

# TWISTING COCHAINS AND HIGHER TORSION 091104

KIYOSHI IGUSA

*This paper is dedicated to Edgar Brown.*

ABSTRACT. This paper gives a short summary of the central role played by Ed Brown’s “twisting cochains” in higher Franz-Reidemeister (FR) torsion and higher analytic torsion. Briefly, any fiber bundle gives a twisting cochain which is unique up to fiberwise homotopy equivalence. However, when they are based, the difference between two of them is a higher algebraic K-theory class measured by higher FR torsion. Flat superconnections are also equivalent to twisting cochains.

## CONTENTS

Introduction	1
1. $A_\infty$ functors	3
2. Twisting cochain	5
3. Volodin K-theory	7
4. Higher FR torsion	10
5. Flat superconnections	12
6. Supercommutator rules	14
7. Chen’s iterated integrals	15
8. Another method	18
References	19

## INTRODUCTION

About 49 years ago Ed Brown [Bro59] constructed a small chain complex giving the homology of the total space  $E$  of a fiber bundle

$$F \rightarrow E \rightarrow B$$

whose base  $B$  and fiber  $F$  are finite cell complexes. It is given by the tensor product of chain complexes for  $F$  and  $B$  with the usual tensor product boundary map modified by a “twisting cochain.” There are many ways to understand the meaning of the twisting cochain.

- (1) It is the difference between two  $A_\infty$  functors.
- (2) It is a combinatorial flat superconnection.

---

2000 *Mathematics Subject Classification.* Primary 57R22, Secondary 19J10, 55R40.

*Key words and phrases.* higher Franz-Reidemeister torsion, higher analytic torsion, A-infinity functors, superconnections, algebraic K-theory, Volodin K-theory, Chen’s iterated integrals.

Research for this paper was supported by NSF.

- (3) It is a family of chain complexes homotopy equivalent to  $F$  and parametrized by  $B$ .

If  $F \rightarrow E \rightarrow B$  is a smooth bundle with compact manifold fiber and simply connected base then we get another twisting cochain given by fiberwise Morse theory. Comparison of these two twisting cochains gives an algebraic K-theory invariant of the bundle called the *higher Franz-Reidemeister (FR) torsion*. The purpose of this paper is to explain some of the basic properties of these constructions and unify them using a simplified version of Ed Brown's construction. A longer exposition can be found in [Igu05] which, in turn, gives a summary of the contents of [Igu02].

We summarize the contents of this paper. In the first section we review the definition of an  $A_\infty$  functor.  $A_\infty$  structures were first constructed by Stasheff [Sta63] and  $A_\infty$  categories first appeared in [Fuk93]. But here we take  $A_\infty$  functors only from ordinary categories to the category of projective complexes over a ring  $R$ . We use an old construction of Eilenberg and MacLane [EM53] to make homology into an  $A_\infty$  functor when it is projective.

In the second section we define twisting cochains. These are special cases of  $A_\infty$  functors which appear when there is an underlying functor whose induced maps are all chain isomorphisms. Then this functor gives a coefficient sheaf  $F(X, Y)$  over the category  $\mathcal{X}$  and the higher homotopies in the  $A_\infty$  functor are cochains on  $\mathcal{X}$  with values in  $F$ . We give Brown's definition of the *twisted tensor product*

$$C_*(B) \otimes_\psi C_*(F)$$

whose homology is equal to the homology of the total space of a fiber bundle  $F \rightarrow E \rightarrow B$ . Then we give our version of the same thing. This is a special case of a general theorem of Kadeishvili [Kad80].

Section 3 describes Volodin K-theory [Vol71] and its relationship to twisting cochains. Basically, there is a canonical twisting cochain on the Volodin category and if a twisting cochain on a category  $\mathcal{X}$  is a family of finitely generated (f.g.) contractible based free  $R$ -complexes, it gives a functor from  $\mathcal{X}$  into a generalization of the Volodin category which we call the *Whitehead category*. This is a variation of the old theorem of Vasserstein and Wagoner equating Volodin K-theory with Quillen K-theory. (See [Sus81].)

