.0.8 (Happel [11])**.** *The stable category of a Frobenius category is triangulated.*

We denote the stable categories of $\mathcal{MF}(X)$ and $\mathcal{MF}_\phi(X)$ by $\mathcal{C}(X) = \underline{\mathcal{MF}}(X)$ and $\mathcal{C}_\phi(X) = \underline{\mathcal{MF}}_\phi(X)$. These triangulated categories will have cluster structures if X and ϕ are carefully chosen. *Cluster categories* were first constructed by Buan, Marsh, Reineke, Reiten and Todorov [4] as orbit categories of the derived category of bounded complexes over a hereditary algebra. This construction is an alternate construction in type A.

2.1. **Krull-Schmidt theorem.** If X is cyclically ordered we will show that $\mathcal{MF}(X)$ is a Krull-Schmidt R-category with indecomposable objects $E(x,y)$ defined below with $x,y \in X$. This will imply that $\mathcal{C}(X)$ is a Krull-Schmidt $\mathbf{k}$-category with indecomposable objects $E(x,y)$ where $y \not\approx x$. The following easy proposition implies that $\mathcal{MF}_\phi(X)$ and thus $\mathcal{C}_\phi(X)$ will also be Krull-Schmidt categories.

Proposition 2.1.1. *The subcategory $\mathcal{MF}_\phi(X)$ is closed under direct summands in $\mathcal{MF}(X)$.*

Proof. Suppose that $(P_1,d_1)\oplus(P_2,d_2)$ lies in $\mathcal{MF}_\phi(X)$. Then $d_1\oplus d_2 : P_1\oplus P_2 \to P_1\oplus P_2$ factors through $\eta_1\oplus\eta_2 : P_1\oplus P_2 \to \phi P_1\oplus\phi P_2$. Then d_1 factors through η_1 and d_2 factors through η_2 making (P_1,d_1) and (P_2,d_2) objects of $\mathcal{MF}_\phi(X)$. □

To construct the objects $E(x,y)$, we first recall that $c(xyx)=c(yxy)=0$ if and only if $x\approx y$ if and only if $P_x\cong P_y$.

Definition 2.1.2. We define a sequence of three not necessarilty distinct elements (x,y,z) in X to be in *cyclic order* if there exist liftings $\tilde{x},\tilde{y},\tilde{z}\in\tilde{X}$ (possibly $\tilde{x}\neq\tilde{z}$ when $x=z$) so that $\tilde{x}\le\tilde{y}\le\tilde{z}\le\sigma\tilde{x}$. For example, $(x,\phi x,\phi^2 x)$ is in cyclic order for any admissible automorphism ϕ of X.

It is easy to see that, if $(xyz),(zwx)$ are in cyclic order and $x\not\approx z$ then (yzw) is also in cyclic order since there is a lifting $\tilde{w}\in\tilde{X}$ of w so that $\tilde{x}<\tilde{z}\le\tilde{w}\le\sigma^n\tilde{x}<\sigma\tilde{z}<\sigma^2\tilde{x}$ forcing $n=1$. We call this *composition of cyclic order*. When the entire set X is cyclically ordered, (xyz) is in cyclic order if and only if either $c(xyz)=0$ or $x\approx z$.

Definition 2.1.3. Let (X,c) be a cyclic poset and let $x,y\in X$ so that $c(xyx)=c(yxy)=1$. Then we define $E(x,y)$ to be the object in $\mathcal{MF}(X)$ given by

$$E(x,y) := \left(P_x\oplus P_y, \begin{bmatrix} 0 & f_{yx} \\ f_{xy} & 0 \end{bmatrix}\right).$$

If $x\approx y$ then we define $E(x,y)$ and $E(x,y)'$ by

$$E(x,y) := \left(P_x\oplus P_y, \begin{bmatrix} 0 & tf_{yx} \\ f_{xy} & 0 \end{bmatrix}\right), \qquad E(x,y)' := \left(P_x\oplus P_y, \begin{bmatrix} 0 & f_{yx} \\ tf_{xy} & 0 \end{bmatrix}\right)$$

For example, $E(x,\phi x) = \left(P_x\oplus P_{\phi x}, \begin{bmatrix} 0 & \xi_x \\ \eta_x & 0 \end{bmatrix}\right) = G_\phi P_x$ is the projective-injective object in $\mathcal{MF}_\phi(X)$.

We observe that $E(x,y)'\cong E(y,x)$ by the isomorphism which switches P_x,P_y. This second copy of the same object is convenient for notational symmetry and is used in the proof of Corollary 2.2.2 below. (In the notation of [14],[15], $E(x,x)'=E(x,x+2\pi)$.)

Lemma 2.1.4. *$E(x,y)$ is isomorphic to $E(a,b)$ in $\mathcal{MF}(X)$ if and only if they are isomorphic in $\mathcal{P}(X)$. In particular, $E(x,y)\cong E(y,x)$.*

Proof. We only need to prove sufficiency. Suppose that $P_x\oplus P_y\cong P_a\oplus P_b$. Then either $x\approx a$ and $y\approx b$ in X or $x\approx b$ and $y\approx a$. In the first case, it is clear that $E(x,y)\cong E(a,b)$. In the second case we have $x\not\approx y$ (otherwise we are in Case 1). Then the transposition isomorphism $P_x\oplus P_y\cong P_y\oplus P_x$ commutes with the operator d and therefore gives an isomorphism $E(x,y)\cong E(y,x)$. But $E(y,x)\cong E(a,b)$ as in Case 1. So, $E(x,y)\cong E(a,b)$ in both cases. □

Lemma 2.1.5. *The endomorphism ring of $E(x,y)$ is isomorphic to the local ring $R[\sqrt{t}]$.*

Proof. This is true for $y = \phi x$ by Proposition 1.4.5 and the same proof works for any x, y. Alternatively, note that the condition $c(xyx) \leq 1$ implies that the cyclic subposet $\{x, y\}$ of X is cyclically ordered and therefore can be embedded in the circle S^1. Then the endomorphism ring of $E(x, y)$ is isomorphic to the endomorphism ring of the corresponding object of $\mathcal{MF}(S^1)$ which was shown to have endomorphism ring $R[\sqrt{t}]$ in [14]. □

Lemma 2.1.6. *The object $E(x, y)$ of $\mathcal{MF}(X)$ lies in $\mathcal{MF}_\phi(X)$ if and only if $(\phi x, y, \phi^{-1}x)$ is in cyclic order.*

Proof. Suppose that $E(x, y)$ lies in $\mathcal{MF}_\phi(X)$. Then the condition that $f_{xy} : P_x \to P_y$ factors through $\eta : P_x \to P_{\phi x}$ is equivalent to the condition that $c(x, \phi x, y) = 0$. The other condition, that $t^n f_{yx} : P_y \to P_x$ factors through $\eta : P_y \to P_{\phi y}$ is equivalent to either $x \approx y$ or $x \not\approx y$ and $c(y, \phi^{-1}x, x) = 0$. In the second case we conclude that $(\phi x, y, \phi^{-1}x)$ is in cyclic order. In the first case we must have $x \approx \phi x \approx y$ which also implies that $(\phi x, y, \phi^{-1}x)$ is in cyclic order.

Conversely, suppose that $(\phi x, y, \phi^{-1}x)$ is in cyclic order. Then, either $\phi x \approx \phi^{-1}x$, which implies $x \approx \phi x$ making $c(x, \phi x, y) = 0 = c(y, \phi^{-1}x, x)$, so that $E(x, y)$ lies in $\mathcal{MF}_\phi(X)$ or $\phi x \not\approx \phi^{-1}x$, which implies that $x \not\approx y$ and, since $(\phi^{-1}x, x, \phi x)$ is in cyclic order, it also implies that $(x, \phi x, y)$ and $(y, \phi^{-1}x, x)$ are in cyclic order by composition of cyclic order. □

Theorem 2.1.7. *Suppose that (X, c) is cyclically ordered and ϕ is an admissible automorphism of X. Then every object of $\mathcal{MF}_\phi(X)$ is isomorphic to a finite direct sum of objects of the form $E(x, y)$ where $(\phi x, y, \phi^{-1}x)$ is in cyclic order. Furthermore, each $E(x, y)$ is indecomposable and the indecomposable direct summands of any object of $\mathcal{MF}_\phi(X)$ are uniquely determined up to isomorphism. Finally, $E(x, y)$ is projective-injective in $\mathcal{MF}_\phi(X)$ if and only if either $y \approx \phi x$ or $x \approx \phi y$.*

Proof. We use the fact that any finite cyclically ordered set X_0 has a cyclic order preserving embedding $X_0 \hookrightarrow S^1$ and the fact that the theorem holds for $X = S^1$ and $\phi = id_{S^1}$ by [14]. If (P, d) is any object of $\mathcal{MF}_\phi(X)$ then $P = \bigoplus P_{x_i}$. Let $X_0 = \{x_i\}$. Since this is a finite cyclically ordered set, we conclude that (P, d), as an object of $\mathcal{MF}(X)$, decomposes into objects $E(x, y)$ with $x, y \in X_0$. By Proposition 2.1.1, the components of (P, d) also lie in $\mathcal{MF}_\phi(X)$. So, $(\phi x, y\phi^{-1}x)$ is in cyclic order by Lemma 2.1.6 above.

The uniqueness of decomposition follows from Lemma 2.1.5.

