Dear Jean-Louis,

The answer to your question lies in Quillen's paper on the algebraic K-theory of finite fields (Annals 96 (1972)).

We want to look at the generator of $K_3 \mathbb{Z}_2$ and lift it to $K_3 \mathbb{Z}$.

But $K_3 \mathbb{Z}_2 \cong H_3 \text{GL}(\mathbb{Z}_2) \cong \mathbb{Z}_3$.

Thus the nontrivial element is detected by a cohomology class in $H^3(\text{GL}(\mathbb{Z}_2); \mathbb{Z}_3) \cong \mathbb{Z}_3$.

Now look in Quillen's paper page 577.

Set $p = 2$, $e = 3$, $k = \mathbb{F}_2 = \mathbb{Z}_2$.

$r = \text{degree of } k(\mu_3)/k$

take $r = 2$ since $k(\mu_3) = \mathbb{F}_4$.

Therefore $e_2$ has degree 3 and $e_2$ is nontrivial as an element of $H^3(\text{GL}_2(\mathbb{Z}_2); \mathbb{Z}_3)$.

Now use Quillen's Lemma 12 and 13 on page 573.
We see that $e_3$ restricts to a nontrivial element of $H^3(C; \mathbb{Z}_3)$.

On page 570 the group $C$ is defined:

$$C = k(\mu_3)^* \cong \mathbb{Z}_3$$

The map $C \to GL_2(\mathbb{Z}_2)$ is given by considering $k(\mu_3)$ as a $k[C]$-module. In any case $GL_2(\mathbb{Z}_2)$ has only 6 elements so there is no choice.

The generator of $C$ goes to $(0,1)$.

But $H^3(C; \mathbb{Z}_3) \cong \text{Hom}(H_3(C); \mathbb{Z}_3)$.

So the composition

$$H_3 C \to H_3 GL_2 \mathbb{Z}_2 \to H_3 GL \mathbb{Z}_2 \cong \mathbb{Z}_3 \mathbb{Z}_2$$

is nontrivial.

Now consider $GL_3 \mathbb{Z}_2$.

This group has $(2^3-1)(2^2-2)(2^3-4) = 2^3 \cdot 7$ elements. Thus the matrix $egin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is conjugate to $egin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ or its inverse.
This gives a lifting of $\mathbb{C} \xrightarrow{\phi} \text{Gl}_2 \mathbb{Z}$ to a representation $\mathbb{C} \xrightarrow{\phi'} \text{Gl}_3 \mathbb{Z}$.

To lift $\phi'$ to $\text{SL}(2)$ we need to express $(0 1 0 \atop 0 0 1 \atop 1 0 0)$ as a product of elementary matrices: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \bar{w}_{12} \bar{w}_{31} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Unfortunately $(\bar{w}_{12} \bar{w}_{31})^3 \neq 1$ in $\text{SL}(2)$ so we must take $\bar{w}_{12}^3 \bar{w}_{31}$. This gives a representation $\mathbb{C} \xrightarrow{\phi'\prime} \text{SL}(2)$.

The element of order 3 in $K_3 \mathbb{Z}$ is the image of the picture

in $H_3 \text{SL}(2)$ under the map $\phi'\prime : t \mapsto \bar{w}_{12}^3 \bar{w}_{31}$.

The picture is given as follows:

(One can make this look like the generator of $H_3 \mathbb{C}$ by choosing a trivialization of $\nu^3$. )
\[ w = w_{21}^3 w_{31} \]

\[ x = w_{14} w_{25} w_{36} \]

\[ y = w_{16} w_{24} w_{35} \]

\[ z = w_{15} w_{26} w_{34} \]

\[ r = w_{34}^3 w_{64} \]
A similar argument is probably possible for other finite fields (Start with the first nontrivial cohomology class in $H^*(G_l(\mathbb{Z}_p); \mathbb{Z}_e)$.) I do not have time to think about this. I am busy writing up the proof of stability of pseudoisotopies and rewriting my "Higher singularities" paper.

On Dec 6 we are leaving France to visit Bielefeld, Oslo, Yugoslavia. We will be back in Waltham on Jan 11.

I am sending you:
1. Copy of a letter from a student (Ogle)
2. Outline of my proof of stability.