Higher Franz-Reidemeister torsion is in section 4. We show that, under the right conditions, a smooth fiber bundle with compact manifold fiber  $M \rightarrow E \rightarrow B$  gives two canonical twisting cochains, the topologically defined twisting cochain of Ed Brown and a new twisting cochain obtained by fiberwise Morse theory. The fiberwise mapping cone of the comparison map is fiberwise contractible. It gives a mapping into the Whitehead category provided that we have a basis for the topological twisting cochain. (The Morse theoretic twisting cochain has a basis coming from the critical points.) Such a basis can be chosen in the special case when  $\pi_1 B$  acts trivially on the rational homology of  $M$ . The higher FR torsion has been computed in many cases but here we give only one example: the case when the fiber is a closed oriented even dimensional manifold (Theorem 4.6).

The rest of the paper contains an elementary discussion of flat superconnections. The aim is to show that they are equivalent to twisting cochains. Section 5 derives a definition of an *infinitesimal twisting cochain*. This is basically a twisting cochain on very small simplices expressed in terms of differential forms. Next, in section 6,

we incorporate the supercommutator rules and we show in the last section 8 that a flat superconnection is the differential in a cohomology complex which is dual to Brown's twisted tensor product. Going backwards, the second to last section 7 explains the first step of how superconnections can be integrated using Chen's iterated integrals to give a simplicial twisting cochain. Complete details for integration of superconnections can be found in [Igu09].

This completes this summary of this paper which is a summary of the longer paper [Igu05] which summarizes the book [Igu02].

I would like to thank Bernard Keller whose lectures and comments lead me the work of my colleague Ed Brown. I thank Ed Brown for many useful conversations about the topics in this paper. I also thank Jim Stasheff for his support and encouragement and for helpful comments on the first version of this paper and I also want to thank Jonathan Block for explaining his sign conventions to me. This was very helpful for getting the correct signs in [Igu09] and I have changed the signs in this paper accordingly.

## 1. $A_\infty$ FUNCTORS

In this paper we consider chain complexes of projective right  $R$ -modules over a ring  $R$ . This gives a differential graded category  $\mathcal{C}(R)$  with graded hom sets

$$\mathrm{HOM}(C_*, D_*) = \bigoplus \mathrm{HOM}_n(C_*, D_*) = \bigoplus_n \prod_k \mathrm{Hom}_R(C_k, D_{n+k})$$

with differential  $m_1$  given by

$$m_1(f) = df - (-1)^n fd$$

for all  $f \in \mathrm{HOM}_n(C_*, D_*)$ . We will be considering functors from an ordinary category  $\mathcal{X}$  into this differential graded category.

To fix a problem which arises in the notation we will use the opposite of the usual nerve (or more precisely, the nerve of the opposite category). Thus a  $p$ -simplex in  $\mathcal{X}$  will be a sequence of morphisms of the form:

$$(1) \quad X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} \cdots \xleftarrow{f_p} X_p$$

The main purpose of this is to make the domain  $X_j$  of the front  $j$ -face of a  $j+k$  simplex equal to the range of the back  $k$ -face.

**Definition 1.1.** An  $A_\infty$  functor  $\Phi : \mathcal{X} \rightarrow \mathcal{C}(R)$  on an ordinary category  $\mathcal{X}$  is an operation which assigns to each object  $X \in \mathcal{X}$  a projective  $R$ -complex  $\Phi X$  and to each sequence of composable morphisms (1) a morphism

$$\Phi_p(f_1, f_2, \dots, f_p) : \Phi X_p \rightarrow \Phi X_0$$

of degree  $p-1$  which is equal to the boundary map  $\Phi_0 = d$  of  $\Phi X_0$  when  $p=0$  and satisfies the following cocycle condition for  $p \geq 1$ .

$$(2) \quad \sum_{i=0}^p (-1)^i \Phi_i(f_1, \dots, f_i) \Phi_{p-i}(f_{i+1}, \dots, f_p) \\ = \sum_{i=1}^{p-1} (-1)^i \Phi_{p-1}(f_1, \dots, f_i f_{i+1}, \dots, f_p)$$

This can be written as follows where  $m_2(f, g) = f \circ g$ .

$$m_1(\Phi_p) + \sum_{i=1}^{p-1} (-1)^i m_2(\Phi_i, \Phi_{p-i}) = \sum_{i=1}^{p-1} (-1)^i \Phi_{p-1}(1_{i-1}, m_2, 1_{p-i-1}).$$

For  $p = 0, 1, 2, 3$  this equation has the following interpretation.

$$(p = 0) \quad \Phi_0 \Phi_0 = 0$$

$$(p = 1) \quad d\Phi_1(f) = \Phi_1(f)d, \text{ i.e.,}$$

$$\Phi_1(f) : \Phi X_0 \rightarrow \Phi X_1$$

is a chain map.