By Theorem 1.4.7, the indecomposable projective-injective objects of $\mathcal{MF}_\phi(X)$ are $G_\phi P_z = E(z, \phi z)$. By Lemma 2.1.4, the object $E(x, y)$ is isomorphic to such an object if and only if either $y \approx \phi x$ or $x \approx \phi y$. □

2.2. **Frobenius cyclic posets.** We reformulate the Krull-Schmidt property of our Frobenius category into the notion of a “Frobenius cyclic poset” (Definition 2.2.3) which is used in the next paper [16]. The idea is to reverse the two steps: instead of constructing a twisted matrix factorization category of a linearization of a $t^{\mathbb{N}}$-category corresponding to a cyclically ordered set Z, we take “discrete twisted matrix factorizations” of Z which form a cyclic poset $\mathcal{X}(Z, \phi)$ (Definition 2.2.1 below). By Theorem 2.1.7, the corresponding $t^{\mathbb{N}}$-category is the Frobenius category we already constructed (Corollary 2.2.2)

Definition 2.2.1. Suppose that Z is a cyclically ordered set with totally ordered covering poset $\tilde{Z}$ and ϕ is an admissible automorphism of Z with corresponding covering automorphism $\tilde{\phi} : \tilde{Z} \to \tilde{Z}$ so that $a \leq \tilde{\phi}a < \sigma\tilde{\phi}^{-1}a \leq \sigma a$ for all $a \in \tilde{Z}$. Then we define $\mathcal{X}(Z, \phi)$ to be the cyclic poset given as follows. The covering poset of $\mathcal{X}(Z, \phi)$ is the set

$$\tilde{\mathcal{X}} = \{(a, b) \in \tilde{Z} \times \tilde{Z} \mid \tilde{\phi}a \leq b \leq \sigma\tilde{\phi}^{-1}a\}$$

with partial ordering $(a,b)\le(a',b')$ if and only if $a\le a'$ and $b\le b'$ and with automorphism σ defined by $\sigma(a,b)=(b,\sigma a)$. Let $\mathcal{X}_0(Z,\phi)$ be the cyclic subposet of $\mathcal{X}(Z,\phi)$ given by the covering subposet

$$\tilde{\mathcal{X}}_0=\{(a,b)\,|\,b\approx\phi a \text{ or } b\approx\phi^{-1}a\}.$$

Corollary 2.2.2. *Let $R=\mathbf{k}[[t]]$. Then the additive R-category $\mathcal{P}(\mathcal{X}(Z,\phi))=add\,\mathcal{R}(\mathcal{X})$ has the structure of a Frobenius category with $\mathcal{P}(\mathcal{X}_0(Z,\phi))$ being the full subcategory of projective-injective objects.*

In other words, the pair $(\mathcal{X}(Z,\phi),\mathcal{X}_0(Z,\phi))$ satisfies the following definition.

Definition 2.2.3. A *Frobenius cyclic poset* is defined to be a pair (X,X_0) where X is a cyclic poset and X_0 is a subset of X with the induced cyclic poset structure such that, for $R=\mathbf{k}[[t]]$ with $\mathbf{k}$ any field, $\mathcal{P}(X)=add\,\mathcal{R}(X)$ has the structure of a Frobenius category with projective-injective objects forming the full subcategory $\mathcal{P}(X_0)$.

We will see in [16] that the exact structure of such a Frobenius category is uniquely determined. So the words "has the structure of a Frobenius category" in the above definition can be replaced with "has a uniquely determined structure of a Frobenius category".

In order to prove the corollary it will be useful to "double" the cyclic poset $\mathcal{X}$ since there are two objects in the Frobenius category $\mathcal{MF}_\phi(Z,R_0)$ (where $R_0=\mathbf{k}[[t^2]]$) for every object in $\mathcal{R}(\mathcal{X})$.

Definition 2.2.4. For $Z,\tilde{Z},\phi,\tilde{\phi}$ as in Definition 2.2.1, let $\mathcal{X}^{(2)}=\mathcal{X}^{(2)}(Z)$ denote the cyclic poset given by the covering poset $\tilde{\mathcal{X}}^{(2)}$ which consists of two copies of $\tilde{\mathcal{X}}$:

$$\tilde{\mathcal{X}}^{(2)}=\tilde{\mathcal{X}}\times\{+,-\}=\{(a,b)_\epsilon\,|\,a,b\in\tilde{Z},\epsilon=\pm,\tilde{\phi}a\le b\le\sigma\tilde{\phi}^{-1}a\}$$

with partial ordering disregarding the sign: $(a,b)_\epsilon\le(a',b')_{\epsilon'}$ if $a\le a'$ and $b\le b'$. In particular, $(a,b)_+\approx(a,b)_-$. The automorphism σ is given by $\sigma(a,b)_\epsilon=(b,\sigma a)_{-\epsilon}$. Let $\mathcal{X}^{(2)}$ be the set of σ-orbits in $\tilde{\mathcal{X}}^{(2)}$ and let $\mathcal{X}_0^{(2)}$ be the subset of σ-orbits of $\tilde{\mathcal{X}}_0^{(2)}=\tilde{\mathcal{X}}_0\times\{\pm\}$.

The $\mathbb{N}$-category $\mathbb{N}\tilde{\mathcal{X}}_0^{(2)}$ has *even* and *odd* morphisms. In the notation of Definition 1.1.17, the *even* morphisms have the form $[(x,y)_+,(\sigma^n a,\sigma^n b)_+]$ when $x\le\sigma^n a$ and $y\le\sigma^n b$ and the *odd* morphisms are $[(x,y)_+,(\sigma^m b,\sigma^{m+1}a)_-]$ when $x\le\sigma^m b$ and $y\le\sigma^{m+1}a$.

Clearly, $\mathcal{R}(\mathcal{X}^{(2)})$ is equivalent to $\mathcal{R}(\mathcal{X})$ and this equivalence sends $\mathcal{R}(\mathcal{X}_0^{(2)})$ to $\mathcal{R}(\mathcal{X}_0)$.

Proof of Corollary 2.2.2. Let $R_0=\mathbf{k}[[t^2]]$. This is a discrete valuation ring with uniformizer $v=t^2$ and $R=R_0[\sqrt{v}]$. Let $\mathcal{MF}_\phi(Z,R_0)$ be the Frobenius category constructed from the cyclic poset Z using (R_0,v) instead of (R,t) in Definitions 1.3.1, 1.4.2. Let $\mathcal{M}(Z)$ be the full subcategory of $\mathcal{MF}_\phi(Z,R_0)$ with objects $E(x,y)$ (where $(\phi x,y,\phi^{-1}x)$ are in cyclic order) and $E(x,y)'$ (where $x\approx\phi x\approx y$) and let $\mathcal{M}_0(Z)$ be the full subcategory of $\mathcal{M}(Z)$ of the projective-injective objects $E(x,y)$ (where either $y\approx\phi x$ or $x\approx\phi y$) and $E(x,y)'$ (where $x\approx\phi x\approx y$).

We claim that $\mathcal{M}(Z)$ is isomorphic to $\mathcal{R}(\mathcal{X}^{(2)})$ as R-categories and that this isomorphism sends $\mathcal{M}_0(Z)$ to $\mathcal{R}(\mathcal{X}_0^{(2)})$. This will imply that the additive categories that they generate are also equivalent and this equivalence will give the structure of a Frobenius category to $add\,\mathcal{R}(\mathcal{X}^{(2)})$ with $add\,\mathcal{R}(\mathcal{X}_0^{(2)})$ being the full subcategory of projective-injective objects, thereby proving the corollary.

A bijection between the objects of $\mathcal{M}(Z)$ and the objects of $\mathcal{R}(\mathcal{X}^{(2)})$ which are the elements of $\mathcal{X}^{(2)}$ is given as follows. Let $\psi : \tilde{\mathcal{X}}^{(2)} \to Ob\mathcal{M}(Z)$ be defined by

$$\psi(x,y)_+ = \begin{cases} E(\pi(x),\pi(y))' & \text{if } y \approx \sigma\tilde{\phi}x \approx \sigma x \\ E(\pi(x),\pi(y)) & \text{otherwise} \end{cases}$$

and $\psi(x,y)_- = \psi(y,\sigma x)_+$. Then we claim that ψ induces a bijection Ψ between the set of σ-orbits in $\tilde{\mathcal{X}}^{(2)}$ and the set of objects of $\mathcal{M}(Z)$. The inverse of Ψ is given as follows. If $x \not\approx y$ then $\Psi^{-1}E(x,y)$ is the σ-orbit of $(\tilde{x},\tilde{y})_+$ where $\tilde{x},\tilde{y} \in \tilde{Z}$ are any lifting of x,y so that $\tilde{\phi}\tilde{x} \le \tilde{y} \le \tilde{\phi}^{-1}\tilde{x}$. If $x \approx y$ then this is not well defined since there are two such σ orbits given by $(\tilde{x},\tilde{y})_+$ and $(\tilde{x},\sigma\tilde{y})_+$ where $\tilde{x} \approx \tilde{y}$. In this case we let $\Psi^{-1}E(x,y) = (\tilde{x},\tilde{y})_+$ and $\Psi^{-1}E(x,y)' = (\tilde{x},\sigma\tilde{y})_+$. It is straightforward to show that these are inverse maps.

We will use the completed linearization $\widehat{\mathbf{k}\mathcal{X}^{(2)}}$ of $\mathbb{N}\tilde{\mathcal{X}}^{(2)}$ instead of $\mathcal{R}(\mathcal{X}^{(2)})$ since they are isomorphic. (Propositions 1.1.18, 1.2.6.) An isomorphism $\Psi : \widehat{\mathbf{k}\mathcal{X}^{(2)}} \to \mathcal{M}(Z)$ is given on objects above and on morphisms below.

Since morphisms on both sides are given by infinite series of monomials, we will describe the isomorphism only on monomial morphisms. There are two kinds of monomial morphisms: even and odd. An even monomial morphism from the σ orbit of $(x,y)_+$ to that of $(a,b)_+$ is given by $f = r[(x,y)_+,(\sigma^n a,\sigma^n b)_+]$ where $r \in \mathbf{k}$ and an odd morphism between the same two σ-orbits is given by $g = s[(x,y)_+,(\sigma^m b,\sigma^{m+1}a)_-]$.

The functor Ψ takes the σ-orbit of $(x,y)_+$ to $\Psi(x,y)_+ = (P_{\pi(x)} \oplus P_{\pi(y)}, d)$ where

$$d = \begin{bmatrix} 0 & [y,\sigma x] \\ [x,y] & 0 \end{bmatrix}.$$

The functor Ψ takes the even morphism f to the diagonal morphism

$$\Psi f = \begin{bmatrix} r[x,\sigma^n a] & 0 \\ 0 & r[y,\sigma^n b] \end{bmatrix} : \Psi(x,y)_+ \to \Psi(a,b)_+$$

and the odd morphism g to

$$\Psi g = \begin{bmatrix} 0 & s[y,\sigma^{m+1}a] \\ s[x,\sigma^m b] & 0 \end{bmatrix} : \Psi(x,y)_+ \to \Psi(b,\sigma a)_- = \Psi(a,b)_+.$$

The functor Ψ commutes with composition since composition is defined component-wise on both source and target. I.e., Ψ lies over the isomorphism $add\,\widehat{\mathbf{k}Z} \to add\,\mathcal{R}(Z)$. Therefore, Ψ gives an isomorphism of categories $\Psi : \widehat{\mathbf{k}\mathcal{X}^{(2)}} \cong \mathcal{M}(Z)$. □

2.3. **Continuous Frobenius categories.** The continuous Frobenius categories [14], [15] are example of the formalism of Frobenius cyclic posets.