$$(p = 2) \quad d\Phi_2(f_1, f_2) + \Phi_2(f_1, f_2)d = \Phi_1(f_1)\Phi_1(f_2) - \Phi_1(f_1 f_2), \text{ i.e.,}$$

$$\Phi_2(f_1, f_2) : \Phi_1(f_1 f_2) \simeq \Phi_1(f_1)\Phi_1(f_2)$$

is a chain homotopy. (In particular, if  $\Phi_2 = 0$  then  $\Phi_1$  is a functor assuming that  $\Phi_1(id) = id$ , which follows if it takes isomorphisms to isomorphisms.)

$$(p = 3) \quad d\Phi_3(f_1, f_2, f_3) - \Phi_3(f_1, f_2, f_3)d =$$

$$\Phi_2(f_1, f_2 f_3) - \Phi_2(f_1, f_2)\Phi_1(f_3) + \Phi_1(f_1)\Phi_2(f_2, f_3) - \Phi_2(f_1 f_2, f_3).$$

In other words,  $\Phi_3$  is a null homotopy of the coboundary of  $\Phi_2$ .

A *natural transformation* of  $A_\infty$  functors is an  $A_\infty$  functor on the product category  $\mathcal{X} \times I$  where  $I$  is the category with two objects  $0, 1$  and one nonidentity morphism  $0 \rightarrow 1$ . This is a family of chain maps  $\Phi X \rightarrow \Phi' X$  which are natural only up to a system of higher homotopies. If these chain maps are homotopy equivalences we say that  $\Phi, \Phi'$  are  $A_\infty$  *homotopy equivalent* or *fiber homotopy equivalent*. (We view an  $A_\infty$  functor as a family of chain complexes over the nerve of  $\mathcal{X}^{op}$ .)

One way to construct an  $A_\infty$  functor on  $\mathcal{X}$  is to start with an actual functor  $C$  from  $\mathcal{X}$  to the category of projective  $R$ -complexes and degree zero chain maps, then replace each  $C(X)$  with a homotopy equivalent projective  $R$ -complex  $\Phi X$ . Following Eilenberg and MacLane [EM53], the higher homotopies are given by

$$\Phi_p(f_1, \dots, f_p) = q_0 C(f_1) \eta_1 C(f_2) \eta_2 \cdots \eta_{p-1} C(f_p) j_p$$

where  $j_i : \Phi X_i \rightarrow C(X_i)$  and  $q_i : C(X_i) \rightarrow \Phi X_i$  are homotopy inverse chain maps and  $\eta_i : C(X_i) \rightarrow C(X_i)$  is a chain homotopy  $id \simeq j_i \circ q_i$ .

In the special case when the homology of  $C(X)$  is projective (e.g., if  $R$  is a field), the *homology complex*

$$\Phi X = H_*(C(X))$$

(with zero boundary map) gives an example of a homotopy equivalent chain complex. Using the construction above, we obtain the  $A_\infty$  *homology functor*.

Suppose that  $p : E \rightarrow B$  is a fiber bundle where the base  $B$  is (the geometric realization of) a simplicial complex. Then for each simplex  $\sigma$  in  $B$  the inverse image  $E|\sigma = p^{-1}(\sigma)$  is homeomorphic to  $F \times \sigma$  and thus homotopy equivalent to the fiber  $F$ . Taking either a cellular chain complex or the total singular complex, we get a functor from the category  $\text{simp } B$  of simplices in  $B$ , with inclusions as morphisms, to the category of augmented chain complexes. When the homology of  $F$  is projective, the construction above gives an  $A_\infty$  homology functor on  $\text{simp } B$ .

We will see that this  $A_\infty$  functor gives a twisting cochain on  $B$  which, by Ed Brown's twisted tensor product construction, gives a chain complex for the homology of  $E$ . But first we want to point out that the dual of an  $A_\infty$  functor is also an  $A_\infty$  functor.

Suppose that  $R = K$  is a field. Then we have the duality functor  $V^* = \text{Hom}(V, K)$  which transforms a chain complex  $C$  into a cochain complex  $C^*$ . Duality also reversed the order of composition and therefore gives a functor  $\mathcal{C}(K) \rightarrow \mathcal{C}(K)^{op}$ . So, given an  $A_\infty$  functor  $\Phi : \mathcal{X} \rightarrow \mathcal{C}(K)$  we can compose with the duality functor

$$\mathcal{X} \rightarrow \mathcal{C}(K) \rightarrow \mathcal{C}(K)^{op}$$

This is the same as a functor  $\Phi^* : \mathcal{X}^{op} \rightarrow \mathcal{C}(K)$ .

**Proposition 1.2.** *The composition of an  $A_\infty$  functor  $\Phi : \mathcal{X} \rightarrow \mathcal{C}(K)$  with the duality functor on  $\mathcal{C}(K)$  gives an  $A_\infty$  functor  $\Phi^*$  on  $\mathcal{X}^{op}$ .*

*Proof.* This is very straightforward. The interesting point is that there is no change in signs. Apply duality to Equation (2) and reverse the order of the morphisms  $f_i^*$  (since they are begin composed in the opposite order in  $\mathcal{X}^{op}$ ). We get:

$$\sum_{i=0}^p (-1)^i \Phi_{p-i}^*(f_p^*, \dots, f_{i+1}^*) \Phi_i^*(f_i^*, \dots, f_1^*) = \sum_{i=1}^{p-1} (-1)^i \Phi_{p-1}^*(f_p^*, \dots, f_{i+1}^* f_i^*, \dots, f_1^*)$$

Now multiply both sides by  $(-1)^p$ , replace  $i$  by  $p-i$  and make the notation change:  $g_i = f_{p-i+1}^*$  to see that  $\Phi^*$  satisfies the definition of an  $A_\infty$  functor.  $\square$

## 2. TWISTING COCHAIN

Twisting cochains arise from  $A_\infty$  functors  $\Phi$  for which  $\Phi_1$  is a natural isomorphism, i.e.,  $\Phi_1(f)$  is a chain isomorphism for all  $f : X \rightarrow Y$  and  $\Phi_1(fg) = \Phi_1(f)\Phi_1(g)$ . In that case, the graded bifunctor

$$(3) \quad F(X, Y) = \text{HOM}(\Phi X, \Phi Y)$$

gives a locally trivial coefficient system on the category  $\mathcal{X}$ .

Since we are using the nerve of the opposite category, a  $p$ -cochain on  $\mathcal{X}$  with coefficients in a bifunctor  $F$  is a mapping  $\psi$  which assigns to each  $p$ -simplex

$$X_* = (X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} \dots \xleftarrow{f_p} X_p)$$

in  $\mathcal{X}$  an element  $\psi(X_*) \in F(X_p, X_0)$ . The *coboundary* of  $\psi$  is the  $p+1$  cochain given by

$$(4) \quad \delta\psi(X_0, \dots, X_{p+1}) = (f_1)_* \psi(X_1, \dots, X_{p+1}) \\ + \sum_{i=1}^p (-1)^i \psi(X_0, \dots, \widehat{X}_i, \dots, X_{p+1}) + (-1)^p (f_{p+1})^* \psi(X_0, \dots, X_p).$$

**Definition 2.1.** A *twisting cochain*  $\psi$  on  $\mathcal{X}$  with coefficients in a functor  $(\Phi, \Phi_1)$  is a sum of cochains  $\psi = \sum_{p \geq 0} \psi_p$  where  $\psi_p$  is a  $p$ -cochain on  $\mathcal{X}$  with coefficients in the degree  $p-1$  part  $F_{p-1}$  of the  $\text{Hom}(\Phi, \Phi)$  bifunctor  $F$  of (3) so that the following condition is satisfied.

$$\delta\psi = \psi \cup' \psi.$$

Here  $\cup'$  is the cup product using the Koszul sign rule:

$$\psi_p \cup' \psi_q(X_0, \dots, X_{p+q}) = (-1)^p \psi_p(X_0, \dots, X_p) \psi_q(X_p, \dots, X_{p+q}).$$

since  $\psi_q$  has total odd degree.

By comparison of definitions we have the following.

**Proposition 2.2.**  $\psi$  is a twisting cochain on  $\mathcal{X}$  if and only if

$$(\Phi, \psi_0, \Phi_1 + \psi_1, \psi_2, \psi_3, \dots)$$

is an  $A_\infty$  functor.

Note that the boundary map  $\psi_0(X)$  may be completely unrelated to the original boundary map for  $\Phi X$ .

An example is the  $A_\infty$  homology functor

$$\sigma \mapsto H_*(E_\sigma)$$

of a fiber bundle  $E \rightarrow B$  over a triangulated space  $B$  whose fiber  $F$  has projective homology. In this case  $\psi_0$  and  $\psi_1$  are both zero and the higher homotopies  $\psi_p$  for  $p \geq 2$  are unique up to simplicial homotopy (over  $B \times I$ ).