In this family of examples, X is the cyclically ordered set $X = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ with covering poset $\tilde{X} = \mathbb{R}$ with $\sigma(x) = x+2\pi$. For any fixed $0 \le \theta < \pi$, let ϕ be the automorphism of S^1 with lifting $\tilde{\phi}(x) = x+\theta$. Then ϕ is admissible and the Frobenius category $\mathcal{MF}_\phi(S^1)$ is equal to the continuous Frobenius category $\mathcal{F}_{\pi-\theta}$ with stable category $\mathcal{C}_{\pi-\theta}$. (See [14, 15].) In subsequent papers [16, 17] we use the fact (Corollary 2.2.2) that, for $R = \mathbf{k}[[t]]$ and $R_0 = \mathbf{k}[[t^2]]$, the Frobenius category $\mathcal{MF}_\phi(S^1,R_0)$ is isomorphic to the completed linearization of the Frobenius cyclic poset $(\mathcal{X},\mathcal{X}_0)$ where

$$\tilde{\mathcal{X}} = \{(x,y)_\epsilon \in \mathbb{R}^2 \times \{\pm\} \,|\, x+\theta \le y \le x+2\pi-\theta\}$$

partially ordered by $(x,y)_\epsilon \le (x',y')_{\epsilon'}$ if $x \le x'$ and $y \le y'$ and $\sigma(x,y)_\epsilon = (y,x+2\pi)_{-\epsilon}$.

The indecomposable objects of $\mathcal{C}_{\pi-\theta}$ are $E(x,y)$ where x,y are distinct points on the circle subtending an angle more than θ. This triangulated category has a cluster structure if and only if $\theta = 2\pi/(n+3)$ for some positive integer n (Cor. 5.4.4 in [15]).

2.4. **Discrete cluster category of type** A. The basic examples of discrete cluster categories of type A are given by the cyclic posets $Z_n = \{1, 2\cdots, n\}$ where $n \geq 3$ with $\phi(i) = i+1$ modulo m which give the cluster category of type A_{n-3} and the cyclic poset $\mathbb{Z}$, with $\phi(i) = i+1$, which gives the ∞-gon of [12]. Both are examples of the following construction.

Theorem 2.4.1. *Let Z be a cyclically ordered set having at least four elements with totally ordered covering $\tilde{Z}$ and let ϕ be an admissible automorphism of Z satisfying the following.*

(1) $x < \tilde{\phi}x < \tilde{\phi}^2 x < \sigma x$ *for all* $x \in \tilde{Z}$.
(2) *There are no elements* $x, z \in \tilde{Z}$ *so that* $x < z < \tilde{\phi}x$.

Then the stable category of the Frobenius category $\mathcal{MF}_\phi(Z)$ is a 2-Calabi-Yau triangulated category with a cluster structure.

We show in Theorem 2.4.5 that $\mathcal{C}_\phi(Z)$ is a 2-CY triangulated category. After reviewing the definition of a cluster structure in 2.4.10, we complete the proof of Theorem 2.4.1 above. We also give a description of all possible cases of this construction in Lemma 2.4.12 below and construct the Auslander-Reiten quiver of the general example (Theorem 2.4.13).

This theorem is a mild generalization of the results of [18], [12]. We refer to [12] for many of the proofs. We will give the definitions and statements so that the reader can make the comparison and we illustrate one new example: when $Z = \mathbb{Z} * \mathbb{Z}$ in Figure 1.

2.4.1. 2-*Calabi-Yau property.* For convenience of notation we assume that Z is infinite and that no two distinct elements of Z are equivalent. Then we can take Z to be a totally ordered set and ϕ to be an automorphism of Z so that ϕx is the smallest element of Z which is greater than x for every $x \in Z$. We call ϕx the *successor* of x and denote it by x^+. Similarly, we call $\phi^{-1}x$ the *predecessor* of x and we denote it by x^-.

The indecomposable objects of $\mathcal{MF}_\phi(Z)$ are $E(x,y) \cong E(y,x)$ where x,y are distinct elements of Z. We denote the one with first coordinate smaller than the second by the set $X = \{x,y\}$. (Thus $X = E(x,y)$ if $x < y$.) Such a subset X will denote a projective-injective object if and only if the larger element of X is the successor of the smaller element.

Finally, we observe that the stable category of $\mathcal{MF}_\phi(Z)$ depends only on $\mathbf{k} = R/\mathfrak{m}$ and is independent of the choice of R since multiplication by t always factors through a projective-injective object. Therefore, we may assume that $R = \mathbf{k}[[t]]$ and $\mathcal{MF}_\phi(Z) = \mathcal{P}(\mathcal{X})$ and $\mathcal{C}_\phi(Z) = \mathcal{P}(\mathcal{X})/\mathcal{P}(\mathcal{X}_0)$ with $\mathcal{X} = \mathcal{X}(Z), \mathcal{X}_0 = \mathcal{X}_0(Z)$ given by Definition 2.2.1. Since $t = 0$ in the stable category, $\mathcal{C}_\phi(X,Y) = \mathbf{k}$ or 0 for all indecomposable objects X, Y.

Lemma 2.4.2. *If X, Y are 2-element subsets of Z which are not projective-injective then, in the stable category $\mathcal{C}_\phi(Z)$ of $\mathcal{MF}_\phi(Z)$, we have $\mathcal{C}_\phi(Z)(X,Y) = \mathbf{k}$ if*

$$x_0 \leq y_0 < x_1^-, \quad x_1 \leq y_1 < \sigma x_0^- \tag{2.1}$$

for some liftings $x_i, y_i \in \tilde{Z}$ of the elements of X, Y and $\mathcal{C}_\phi(Z)(X,Y) = 0$ otherwise. Furthermore, any nonzero morphism $X \to Y$ factors through a 2-element subset S of Z if and only if

$$x_0 \leq s_0 \leq y_0, \quad x_1 \leq s_1 \leq y_1$$

for some liftings $s_i \in \tilde{Z}$ of the elements of S (with the x_i, y_i satisfying (2.1) *above).*

Proof. Choose liftings $x_0, x_1 \in \tilde{Z}$ for the elements of X so that $x_0 < x_1 < \sigma x_0$. If $\pi^{-1}Y$ has no elements in the half open interval $[x_0, x_1^-)$ then any morphism $X \to Y$ in $\mathcal{MF}_\phi(Z)$ will factor through the projective-injective object $\pi\{x_0, x_1^-\}$. So, in order to have a nonzero morphism $X \to Y$ in $\mathcal{C}_\phi(Z)$, there must be a lifting $y_0 \in [x_0, x_1^-)$ of one of the elements of Y. Similarly, there must be a lifting $y_1 \in [x_1, \sigma x_0^-)$ of the other element of Y. Therefore, (2.1) is necessary to have a nonzero morphism $X \to Y$.

The statement about when a morphism $X \to Y$ factors through S is clear and, assuming (2.1), S cannot be projective-injective. Therefore $\mathcal{C}_\phi(Z)(X, Y) \neq 0$ when (2.1) holds. □

Lemma 2.4.3. *In the triangulated category $\mathcal{C}_\phi(Z)$ we have $E(x_0, x_1)[1] = E(x_1^-, x_0^-)$. Furthermore, the shift functor $[1]$ takes basic even morphisms to basic even morphisms and basic odd morphisms to negative basic odd morphisms.*

Proof. For any $x_0 \neq x_1$ in Z we choose the exact sequence in $\mathcal{MF}_\phi(Z)$:

$$E(x_0,x_1) \xrightarrow{\begin{pmatrix}-1\\ 1\end{pmatrix}} \begin{matrix} E(x_0, x_0^-) \\ \oplus \\ E(x_1^-, x_1)\end{matrix} \xrightarrow{(1\ \ 1)} E(x_1^-, x_0^-)$$

where the middle term is the injective envelope of $X = E(x_0, x_1)$ and the projective cover of $E(x_1^-, x_0^-)$. All four morphisms are even morphisms which are plus or minus a basic morphism as indicated. With this choice of injective envelopes for all indecomposable objects of $\mathcal{MF}_\phi(Z)$ we obtain $E(x_1^-, x_0^-) = E(x_0, x_1)[1]$ in the stable category $\mathcal{C}_\phi(Z)$. It is clear that this is natural with respect to even morphisms, i.e., that $[1]$ takes basic even morphisms to basic even morphisms.

If we switch the order of x_0, x_1 then the signs on the two morphisms on the left will change. Therefore $\eta[1] = -\eta$ where $\eta : E(x_0, x_1) \cong E(x_1, x_0)$ is the basic odd isomorphism. This implies that $[1]$ changes the sign of all odd morphisms. □

Lemma 2.4.4. $\operatorname{Ext}^1(X, Y) \neq 0$ *if and only if the subsets X, Y of Z are "crossing" in the sense that*

$$x_0 < y_0 < x_1 < y_1 < \sigma x_0$$

for some liftings $x_i, y_i \in \tilde{Z}$ of the elements of X, Y. In particular, $\operatorname{Ext}^1(X, X) = 0$.

Proof. This follows from the previous two lemmas. □

Theorem 2.4.5. *$\mathcal{C}_\phi(Z)$ is 2-Calabi-Yau, i.e., there is a natural isomorphism*

$$\operatorname{Ext}^1(X, Y) \cong D \operatorname{Ext}^1(Y, X)$$

Proof. The lemma above implies that $Ext^1(X, Y) \cong DExt^1(Y, X)$ for any two indecomposable objects X, Y. It remains to show that the isomorphism is natural. Equivalently, we need to define a natural nondegenerate pairing

$$\langle \cdot, \cdot \rangle : \operatorname{Ext}^1(X, Y) \otimes \operatorname{Ext}^1(Y, X) \to \mathbf{k}$$

Since $\operatorname{Ext}^1(X, Y) = (X, Y[1])$ by definition, this pairing can be defined as the composition

$$(X, Y[1]) \otimes (Y, X[1]) \xrightarrow{[1]\otimes id} (X[1], Y[2]) \otimes (Y, X[1]) \xrightarrow{\circ} (Y, Y[2]) \xrightarrow{\operatorname{Tr}} \mathbf{k}$$

where the trace map $\operatorname{Tr} : (Y, Y[2]) \to \mathbf{k}$ is given by choosing a decomposition of Y into indecomposable objects $Y = \bigoplus Y_i$ and letting $\operatorname{Tr}(f) = \sum f_{ii}$ where $f_{ij} \in \mathbf{k}$ is the scalar corresponding to the ij component of f (since $(Y_j, Y_i) = \mathbf{k}$ or 0). Thus

$$\langle f, g\rangle = \mathrm{Tr}(f[1] \circ g)$$

It is clear that this pairing is natural in X. Naturality in Y comes from the fact that [2] takes basic morphisms to basic morphisms (without changing sign) and thus $\mathrm{Tr}(g \circ h) = \mathrm{Tr}(h[2] \circ g)$. Naturality in Y also follows from the antisymmetry of the pairing:

$$\langle f, g\rangle = -\langle g, f\rangle$$

which follows from the fact that, for indecomposable X, any morphism $X \to X[2]$ must be odd. So, one of the morphisms f or g is odd and the other is even and $[1], [-1]$ will reverse the sign of the odd morphism and preserve the sign of the even morphism making $\mathrm{Tr}(f[1] \circ g) = -\mathrm{Tr}(g[1] \circ f)$. □

2.4.2. *Clusters.* Clusters in $\mathcal{C}_\phi(Z)$ are maximal compatible sets of indecomposable objects which satisfy a certain continuity property which we now explain.

We recall that a *Dedekind cut* in $\tilde{Z}$ is a nonempty proper subset S of $\tilde{Z}$ so that if $x < y$ and $y \in S$ then $x \in S$. We consider only *proper* Dedekind cuts, i.e., those which do not have suprema in $\tilde{Z}$. Note that $\sigma S \backslash S$ is a fundamental domain for σ and therefore maps bijectively to Z and induces a total ordering on the set Z so that Z does not have a maximum or minimum element. We call this a *Dedekind ordering* on Z. We say that a family of elements z_α in Z *converges to* ∞ with respect to such an ordering if for all $x \in Z$ there exists α so that $z_\alpha > x$. Convergence to $-\infty$ is defined similarly.

Note that any nonempty subset T of Z either has a supremum or converges to ∞ with respect to another Dedekind ordering of Z. (Take the Dedekind cut S consisting of all $z \in Z$ which are less than some element of T.)

For indecomposable $X \cong E(x, y)$, the points x, y are called the *endpoints* of X.

Definition 2.4.6. A *cluster* if $\mathcal{C}_\phi(Z)$ is defined to be a maximal collection $\mathcal{T}$ of nonisomorphic indecomposable objects T_i satisfying two properties:

1. (Compatibility) $\mathrm{Ext}^1(T_i, T_j) = 0$ for all i, j.
2. (Limit Condition) Suppose that T_α is a transfinite sequence of objects in $\mathcal{T}$ so that one end x_α of T_α converges to ∞ with respect to some Dedekind ordering of Z and the other end of T_α is fixed at, say x, then there is another transfinite sequence S_β of objects in $\mathcal{T}$ so that one end of S_β is fixed at the same point x and the other endpoint of S_β converges to $-\infty$ with respect to the same Dedekind ordering of Z.

We say that indecomposable X, Y in $\mathcal{C}_\phi(Z)$ are *compatible* if $\mathrm{Ext}^1(X, Y) = 0 = \mathrm{Ext}^1(Y, X)$.

Note that Limit Condition (2) is automatically satisfied if $\mathcal{T}$ is *locally finite* in the sense that, for every $x \in Z$, there are only finitely many objects in $\mathcal{T}$ with one end at x. In the case when $Z = \mathbb{Z}$, this definition is equivalent to the condition in [12] Theorem B and the proof that such sets satisfy the definition of a cluster is analogous. So, we omit the proof. However, we need to prove the symmetry of the above Limit Condition:

Lemma 2.4.7. *The Limit Condition implies its converse, i.e., the same statement holds if* ∞ *and* $-\infty$ *are reversed.*

Proof. Suppose that T_α is a collection of objects in $\mathcal{T}$ with one end fixed at x and the other end converging to $-\infty$.

Claim 1 $\mathcal{T}$ contains an object isomorphic to $E(x, y)$ for some $y > x$.

If not then, in particular, $E(x, x^{++})$ is not in $\mathcal{T}$ which is equivalent to saying that $\mathcal{T}$ has an object isomorphic to $E(x^+, y)$ where $y > x^+$. But the set of all such y must have a supremum.

Otherwise, by the Limit Condition, the y's must converge to ∞ which is impossible since $E(x^+, z)$ is not compatible with the objects T_α for any $z < x^+$. Let y_0 be this supremum. Then $E(x, y_0)$ is compatible with all objects in $\mathcal{T}$ and therefore an object in $\mathcal{T}$ (up to isomorphism). This proved Claim 1.

We want to show that the set of all $y > x$ so that $E(x, y)$ is in $\mathcal{T}$ (up to isomorphism) converges to ∞. If this is not the case then this set must have a supremum since, by the Limit Condition, if it converges to some Dedekind cut from below then it also converges to it from above. So, let y_1 be the supremum of this set. Then consider the set of all $z > y_1$ so that $\mathcal{T}$ contains an object isomorphic to $E(y_1, z)$. As in Claim 1, this set is nonempty and contains a maximal element z_1. But then $E(x, z_1)$ is compatible with all objects in $\mathcal{T}$ but is not contained in $\mathcal{T}$ since $z_1 > y_1$. This contradicts the maximality of $\mathcal{T}$ and this contradiction proved the lemma. □

Lemma 2.4.8. *$\mathcal{C}_\phi(Z)$ contains at least one cluster. Furthermore, for any object T of any cluster $\mathcal{T}$, there is, up to isomorphism, a unique object T^* so that $\mathcal{T}\backslash T \cup T^*$ is a cluster.*

As in [12], the proof depends on the following.

Lemma 2.4.9. *For any object $T = E(x, y)$ in any cluster $\mathcal{T}$ of $\mathcal{C}_\phi(Z)$, there are unique elements $a, b \in Z$ so that $\tilde{x} < \tilde{a} < \tilde{y} < \tilde{b} < \sigma\tilde{x}$ for some liftings $\tilde{x}, \tilde{y}, \tilde{a}, \tilde{b} \in \tilde{Z}$ of x, y, a, b and so that $E(x, a), E(a, y), E(y, b), E(b, x)$ are either zero or isomorphic to objects in $\mathcal{T}$.* □

This implies that we have exact sequences $E(x, y) \to E(x, b) \oplus E(a, y) \to E(a, b)$ and $E(a, b) \to E(a, x) \oplus E(y, b) \to E(x, y)$ in the Frobenius category $\mathcal{MF}_\phi(Z)$ giving the following distinguished triangles in the triangulated category $\mathcal{C}_\phi(Z)$:

$$T = E(x, y) \to E(x, b) \oplus E(a, y) \to T^* = E(a, b) \to T[1]$$

$$T[-1] \to T^* = E(a, b) \to E(a, x) \oplus E(y, b) \to T = E(x, y)$$

where, up to isomorphism, $B = E(x, b) \oplus E(a, y)$ is a right $add(\mathcal{C}_\phi(Z)\backslash T)$-approximation of T and $B' = E(a, x) \oplus E(y, b)$ is a left $add(\mathcal{C}_\phi(Z)\backslash T)$-approximation of T. As in [12] and as outlined below, this implies that $\mathcal{C}_\phi(Z)$ is a cluster category in the sense that it has a cluster structure according to the following definition from [3].

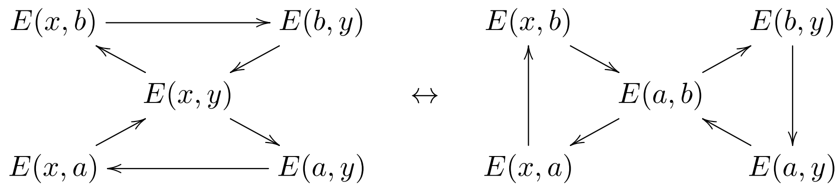
Definition 2.4.10. Suppose that $\mathcal{C}$ is a triangulated Krull-Schmidt category. Then a *cluster structure* on $\mathcal{C}$ is a collection of sets $\mathcal{T}$ called *clusters* of nonisomorphic indecomposable objects called *variables* satisfying the following conditions.

(a) For any cluster variable T in any cluster $\mathcal{T}$ there is, up to isomorphism, a unique object T^* not isomorphic to T so that $\mathcal{T}^* := \mathcal{T}\backslash T \cup T^*$ is a cluster.
(b) There are short exact sequences (or distinguished triangles)

$$T^* \to B \to T, \qquad T \to B' \to T^*$$

so that B is a minimal right add$(\mathcal{T}\backslash T)$-approximation of T and B' is a minimal left add$(\mathcal{T}\backslash T)$-approximation of T. We write $B = B_\mathcal{T}(T)$ and $B' = B'_\mathcal{T}(T)$.
(c) There are no loops or 2-cycles in the quiver of any cluster $\mathcal{T}$. (No loops means that any nonisomorphism $T \to T$ factors through $B_\mathcal{T}(T)$ and no 2-cycles means that there do not exist cluster variables T, S in $\mathcal{T}$ so that S is a summand of $B_\mathcal{T}(T)$ and T is a summand of $B_\mathcal{T}(S)$.
(d) The quiver of $\mathcal{T}^*$ is obtained from the quiver of $\mathcal{T}$ by Fomin-Zelevinski mutation.
(e) If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by replacing each variable with an isomorphic object then $\mathcal{T}'$ is a cluster.

Proof of Theorem 2.4.1. We have already shown conditions (a) and (b) and condition (e) holds by definition. Condition (c) is easy since the only arrows in the quivers of $\mathcal{T}$ starting or ending at $T = E(x, y)$ and the only arrow in the quiver of $\mathcal{T}\backslash T \cup T^*$ starting or ending at $T^* = E(a, b)$ are given in the following diagram.



Some of the terms may be zero and should be deleted. Condition (d) holds by examination of the above diagram. Therefore, $\mathcal{C}_\phi(Z)$ has a cluster structure. By Theorem 2.4.5, it is 2-Calabi-Yau. □

2.4.3. *Example.* We discuss explicit examples of discrete cluster categories.

Lemma 2.4.11. *$\mathcal{C}_\phi(Z)$ has almost split triangles given by*

$$E(x, y) \to E(x, y^+) \oplus E(x^+, y) \to E(x^+, y^+) \to E(y^-, x^-)$$

where $y \neq x^{++}$ and

$$E(x^-, x^+) \to E(x^-, x^{++}) \to E(x, x^{++}) \to E(x, x^{--})$$

Up to isomorphism, the irreducible maps are the basic morphisms $E(x, y) \to E(x, y^+)$.

Proof. The fact that the first sequence is a distinguished triangle (up to the signs of the morphisms) follows from that fact that $E(x, y) \to E(x, y^+) \oplus E(x^+, y) \to E(x^+, y^+)$ is an exact sequence in the Frobenius category. The second sequence is an example of a "positive triangle" [15]. This is a distinguished triangle where all three morphisms are basic.