In his original paper [Bro59], Ed Brown showed that the homology of the total space  $E$  is given by the *twisted tensor product*

$$C_*(B) \otimes_\psi C_*(F)$$

of cellular chain complexes  $C_*(B)$  and  $C_*(F)$  for the base and fiber. Taking  $C_*(F) = H_*(F)$ , this is the usual bicomplex  $C_p(B; H_q(F))$  with boundary map modified by the twisting cochain  $\psi$ :

$$(5) \quad \partial_\psi(x \otimes y) = \partial x \otimes y - \sum_{p+q=\deg x} (-1)^p f_p(x) \otimes \psi_q(b_q(x))(y).$$

Here  $x = (x_0 \supseteq x_1 \supseteq \dots \supseteq x_n)$  is a simplex in the first barycentric subdivision of  $B$ ,  $f_p(x) = (x_0 \supseteq \dots \supseteq x_p)$  is the front  $p$ -face of  $x$  and  $b_q(x) = (x_p \supseteq \dots \supseteq x_n)$  is the back  $q$ -face.

**Theorem 2.3** (Brown[Bro59]). *Assuming that  $F$  has projective homology, the twisted tensor product gives the homology of the total space:*

$$H_*(C_*(B) \otimes_\psi H_*(F)) \cong H_*(E).$$

**Remark 2.4.** The mapping  $f_p \otimes \psi_q b_q$  in (5) has bidegree  $(-q, q-1)$ . It gives the corresponding boundary map in the Serre spectral sequence for  $E$  which is given by filtering the twisted tensor product by reverse filtration of  $H_*(F)$  (by subcomplexes  $H_{*\geq n}(F)$ ).

The twisted tensor product can be chosen to be natural since we can pull back the twisting cochain from a fixed universal  $A_\infty$  homology functor on the entire category of augmented chain complexes with projective homology and chain maps (in some fixed universe since we use the Axiom of Choice on the collection of all chain complexes).

**Corollary 2.5.** *When  $E \rightarrow B$  is an oriented  $n-1$  sphere bundle, the degree  $n$  part  $\psi_n$  of the twisting cochain  $\psi$  is a cocycle representing the Euler class of  $E$ :*

$$[\psi_n] = e^E \in H^n(B; R).$$

*Proof.* Since  $\text{HOM}(H_*(S^{n-1}), H_*(S^{n-1}))$  has elements only in degrees  $0, n-1$ ,  $\psi_k = 0$  for  $k \neq n$ . By definition of a twisting cochain we have

$$\delta\psi_n = (\psi \cup' \psi)_n = 0.$$

Therefore,  $\psi_n$  is an  $n-1$  cocycle on  $B$ . Since it gives the differential in the Serre spectral sequence it must represent the Euler class.  $\square$

In the present setting, Ed Brown's twisted tensor product is equivalent to the following construction.

**Definition 2.6.** The *total complex*  $E(\psi; \Phi)$  of the twisting cochain  $\psi$  with coefficients in the functor  $(\Phi, \Phi_1)$  is given by

$$E(\psi; \Phi) = \bigoplus_{k \geq 0} \bigoplus_{(X_0 \leftarrow \dots \leftarrow X_k)} (X_*) \otimes \Phi X_k$$

with boundary map  $\partial_\psi$  given by (5).

**Remark 2.7.** Note that every simplex  $X_* = (X_0 \leftarrow \dots \leftarrow X_k)$  gives a subcomplex of the total complex  $E(\psi; \Phi)$  by:

$$E(X_*) = \bigoplus_{j \geq 0} \bigoplus_{a: [j] \rightarrow [k]} a^*(X_*) \otimes \Phi X_{a(j)}.$$

This is also the total complex of the  $A_\infty$  functor on  $[k]$  (considered as a category with objects  $0, 1, \dots, k$  and morphisms  $k \rightarrow k-1 \rightarrow \dots \rightarrow 1 \rightarrow 0$ ) given by pulling back  $\psi$  along the functor  $X_* : [k] \rightarrow \mathcal{X}$ .

Note that this gives a functor from the category of simplices in  $\mathcal{X}$  to the category of subcomplexes of the total complex with morphisms being inclusion maps. Thus, just as the  $A_\infty$  homology functor constructs an  $A_\infty$  functor out of an actual functor, the total complex construction gives an actual functor on  $\text{simp } \mathcal{N}\mathcal{X}^{op}$  from an  $A_\infty$  functor on  $\mathcal{X}$ .