It is easy to see that these are almost split triangles: Given any morphism $f : E(a, b) \to E(x^+, y^+)$ which is not an isomorphism, assuming that f is diagonal, either $a \neq x^+$ in which case f factors through $E(x, y^+)$ or $b \neq y$ in which case f factors through $E(x^+, y)$. Similarly, any nonisomorphism $E(x, y) \to E(a, b)$ factors through $E(x, y^+) \oplus E(x^+, y)$. So, the first triangle is almost split. A similar argument shows that the second triangle is almost split. The description of the irreducible morphisms is clear since $x \to x^+$ are the irreducible morphisms in $\mathcal{P}(Z)$. □

Lemma 2.4.12. *Let (Z, ϕ) be as in Theorem 2.4.1. Then either $Z \cong Z_n$ with $\phi(i) = i + 1$ or $Z \cong S * \mathbb{Z}$ where S is cyclically ordered and $\phi(s, i) = (s, i + 1)$.*

Proof. We prove only the infinite case. The automorphism ϕ gives an action of the additive group $(\mathbb{Z}, +)$ on the set Z. When Z is infinite, this must be a free action since, by (2) in Theorem 2.4.1, Z has no elements between x and $\phi(x)$. Let $S \subset Z$ be given by choosing one element from each ϕ-orbit. Then S is cyclically ordered and $x < y$ implies $\tilde{\phi}^n x < \tilde{\phi}^m y$ for all $x, y \in \tilde{S} \subset \tilde{Z}$, $n, m \in \mathbb{Z}$. So, we have isomorphism of cyclically ordered sets $S * \mathbb{Z} \cong Z$ sending (x, i) to $\phi^i x$. □

Theorem 2.4.13. *The Auslander-Reiten quiver of $\mathcal{C}_\phi(Z) \cong \mathcal{C}_\phi(S * \mathbb{Z})$ is a union of $\mathbb{Z}A_\infty$ components C_s indexed by the elements of $s \in S$ and $\mathbb{Z}A_\infty^\infty$ components C_{ab} indexed by unordered pairs of distinct elements of S.*

Proof. For every $s \in S$, the C_s component consists of all objects $E((s,i),(s,j))$ where $i,j \in \mathbb{Z}$ with $|i-j| \geq 2$. Since these contain the simple objects $E((s,i),(s,i+2))$, they are of type $\mathbb{Z}A_\infty$. For $a \neq b \in S$, the component C_{ab} contains all objects $E((a,i),(b,j))$ where $i,j \in \mathbb{Z}$. This is a component of type $\mathbb{Z}A_\infty^\infty$. □

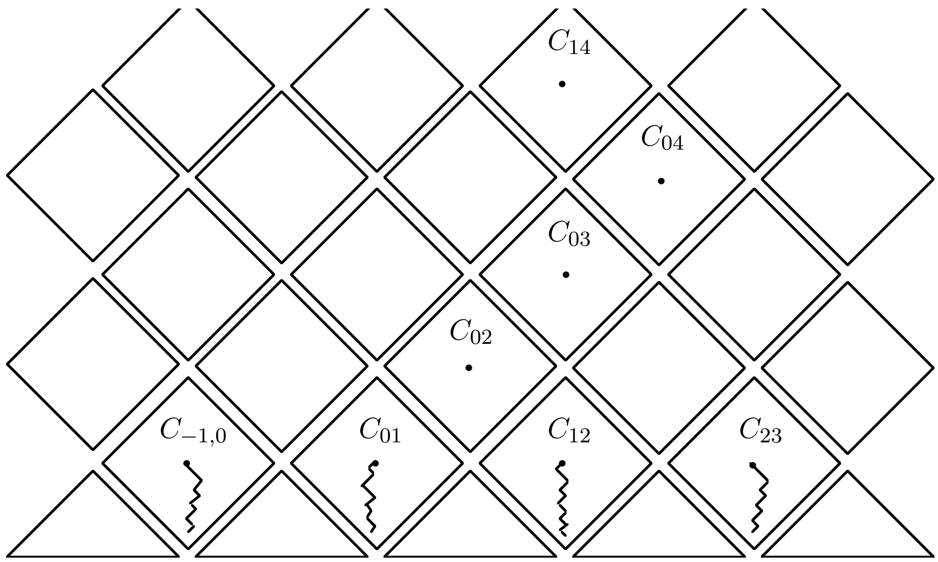


FIGURE 1. Example of a cluster in $\mathcal{C}_\phi(\mathbb{Z}*\mathbb{Z})$. In each $C_{i,i+1}$ component, choose a zig-zag starting at the center point $((0,0)$ followed by an infinite sequence of objects given by adding $(1,0)$ or $(0,-1)$ at each step and converging to $(\infty,-\infty))$. Then take the center points of C_{ij} where (i,j) form another zig-zag patter, such as $(0,2),(0,3),(0,4),(-1,4),\cdots$ (adding $(0,1)$ or $(-1,0)$ at each step and converging to $(-\infty,\infty)$).

An example of a cluster in $\mathcal{C}_\phi(Z)$ for $S=\mathbb{Z}$ ($Z=\mathbb{Z}^2$ with lexicographic order) is given by taking a "zig-zag" in each component $C_{i,i+1}$ starting at the center point $E((i,0),(i+1,0))$ plus a zig-zag pattern of center points. By a *zig-zag* in C_{ab} we mean a sequence of objects $X_0, X_1, \cdots$ so that, if $X_k = E((a,i),(b,j))$ then $X_{k+1} = E((a,i+1),(b,j))$ or $E((a,i),(b,j-1))$ and so that $i \to \infty$ and $j \to -\infty$ as $k \to \infty$. By a *zig-zag pattern of center points* we mean a sequence of objects $Y_0, Y_1, \cdots$ where each $Y_k = E((i,0),(j,0))$ and $Y_{k+1} = E((i-1,0),(j,0))$ or $E((i,0),(j+1,0))$ and so that $Y_0 = E((i,0),(i+1,0))$ for some i and so that $i \to -\infty$ and $j \to \infty$ as $k \to \infty$. See Figure 1.

2.5. **m-cluster category of type A_∞.** As another example, we will construct a triangulated category $\mathcal{C}_\Phi(Z_m * \mathbb{Z})$ whose *standard objects* form a thick subcategory $\mathcal{C}_\infty^m$ which satisfies the definition of an m-cluster category for $m \geq 3$. We will show that the *standard m-clusters* are in bijection with the set of partitions of the ∞-gon into $m+2$-gons. Such a category has already been constructed in [13]. We also show that the triangulated category $\mathcal{C}_\Phi(Z_m * \mathbb{Z})$ is $m+1$-Calabi-Yau and that its $m+1$-rigid objects have a partial cluster structure. We call maximal compatible sets of $m+1$-rigid objects *nonstandard clusters*. These correspond to 2-periodic partitions of the double ∞-gon into $m+2$-gons plus one $m+3$-gon in the middle. The sides of this middle polygon have more than m mutations. These structures can also be obtained by taking m-cluster categories of type A_n as in [1] and taking the limit as n goes to ∞.

2.5.1. *Definitions.* Let $m \geq 3$ and let $Z_\infty^m = Z_m * \mathbb{Z}$ with elements (p,k) denoted by x_k^p where $p = 1, \cdots, m, k \in \mathbb{Z}$. Let $\Phi : Z_\infty^m \to Z_\infty^m$ be the automorphism given by

$$\Phi(x_k^p) = \begin{cases} x_k^{p+1} & \text{if } p \neq m \\ x_{k+1}^1 & \text{if } p = m \end{cases}$$

Then $\Phi^m(x_k^p) = x_{k+1}^p$ which is the successor of x_k^p for any $x_k^p \in Z_\infty^m$. Since $x, \Phi(x), \Phi^2(x)$ are in cyclic order, Φ is an admissible automorphism of Z_∞^m. To simplify notation we use the convention that $x_k^{p+m} = x_{k+1}^p$ for all integers p, k and, more generally, $x_k^p = x_j^q$ if $mk + p = mj + q$. In other words, $\lambda(x_k^p) = mk + p$ gives a bijection $\lambda : Z_\infty^m \to \mathbb{Z}$. For example, the formula for Φ is $\Phi(x_k^p) = x_k^{p+1}$ for all integers k, m. Note that λ does not preserve order.

It is important to observe that $\Phi^m(x_k^p) = x_k^{p+m} = x_{k+1}^p$ is the *successor* of x_k^p.

By Theorem 2.1.7 we have:

Proposition 2.5.1. *$\mathcal{MF}_\Phi(Z_\infty^m)$ is a Frobenius category with indecomposable objects $E(x,y)$ where $x, y \in Z_\infty^m$ so that $(\Phi(x), y, \Phi^{-1}(x))$ is in cyclic order, i.e., either $\Phi^{-1}(x) < \Phi(x) \leq y$, $y \leq \Phi^{-1}(x) < \Phi(x)$ or $\Phi(x) \leq y \leq \Phi^{-1}(x)$. Also, we have:*

$$E(x_i^p, x_j^q)[1] = E(\Phi^{-1}x_j^q, \Phi^{-1}x_i^p) = E(x_j^{q-1}, x_i^{p-1})$$

The shift operator [1] takes basic even morphism to basic even morphisms and basic odd morphisms to -1 times basic odd morphisms.

Since $\Phi^{-m}(x_i^p) = x_i^{p-m} = x_{i-1}^p$ is the predecessor of x_i^p, we conclude that

$$E(x,y)[m] = E(x^-, y^-)$$

Lemma 2.4.11 still holds, except that the formula for the shift [1] has changed. The new lemma is:

Lemma 2.5.2. *The stable category $\underline{\mathcal{MF}}_\Phi(Z_\infty^m) = \mathcal{C}_\Phi(Z_\infty^m)$ has almost split triangles*

$$\tau E(x,y) \to E(x^-, y) \oplus E(x, y^-) \to E(x,y) \to \tau E(x,y)[1]$$

where $\tau E(x,y) = E(x^-, y^-) \cong E(x,y)[m]$.

Theorem 2.5.3. *The triangulated category $\mathcal{C}_\Phi(Z_\infty^m)$ is $m{+}1$-Calabi-Yau, i.e., there is a natural isomorphism*

$$\operatorname{Ext}^k(X,Y) \cong D\operatorname{Ext}^{m+1-k}(Y,X)$$

for all objects X, Y.

Proof. The lemma implies that the basic morphism $E(x,y) \to \tau E(x,y)[1] = E(x,y)[m+1]$ is a *hammock* which means that for any morphism $E(x,y) \to Y$ for any object Y, there is a morphism $Y \to E(x,y)[m+1]$. The duality is given by the fact that the nondegenerate pairing

$$(X,Y) \otimes (Y, X[m+1]) \to (X, X[m+1]) \cong \mathbf{k}$$

As in the proof of Theorem 2.4.5, naturality of the pairing follows from the fact that it is symmetric up to sign. □

2.5.2. *Auslander-Reiten quiver of $\mathcal{C}_\Phi(Z^m_\infty)$.* As in the previous section, we have the following.

Theorem 2.5.4. *The Auslander-Reiten quiver of $\mathcal{C}_\Phi(Z^m_\infty)$ is a union of $\binom{m}{2}$ components $C_{pq} = C_{qp}$ where p, q are distinct integers modulo m. Of these, the m components $C_{p,p+1}$ are of type $\mathbb{Z}A_\infty$ and the others are of type $\mathbb{Z}A^\infty_\infty$.*

Proof. The points in $Z_m * \mathbb{Z}$ form m *blocks* given by fixing the first coordinate. Objects $E(x, y)$ must have ends x, y in different blocks. We let C_{pq} be the set of all $E(x^p_i, x^q_j)$. These form a component of the Auslander-Reiten quiver since irreducible maps will change the subscripts by 1. If $q = p + 1$ then we must have $j > i$. So, $C_{p,p+1}$ is of type $\mathbb{Z}A_\infty$. The others have type $\mathbb{Z}A^\infty_\infty$ since i, j are arbitrary. □

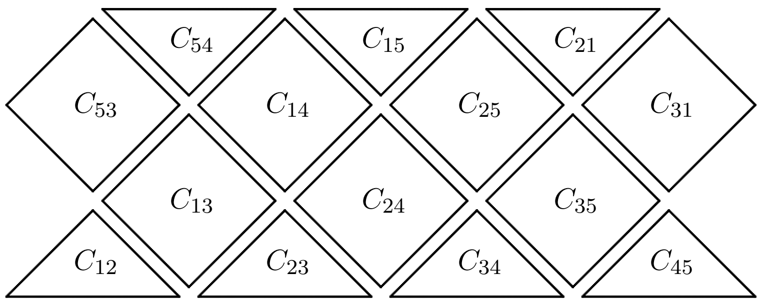


FIGURE 2. When $m = 5$, there are $\binom{m}{2} = 10$ components in the AR-quiver of $\mathcal{C}_\Phi(Z^m_\infty)$. They are labeled $C_{pq} = C_{qp}$. When p, q are consecutive, C_{pq} has type $\mathbb{Z}A_\infty$. Otherwise C_{pq} has type $\mathbb{Z}A^\infty_\infty$. Four components are repeated.

2.5.3. *Standard objects and m-clusters.* We define the *standard objects* of $\mathcal{C}_\Phi(Z^m_\infty)$ to be those objects all of whose indecomposable summands lie in the $\mathbb{Z}A_\infty$ components $C_{p,p+1}$ of the Auslander-Reiten quiver of $\mathcal{C}_\Phi(Z^m_\infty)$. In other words, the standard indecomposable objects are $E(x^p_i, x^{p+1}_j)$ where $i < j$.

The main properties of standard objects are given below. We recall that an indecomposable object X in an $m+1$-Calabi-Yau triangulated category is $m + 1$-*rigid* if $\mathrm{Ext}^k(X, X) = 0$ for $1 \le k \le m$. Two such objects are *compatible* if $\mathrm{Ext}^k(X, Y) = 0$ for $1 \le k \le m$.

Theorem 2.5.5. *The full subcategory of $\mathcal{C}_\Phi(Z^m_\infty)$ consisting of standard objects is a thick subcategory. This subcategory is a true m-*cluster category *in the following sense.*

1. *All standard indecomposable objects X are $m + 1$-rigid.*
2. *Define a* standard m-cluster *to be a maximal compatible set of indecomposable standard objects of $\mathcal{C}_\Phi(Z^m_\infty)$. Then for any such set $\mathcal{T}$ and any object $T \in \mathcal{T}$, there are exactly m other standard objects T^*_i up to isomorphism so that $\mathcal{T}\backslash T \cup T^*_i$ is a standard m-cluster.*
3. *The objects T^*_i with $T^*_0 = T$ are uniquely determined up to isomorphism by the fact that there is a distinguished triangle*
$$T^*_i \to B_i \to T^*_{i+1} \to T^*_i[1]$$
where B_i is the left add $\mathcal{T}\backslash T$-approximation of T^*_i.

Furthermore, the Verdier quotient $\mathcal{C}(Z^m_\infty)/\mathcal{C}^m_\infty$ is triangle equivalent to the standard cluster category of type A_{m-3}.

If $X = E(x, y)$ then let $\lambda_1 X < \lambda_2 X$ be the integers $\lambda(x), \lambda(y)$ in increasing order and let $\lambda X = (\lambda_1 X, \lambda_2 X)$. We say that X *crosses* Y if $\lambda_1 X < \lambda_1 Y < \lambda_2 X < \lambda_2 Y$.

Lemma 2.5.6. *Two standard objects X, Y are compatible if and only if they are noncrossing, i.e., they do not cross each other.*

Since X never crosses itself, this implies that standard objects are $m+1$-rigid. Lemma 2.5.6 also implies the following theorem.

Theorem 2.5.7. *Isomorphism classes of standard m-clusters are in 1-1 correspondence with the partitions of the ∞-gon into $m+2$-gons.*

Definition 2.5.8. By the *∞-gon* we mean the convex hull P_∞ of the set V of all points on the unit circle in $\mathbb{C}$ of the form $e^{i\theta}$ where $\cot\frac{\theta}{2} \in \mathbb{Z}$. This gives a bijection $V \cong \mathbb{Z}$. We call V the *vertex set* of P_∞. By a *partition* of P_∞ we mean a set of noncrossing chords in the circle with both endpoints in the vertex set V. (See [12].)

Proof. For any standard object $X = E(x, y)$, take the chord in the ∞-gon with endpoints corresponding to $\lambda(x), \lambda(y) \in \mathbb{Z}$. If X, Y are compatible then the corresponding chords do not cross. Therefore, any standard m-cluster gives a partition of the ∞-gon P_∞ into polygons. Take one of these polygons, say P and let $(a, b) = \lambda X$ be the longest side of P. Since X is standard, $b - a \equiv 1 \mod m$ and $b \geq a + m + 1$. If $b = a + m + 1$ then the other sides of P are sides of P_∞ and P is an $m+2$-gon. Otherwise, take the shortest chord $C = (v, w)$ inside the closed interval $[a, b]$. Then C has length $m + 1$ and bounds an $m+2$-gon. Contract C to the side $(v, v+1)$ on P_∞ and subtract m from the position of all vertices $\geq w$. Then $b - a$ decreases by m and we conclude by induction on $b - a$ that P is an $m+2$-gon. The converse argument is similar. □

To prove Lemma 2.5.6, we examine which standard objects map to each other.

Lemma 2.5.9. *Let X, Y be two standard objects in the same component of the AR-quiver. Then $\mathcal{C}_\Phi(X, Y) \neq 0$ if and only if*

$$\lambda_1 X \leq \lambda_1 Y < \lambda_2 X - 1 \leq \lambda_2 Y - 1 \tag{2.2}$$

The proof is straightforward and left to the reader. Note that these four integers are congruent to each other modulo m.

Lemma 2.5.10. *Let X, Y be standard objects. Then the following two statements are equivalent.*

1. *There exists $1 \leq k \leq m$ so that $Y[k]$ lies in the same AR-component as X and so that $\mathcal{C}_\Phi(X, Y[k]) \neq 0$.*
2. *X crosses Y, i.e., $\lambda_1 X < \lambda_1 Y < \lambda_2 X < \lambda_2 Y$.*

Proof. Without loss of generality we may assume that $\lambda_1 X$ is divisible by m. Thus, when $Y[k]$ lies in the same AR-component as X, $\lambda_1 Y[k]$ will also be divisible by m and $\lambda_1 Y[k]/m = \lfloor(\lambda_1 Y - 1)/m\rfloor$. Similarly, $\lambda_2 Y[k] - 1$ is divisible by m and $(\lambda_2 Y[k] - 1)/m = \lfloor(\lambda_2 Y - 2)/m\rfloor$. So, by (2.2), Condition (1) is equivalent to the integer inequality:

$$\frac{\lambda_1 X}{m} \leq \left\lfloor \frac{\lambda_1 Y - 1}{m} \right\rfloor < \frac{\lambda_2 X - 1}{m} \leq \left\lfloor \frac{\lambda_2 Y - 2}{m} \right\rfloor \tag{2.3}$$

For all integers a, b and real numbers x, y, the integer inequality $a \leq \lfloor x \rfloor < b \leq \lfloor y \rfloor$ is equivalent to the real inequality $a \leq x < b \leq y$. So, (2.3) is equivalent to the condition

$$\lambda_1 X \leq \lambda_1 Y - 1 < \lambda_2 X - 1 \leq \lambda_2 Y - 2$$

which is equivalent to Condition (2). □

Lemma 2.5.11. *Let X, Y be standard objects. Then the following two statements are equivalent.*

(1) *There exists $1 \leq k \leq m$ so that $\mathcal{C}_\Phi(X, Y[k]) \neq 0$ but $Y[k]$ does not lie in the same AR-component as X.*
(2) *Y crosses X, i.e., $\lambda_1 Y < \lambda_1 X < \lambda_2 Y < \lambda_2 X$.*

Proof. Since $\mathcal{C}_\Phi$ is $m+1$-CY, the condition $\mathcal{C}_\Phi(X, Y[k]) \neq 0$ is equivalent to the condition that $\mathcal{C}_\Phi(Y, X[m+1-k]) \neq 0$. In Condition (1) we must also have $\lambda_1 Y[k] = \lambda_1 Y - k \equiv \lambda_1 X - 1$ modulo m. So, $\lambda_1 X[m+1-k] \equiv \lambda_1 X - 1 + k \equiv \lambda_1 Y \bmod m$. Thus, Condition (1) is equivalent to the condition that there exists $1 \leq \ell \leq m$ so that $X[\ell]$ lies in the same AR-component as Y and $\mathcal{C}_\Phi(Y, X[\ell]) \neq 0$. By the previous lemma, this is equivalent to Condition (2). □

Proof of Lemma 2.5.6. There are two ways that X, Y might be not compatible. By Lemmas 2.5.10 and 2.5.11 these correspond exactly to the two different ways that X, Y can cross. Therefore, they are compatible iff they don't cross. □

Lemma 2.5.12. *Suppose X, Y are nonisomorphic compatible standard objects in $\mathcal{C}_\Phi(Z_\infty^m)$. Then $\mathcal{C}_\Phi(X, Y) \neq 0$ if and only if one of the following holds.*

(1) $\lambda_1 X = \lambda_1 Y$ *and* $\lambda_2 X < \lambda_2 Y$.
(2) $\lambda_1 X < \lambda_1 Y$ *and* $\lambda_2 X = \lambda_2 Y$.
(3) $\lambda_1 X = \lambda_2 Y$.