Using the total complex, a twisting cochain on  $\mathcal{X}$  can be viewed as a family of chain complexes parametrized by the nerve of  $\mathcal{X}^{op}$ . With some extra structure, this gives a map from the geometric realization of  $\mathcal{X}$  to the Volodin K-theory space of  $R$ .

### 3. VOLODIN K-THEORY

Algebraic K-theory is related to twisting cochains in the following way. When two *based, upper triangular twisting cochains* are homotopy equivalent, there is an algebraic K-theory obstruction to deforming one into the other. Formally, we take the pointwise mapping cone. This gives a based free acyclic upper triangular twisting cochain on the category  $\mathcal{X}$ . This is equivalent to a mapping from the geometric realization  $|\mathcal{X}|$  of  $\mathcal{X}$  to a fancy version of the Volodin K-theory space of the ring  $R$ .

When the basis is only well-defined up to permutation and multiplication by elements of a subgroup  $G$  of the group of units of  $R$ , the acyclic twisting cochain on  $\mathcal{X}$  defines a mapping from  $|\mathcal{X}|$  into the fiber  $\mathcal{W}h_\bullet^h(R, G)$  of the mapping

$$\Omega^\infty \Sigma^\infty(BG_+) \rightarrow BGL(R)^+.$$

The well-known basic case is the Whitehead group

$$\mathcal{W}h_1(G) = \pi_0 \mathcal{W}h_\bullet^h(\mathbb{Z}[G], G)$$

which is the obstruction to  $G$ -collapse of a contractible f.g. based free  $R$ -complex. In this section we discuss the different versions of the Volodin construction, show how they are related to twisting cochains and identify the homotopy type of two of them.

The basic definition is sometimes called the ‘‘one index’’ case. It is a space of invertible matrices locally varying by upper triangular column operations. When

this definition is expressed as a twisting cochain, the construction seems artificial, with only one term  $\psi_1$  in the twisting cochain  $\psi$ . However, when the missing higher terms  $\psi_p$  are inserted we recover the general Volodin space.

**Definition 3.1.** For every  $n \geq 2$  the *Volodin category*  $\mathcal{V}^n(R)$  is defined to be the category whose objects are pairs  $(A, \sigma)$  consisting of an invertible  $n \times n$  matrix  $A \in GL(n, R)$  and a partial ordering  $\sigma$  of  $\{1, 2, \dots, n\}$ . A morphism  $(A, \sigma) \rightarrow (B, \tau)$  is a matrix  $T$  so that

- (1)  $\sigma \subseteq \tau$ , i.e.,  $\tau$  is a refinement of  $\sigma$ .
- (2)  $AT = B$ . (So the morphism  $T = A^{-1}B$  is unique if it exists.)
- (3)  $T = (t_{ij})$  is  $\tau$ -upper triangular in the sense that
  - (a)  $t_{ii} = 1$  for  $i = 1, \dots, n$
  - (b)  $t_{ij} = 0$  unless  $i \leq j$  in the partial ordering  $\tau$  ( $i < j \Leftrightarrow (i, j) \in \tau$ ).

Note that composition is reverse matrix multiplication:

$$S \circ T = TS.$$

There is a *simplicial Volodin space*  $V_{\bullet}^n(R)$  without explicit partial orderings. A  $p$ -simplex  $g \in V_p^n(R)$  consists of a  $p + 1$  tuple of invertible  $n \times n$  matrices

$$g = (g_0, g_1, \dots, g_p)$$

so that for some partial ordering  $\sigma$  of  $\{1, \dots, n\}$  the matrices  $g_i^{-1}g_j$  are all  $\sigma$ -upper triangular. There is a simplicial map

$$(6) \quad \mathcal{N}_{\bullet} \mathcal{V}^n(R) \rightarrow V_{\bullet}^n(R)$$

from the nerve of the Volodin category  $\mathcal{V}^n(R)$  to the simplicial set  $V_{\bullet}^n(R)$  given by forgetting the partial orderings. However, the collection of admissible partial orderings on any  $g \in V_p^n(R)$  has a unique minimal element and therefore forms a contractible category. Consequently, (6) induces a homotopy equivalence

$$|\mathcal{V}^n(R)| \simeq |V_{\bullet}^n(R)|$$

If we stabilize matrices in the usual way by adding a 1 in the lower right corner we get the stable Volodin category

$$\mathcal{V}(R) = \varinjlim \mathcal{V}^n(R)$$

which is related to Quillen K-theory by the following well-known theorem due to Vasserstein and Wagoner but best explained by Suslin [Sus81].