Proof. If X, Y are in the same AR-component, $\mathcal{C}_\Phi(X, Y) \neq 0$ is equivalent to (1) or (2) by Lemma 2.5.9 given that X, Y are compatible and thus noncrossing. If X, Y are not in the same AR-component then $\mathcal{C}_\Phi(X, Y) \neq 0$ is equivalent to $\mathcal{C}_\Phi(Y, X[m+1]) \neq 0$ with $Y, X[m+1]$ being in the same AR-component. By Lemma 2.5.9 this is equivalent to:

$$\lambda_1 Y \leq \lambda_1 X - m - 1 < \lambda_2 Y - 1 \leq \lambda_2 X - m - 2.$$

Since these integers are congruent to each other modulo m, this is equivalent to:

$$\lambda_1 Y < \lambda_1 X - 1 \leq \lambda_2 Y - 1 < \lambda_2 X - 2,$$

a condition equivalent to (3) given that X, Y are noncrossing. □

Proof of Theorem 2.5.5. We are now ready to prove the main theorem about standard m-clusters.

(1) We have already observed that standard objects are $m+1$-rigid since they do not cross themselves.

(2) Every element T of every standard m-cluster $\mathcal{T}$ represents a chord in P_∞ which separates two $m+2$-gons. When the chord is removed, we have a $2m+2$-gon G which has $m+1$ diagonals. So, there are m other diagonals corresponding to m other standard objects T_i^* so that $\mathcal{T} \backslash T \cup T_i^*$ forms a standard m-cluster.

(3) By Lemma 2.5.12, morphisms between compatible standard objects correspond to counterclockwise pivots of the corresponding chords in P_∞. So, the left $add\,\mathcal{T} \backslash T$-approximation of T_i^* is given by the direct sum of the two objects B_i', B_i'' (corresponding to chords) which form the sides of the $2m+2$-gon G which are adjacent to T_i^* and are clockwise from the endpoints of T_i^* as shown in the figure. The sequence

$$T_i^* \xrightarrow{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} B_i' \oplus B_i'' \xrightarrow{[1,1]} T_{i+1}^*$$

is exact in $\mathcal{MF}_\Phi(Z^m_\infty)$ and therefore forms a distinguished triangle in the stable category $\mathcal{C}_\Phi(Z^m_\infty)$. Thus the objects T_i^* form the diagonals of the $2m+2$-gon G ordered clockwise.

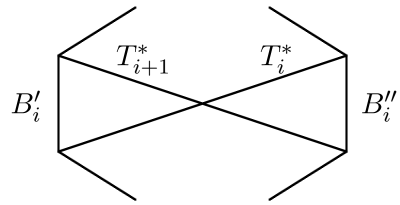


FIGURE 3. The objects B'_i and B''_i share an endpoint with T_i^* and are the objects in $\mathcal{T}\backslash T$ closest to T_i^* in the counterclockwise direction. So, their sum $B_i = B'_i \oplus B''_i$ forms the left $\mathcal{T}\backslash T$-approximation to T_i^* and $T_i^* \to B'_i \oplus B''_i \to T^*_{i+1}$ is a distinguished triangle in $\mathcal{C}_\Phi(Z^m_\infty)$.

To show that the standard objects form a thick subcategory $\mathcal{C}^m_\infty$ of $\mathcal{C}_\Phi(Z^m_\infty)$, it suffices to show that $\mathcal{C}^m_\infty$ is a triangulated full subcategory, in other words, for any morphism of standard objects $f : X \to Y$, the third object in the distinguished triangle $X \to Y \to Z \to X[1]$ is also standard. By the octahedral axiom it suffices to consider the case when X is indecomposable. So, suppose that $X \in C_{p,p+1}$. Then, dropping all of the components of the map $f : X \to Y$ which are zero, we may assume that Y lies in $C_{p,p+1} \cup C_{p-1,p}$. Since $X[1]$ lies in $C_{p-1,p}$, all of the endpoints of the object Z must lie in blocks $p-1, p, p+1$. Therefore, the only possible nonstandard components of Z lie in $C_{p-1,p}$. But this is impossible since none of these objects map nontrivially to $X[1] \in C_{p-1,p}$. So, all components of Z are standard.

Finally, to identify the Verdier quotient category $\mathcal{C}_\Phi(Z^m_\infty)/\mathcal{C}^m_\infty$, we claim that the objects of this quotient are the components of the AR-quiver of $\mathcal{C}_\Phi(Z^m_\infty)$. This follows from two considerations:

(1) Any two indecomposable objects in the same component C_{pq} are equivalent in $\mathcal{C}_\Phi(Z^m_\infty)/\mathcal{C}^m_\infty$.
(2) There is an exact morphism of Frobenius categories $\mathcal{MF}_\Phi(Z^m_\infty) \to \mathcal{MF}_\phi(Z_m)$ so that the inverse image of the indecomposable objects of $\mathcal{MF}_\phi(Z_m)$ are subcategories of $\mathcal{MF}_\Phi(Z^m_\infty)$ which map onto the AR-components of $\mathcal{C}_\Phi(Z^m_\infty)$ and the inverse image of the projective-injective objects are exactly the standard objects (and the projective-injective objects) of $\mathcal{MF}_\Phi(Z^m_\infty)$.

The first statement is easy to see. For example, there is an exact sequence

$$E(x_i^p, x_j^q) \to E(x_k^p, x_j^q) \oplus E(x_i^p, x_N^{p+1}) \to E(x_k^p, x_N^{p+1})$$

for sufficiently large N showing that $E(x_i^p, x_j^q) \sim E(x_k^p, x_j^q)$ modulo standard objects and, similarly, $E(x_i^p, x_j^q) \sim E(x_i^p, x_k^q)$.

The exact morphism in the second statement is induced by the morphism of cyclic posets $p : Z_m * \mathbb{Z} \to Z_m$ given by projection to the first coordinate. This morphism is compatible with Φ and ϕ since $\phi \circ p = p \circ \Phi$. Therefore, it induces an exact functor $\mathcal{MF}_\Phi(Z^m_\infty) \to \mathcal{MF}_\phi(Z_m)$ sending projective-injective objects to projective-injective objects and it is easy to see that it has the stated properties. □

We saw that the endpoints of a standard object lie in two consecutive *blocks*: x_*^p and x_*^{p+1}. We now consider nonstandard objects (with blocks a distance of at least 2 apart).

2.5.4. *Nonstandard objects and periodic m-clusters.*

Proposition 2.5.13. *An indecomposable object X of $\mathcal{C}_\Phi(Z_\infty^m)$ is $m+1$-rigid iff it is one of the following.*

(1) *X is standard.*
(2) *$X = E(x_i^p, x_j^{p+2})$ where $i > j$.*

Remark 2.5.14. We call objects of the second kind *nonstandard $m+1$-rigid objects*. Since $\lambda(x_i^p) > \lambda(x_j^{p+2})$, the nonstandard $m+1$ rigid objects can also be described as those objects Y where $\lambda_1 Y - \lambda_2 Y \equiv 2 \mod m$.

Proof. Suppose first that the endpoints of X lie in blocks at least 3 apart: $X = E(x_i^p, x_j^q)$ where $p+3 \le q \le p+m-3$. Then $X[m-1] = E(x_{i-1}^{p+1}, x_{j-1}^{q+1})$ and $\mathcal{C}_\Phi(X, X[m-1]) \neq 0$. So, X is not rigid. Next, suppose that $X = E(x_i^p, x_j^{p+2})$. Then the only possible self-extension of X is $\mathcal{C}_\Phi(X, X[m-1])$. Since $X[m-1] = E(x_{i-1}^{p+1}, x_{j-1}^{p+3})$ we have $\mathcal{C}_\Phi(X, X[m-1]) \neq 0$ iff $i-1 < j$. So we need $j < i$ for $X = E(x_i^p, x_j^{p+2})$ to be rigid. □

Corollary 2.5.15. *All objects of $\mathcal{C}_\Phi(Z_\infty^m)$ are $m+1$ rigid iff $m \le 4$.*

Define a *nonstandard m-cluster* to be a maximal collection of pairwise compatible $m+1$-rigid indecomposable objects of $\mathcal{C}_\Phi(Z_\infty^m)$ containing at least one nonstandard object.

Let $B = \mathbb{R} \times [0,1]$ and let $\partial B^{(2)}$ be the set of all two element subsets of the boundary $\partial B = \mathbb{R} \times \{0,1\}$. Let $\lambda_0, \lambda_1 : Z_\infty^m \to \partial B^{(2)}$ be the two embeddings given by $\lambda_0 x = (\lambda(x), 0)$, $\lambda_1 x = (-\lambda(x), 1)$.

Theorem 2.5.16. *Let Ψ be the mapping which sends any standard object $E(x,y)$ to the pair of pairs $\Psi E(x,y) = \{(\lambda_0 x, \lambda_0 y), (\lambda_1 x, \lambda_1 y)\}$ and any nonstandard $m+1$-rigid object $E(x,z)$ to $\Psi E(x,z) = \{(\lambda_0 x, \lambda_1 z), (\lambda_1 x, \lambda_0 z)\}$. Then a collection of rigid indecomposable objects of $\mathcal{C}_\Phi(Z_\infty^m)$ form a nonstandard m-cluster iff it is mapped by Ψ to a 2-periodic decomposition of the A_∞^2-gon into $m+2$-gons (except for the one in the middle).*

The proof follows the same pattern as the proof of Theorem 2.5.7. We state the corresponding sequence of lemmas without proof. The central polygon is discussed in Proposition 2.5.24.

Lemma 2.5.17. *Suppose that $X = E(x_a^p, x_b^{p+1})$ and $Y = E(x_i^{p-1}, x_j^{p+1})$ with $i > j$. Then the following are equivalent.*

(1) $\mathcal{C}_\Phi(Y, X) \neq 0$
(2) $a < j \le b$
(3) $\lambda_1 X < \lambda_1 Y - 1 \le \lambda_2 X - 1$

Lemma 2.5.18. *Suppose that $X = E(x_a^{p-2}, x_b^{p-1})$ and $Y = E(x_i^{p-1}, x_j^{p+1})$ with $i > j$. Then the following are equivalent.*

(1) $\mathcal{C}_\Phi(Y, X) \neq 0$
(2) $a < i \le b$
(3) $\lambda_1 X + 1 < \lambda_2 Y \le \lambda_2 X$

In both lemmas, the three integers in (3) are congruent to each other modulo m. The condition $i > j$ is equivalent to Y being $m+1$-rigid when $m \ge 5$.

Lemma 2.5.19. *Let $Y = E(x_i^{p-1}, x_j^{p+1})$ with $i > j$ and let X be a standard indecomposable object.*

(1) *There exists* $1 \leq k \leq m$ *so that* $X[k] \in C_{p,p+1}$ *and* $\text{Ext}^k(Y,X) \neq 0$ *if and only if*
$$\lambda_1 X < \lambda_1 Y < \lambda_2 X.$$
(2) *There exists* $1 \leq k \leq m$ *so that* $X[k] \notin C_{p,p+1}$ *and* $\text{Ext}^k(Y,X) \neq 0$ *if and only if*
$$\lambda_1 X < \lambda_2 Y < \lambda_2 X.$$

Proposition 2.5.20. *Suppose that* X, Y *are* $m+1$*-rigid indecomposable objects where* X *is standard and* Y *is nonstandard. Then* X, Y *are compatible if and only if neither* $\lambda_1 Y$ *not* $\lambda_2 Y$ *lies between* $\lambda_1 X$ *and* $\lambda_2 X$*. Equivalently,* X, Y *are compatible if and only if* $\Psi X, \Psi Y$ *do not cross.*

Lemma 2.5.21. *Suppose that* $Y = E(x_i^{p-1}, x_j^{p+1}), Z = E(x_a^{q-1}, x_b^{q+1})$ *where* $i > j$ *and* $a > b$*.*
(1) *Suppose* $q = p$*. Then* $\mathcal{C}_\Phi(Y,Z) \neq 0$ *iff* $\lambda_1 Y \leq \lambda_1 Z$ *and* $\lambda_2 Y \leq \lambda_2 Z$*.*
(2) *Suppose* $q = p+1$*. Then* $\mathcal{C}_\Phi(Y,Z) \neq 0$ *iff* $\lambda_2 Z < \lambda_1 Y - 1$*.*

Lemma 2.5.22. *Suppose* $m \geq 5$ *and* Y, Z *are nonstandard* $m+1$*-rigid indecomposable objects with* $Y \in C_{p-1,p+1}$*. Then there exists* $1 \leq k \leq m$ *so that* $\mathcal{C}_\Phi(Y, Z[k]) \neq 0$ *and* $Z[k] \in C_{p-1,p+1} \cup C_{p,p+2}$ *if and only if either* $\lambda_2 Y < \lambda_2 Z$ *and* $\lambda_1 Y < \lambda_1 Z$ *or* $\lambda_2 Z < \lambda_1 Y$*.*

The same statement holds for $m = 4$ if $C_{p-1,p+1}$ is construed to mean the set of all $E(x_i^{p-1}, x_j^{p+1})$ with $i > j$ and similarly for $C_{p,p+2}$.

Using Serre duality to get the other two cases (when $Z[k] \in C_{p-2,p} \cup C_{p-3,p-1}$), we get that nonstandard Y, Z are compatible if and only if all of the following hold:
(1) $\lambda_2 Y \geq \lambda_2 Z$ or $\lambda_1 Y \geq \lambda_1 Z$
(2) $\lambda_2 Z \geq \lambda_1 Y$
(3) $\lambda_2 Z \geq \lambda_2 Y$ or $\lambda_1 Z \geq \lambda_1 Y$
(4) $\lambda_2 Y \geq \lambda_1 Z$

By manipulating these inequalities we get the following.

Proposition 2.5.23. *Suppose* Y, Z *are nonstandard* $m+1$*-rigid indecomposable objects. Then* Y, Z *are compatible if and only if one of the intervals* $[\lambda_1 Y, \lambda_2 Y], [\lambda_1 Z, \lambda_2 Z]$ *contains the other. Equivalently,* Y, Z *are compatible if and only if* $\Psi Y, \Psi Z$ *do not cross.*

Theorem 2.5.16 follows from Propositions 2.5.20 and 2.5.23.

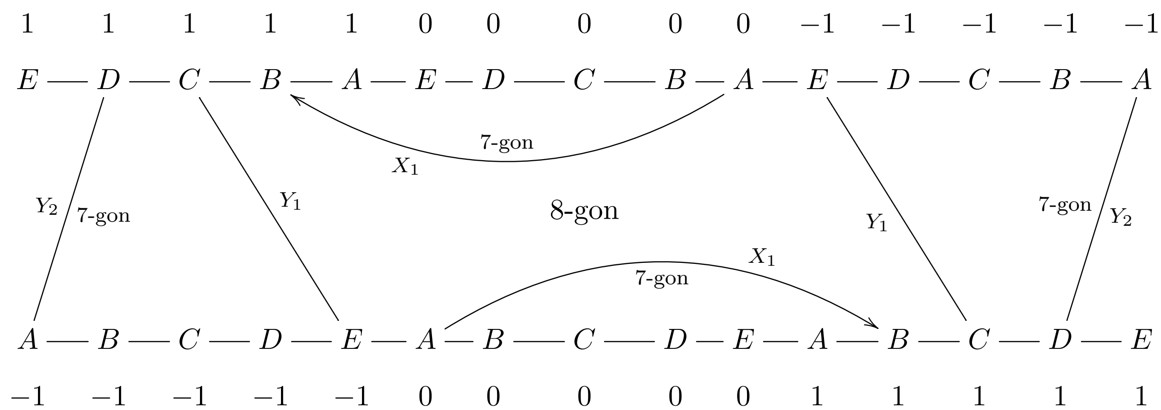
Proposition 2.5.24. *The central polygon in nonstandard* m*-cluster has* $2m-2$ *sides. Given any nonstandard* m*-cluster* $\mathcal{T}$ *and object* $T \in \mathcal{T}$*, if* T *is not a side of the central* $2m-2$*-gon, then there are, up to isomorphism, exactly* m *objects* T^* *so that* $\mathcal{T} \backslash T \cup T^*$ *forms a nonstandard* m*-cluster. When* T *is a side of the central* $2m-2$*-gon then there are, up to isomorphism, exactly* $3m-6$ *objects* T^* *so that* $\mathcal{T} \backslash T \cup T^*$ *forms a nonstandard* m*-cluster. Furthermore, in both cases the objects* T^* *are obtained as a sequence* T_i^* *given by left* $\mathcal{T} \backslash T$*-approximation triangles* $T_i^* \to B_i \to T_{i+1}^*$*.*

Proof. As in the proof of Theorem 2.5.7, we can contract all of the standard objects without changing the number of sides in the polygons having nonstandard sides. Then the central polygon has only two nonzero sides which are given by a single doubled nonstandard object $Y = E(x_i^p, x_j^{p+2})$. We must have $i = j+1$ (otherwise, $E(x_i^p, x_{i-1}^{p+2})$ would be compatible with all objects in $\mathcal{T}$). Then $\lambda_2 Y - \lambda_1 Y = m-2$ which means that the chord connecting the endpoints in P_∞ has $m-1$ sides which means that the doubled object in $P_\infty^{(2)}$ has $2(m-1)$ sides.

When a side of this central $2m-2$-gon is removed then two sides are removed by symmetry. So, we get a $6m-10$-gon. which has $3m-5$ trisections. So, there are $3m-6$ mutations of any side. Figure 3 applies to this case and shows that the objects T_i^* can be obtained by iterated approximation triangles. □

2.5.5. *Example.* For $m = 5$, here is an example of a nonstandard m-cluster. Stardard objects are pairs of horizontal arc connecting x_i^p to x_j^{p+1} (given by consecutive letters in the notation of the figure). Nonstandard objects are pairs of vertical arcs connecting connecting x_i^{p+2} on one side to x_j^p on the other with $i < j$.

There is $2m - 2 = 8$-gon in center. Other regions have $m + 2 = 7$ sides.



Standard: $X_1 = E(A_0, B_1) = E(x_0^1, x_1^2)$ (horizontal).

$Y_1 = E(C_1, E_{-1}) = E(x_1^3, x_{-1}^5)$, $Y_2 = E(D_1, A_{-1}) = E(x_1^4, x_{-1}^1)$ are nonstandard but $(m+1)$-rigid (vertical).

x_j^p is the pth letter of the alphabet with subscript j. For example, $x_j^1 = A_j, x_j^2 = B_j$.

We want to emphasize that the thick subcategory of standard objects in $\mathcal{C}_\Phi(Z_\infty^m)$ forms an m-cluster category in the usual sense. This example is illustrating the properties of the nonstandard objects in our category.

2.6. Chart of examples.

	cyclic poset	automorphism	cluster category	comments
1.	Z_n	$\phi(i) = i + 1$	$\mathcal{C}(A_{n-3})$	cluster category of $\mathbf{k}A_{n-3}$ 2-CY, finite (sec. 2.4)
2.	$\mathbb{Z}$	$\phi(i) = i + 1$	$\mathcal{C}(A_\infty)$	infinity-gon 2-CY, infinite, (sec 2.4)
$3(a)$	S^1	id	$\mathcal{C}_\pi$	continuous cluster category not 2-CY (sec. 2.3)
$3(b)$	S^1	$\phi(x) = x + \theta$	$\mathcal{C}_{\pi-\theta}$	cluster category if $\theta = \frac{2\pi}{n+3}$ (sec. 2.3)
$4(a)$ $4(b)$	$S^1 * \mathbb{Z}$ $S^1 * \mathbb{Z}$	id $\phi(x, i) = (x, i + 1)$	$\tilde{\mathcal{C}}$ $\tilde{\mathcal{C}}'$	not 2-CY ($Y[1] \cong Y$) 2-CY
5.	$Z_m * \mathbb{Z}$	$\Phi(i, j) = (i + 1, j)$ $\Phi(m, j) = (1, j + 1)$	$\mathcal{C}_\Phi(\mathbb{Z}_m * \mathbb{Z})$	contains m-cluster category of type A_∞ $(m + 1)$-CY (sec. 2.5)
6.	$\mathbb{Z} * \mathbb{Z}$	$\phi(x, y) = (x, y + 1)$	$\mathcal{C}_\phi(\mathbb{Z} * \mathbb{Z})$	2-CY cluster category by Theorem 2.4.1

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