**Theorem 3.2.**  $|\mathcal{V}(R)| \simeq |V_{\bullet}^{\infty}(R)| \simeq \Omega BGL(R)^+$  where  $\Omega BGL(R)^+$  is the loop space of the plus construction on the classifying space of  $GL(R) = GL(\infty, R)$ .

The Volodin category  $\mathcal{V}^n(R)$  has a canonical twisting cochain. It comes from the realization that an invertible matrix is the same as a based contractible chain complex with two terms. The definition is given as follows.

- (1) Let  $\Phi(A, \sigma)$  be the same based free chain complex

$$(C_*) : R^n \xrightarrow{0} R^n$$

for every object  $(A, \sigma)$  of  $\mathcal{V}^n(R)$ .

- (2)  $\Phi_1 = (id, id)$  is the identity chain map  $C_* \rightarrow C_*$ .
- (3)  $\psi_0(A, \sigma) = A : R^n \rightarrow R^n$ .

(4)  $\psi_1(T) = (0, T - I)$ . So  $I + \psi_1(T) = (I, T)$  gives a chain isomorphism:

$$\begin{array}{ccc} R^n & \xrightarrow{T} & R^n \\ B \downarrow & & \downarrow A \\ R^n & \xrightarrow{I} & R^n \end{array}$$

The higher homotopies  $\psi_p, p \geq 2$ , are all zero for  $\mathcal{V}^n(R)$ . However, there is a fancier version of the Volodin category with higher homotopies. We call it the “Whitehead category.”

**Definition 3.3.** If  $G$  is a subgroup of the group of units of a ring  $R$  then the *Whitehead category*  $\mathcal{W}h_\bullet(R, G)$  is defined to be the simplicial category whose simplicial set of objects consists of pairs  $(P, \psi)$  where  $\psi$  is an upper-triangular twisting cochain on the category  $[k]$  (as in Remark 2.7) with coefficients in the fixed based chain complex:

$$R^P : R^{P_n} \xrightarrow{0} R^{P_{n-1}} \xrightarrow{0} \dots \xrightarrow{0} R^{P_0}$$

where  $P = \coprod P_i$  is a graded poset (a poset with a grading not necessarily related to the ordering). By *upper-triangular* we mean that  $\psi(\sigma)(x)$  is a linear combination of  $y < x \in P$  for all simplices  $\sigma$  in  $[k]$ . As in the Volodin category,  $\Phi_1$  is the identity mapping on  $R^P$ .

A *morphism*  $(P, \psi) \rightarrow (Q, \phi)$  in  $\mathcal{W}h_k(R, G)$  consists of a graded order preserving monomorphism going the other way:

$$f : Q \rightarrow P$$

so that  $S = P - f(Q)$  is a disjoint union of *expansion pairs* which are, by definition, pairs  $x^+ > x^-$  otherwise unrelated to every other element of  $P$  so that  $\psi_0(x^+) = x^-g$  for some  $g \in G$ , together with a function  $\gamma : Q \rightarrow G$  so that  $\phi$  differs from  $\psi \circ f$  only by multiplication by  $\gamma$ , i.e.,  $f_*(\phi_p(\sigma)(x)) = \psi_p(\sigma)(f(x))\gamma(x)$ .

The following theorem, due to J. Klein and the author, is proved in [Igu02].

**Theorem 3.4** (I-Klein). *There is a homotopy fiber sequence*

$$|\mathcal{W}h_\bullet^h(R, G)| \rightarrow \Omega^\infty \Sigma^\infty(BG_+) \rightarrow BGL(R)^+$$

where  $\mathcal{W}h_\bullet^h(R, G)$  is the simplicial full subcategory of  $\mathcal{W}h_\bullet(R, G)$  consisting of  $(P, \psi)$  so that each chain complex  $(R^P, \psi_0)$  is contractible (i.e., has the homology of the empty set) and  $BG_+ = BG \coprod pt$ .

**Remark 3.5.** If we take  $G$  to be a finite group then  $\Omega^\infty \Sigma^\infty(BG_+)$  is rationally trivial above degree 0 so  $\mathcal{W}h_\bullet^h(R, G)$  has the rational homotopy type of the Volodin space:

$$|\mathcal{W}h_\bullet^h(R, G)| \simeq_{\mathbb{Q}} |\mathcal{V}(R)| \simeq \Omega BGL(R)^+.$$

In particular, if  $R = \mathbb{Q}$ , we get [Bor74]

$$|\mathcal{W}h_\bullet^h(\mathbb{Q}, 1)| \simeq_{\mathbb{Q}} \Omega BGL(\mathbb{Q})^+ \simeq_{\mathbb{Q}} BO.$$

Using the Borel regulator maps

$$K_{4k+1}\mathbb{Q} \rightarrow \mathbb{R}$$

given by continuous cohomology classes in  $H^{2k}(BGL(\mathbb{C}); \mathbb{R})$  we get the *universal real higher Franz-Reidemeister torsion* invariants

$$\tau_{2k} \in H^{4k}(\mathcal{W}h_\bullet^h(\mathbb{Q}, 1); \mathbb{R}).$$

These give characteristic classes for smooth bundles under certain conditions.

#### 4. HIGHER FR TORSION

We will discuss the circumstances under which we obtain well defined algebraic K-theory classes for a fiber bundle. If we have a smooth bundle  $p : E \rightarrow B$  where  $E, B$  and the fiber  $M_b = p^{-1}(b)$  are compact connected smooth manifolds and  $R$  is a commutative ring so that the fiber homology  $H_*(M_b; R)$  is projective then we obtain two canonical twisting cochains on  $B$ .

The first is Brown's twisting cochain  $\psi$  with coefficients in the fiberwise homology bundle

$$\Phi(b) = H_*(M_b; R).$$

Recall that this requires the fiber homology to be projective.

The second is the fiberwise cellular chain complex  $C_*(f)$  associated to a fiberwise *generalized Morse function* (GMF)  $f : E \rightarrow \mathbb{R}$ . These are defined to be smooth functions which, on each fiber  $M_b$ , has only Morse and birth-death singularities (cubic in one variable plus nondegenerate quadratic in the others). The fiberwise GMF is *not* well-defined up to homotopy. However, there is a canonical choice called a “framed function” ([Igu87], [Igu02], [Igu05]) which exists stably and is unique up to framed fiber homotopy. This gives the following.

**Theorem 4.1.** *Any compact smooth manifold bundle  $E \rightarrow B$  gives a mapping*

$$C(f) : B \rightarrow |\mathcal{W}h_\bullet(\mathbb{Z}[\pi_1 E], \pi_1 E)|$$

*which is well-defined up to homotopy and fiber homotopy equivalent to the fiberwise total singular complex of  $E$  with coefficients in  $\mathbb{Z}[\pi_1 E]$ .*

**Remark 4.2.** The *fiberwise total singular complex* of  $E$  is the functor which assigns to each simplex  $\sigma : \Delta^k \rightarrow B$ , the total singular complex of  $E|_\sigma$ . The fiberwise framed function  $f$  is defined on a product space  $E \times D^N$ .

In order to compare the two constructions we need a representation

$$\rho : \pi_1 E \rightarrow U(R)$$

of  $\pi_1 E$  into the group of units of  $R$  with respect to which the fiber homology  $H_*(M_b; R)$  is projective over  $R$ . By the functorial properties of the Whitehead category we get a mapping

$$B \rightarrow |\mathcal{W}h_\bullet(R, G)|$$

where  $G \subseteq U(R)$  is the image of  $\rho$ . By Theorem 4.1, this will be fiberwise homotopy equivalent to the  $A_\infty$  fiberwise homology functor  $\Phi_1 + \psi$ . The fiberwise mapping cone will be fiberwise contractible but it will not give a mapping to  $\mathcal{W}h_\bullet^h(R, G)$  unless the fiberwise homology has a basis. This gives the following.

**Corollary 4.3.** *If  $\pi_1 B$  acts trivially on the fiberwise homology  $H_*(M_b; R)$  then a fiberwise mapping cone construction gives a mapping*

$$C(C(f)) : B \rightarrow |\mathcal{W}h_\bullet^h(R, G)|$$

*which is well-defined up to homotopy.*

**Remark 4.4.** A more precise statement is that we take the direct sum of the fiberwise mapping cone with a fixed contractible projective  $R$ -complex  $P_*$  with the property that  $H_*(M_b; R) \oplus P_*$  is free in every degree.

When  $R = \mathbb{Q}$ , the construction of higher FR torsion extends to the case then  $\pi_1 B$  acts *unipotently* on  $H_*(M; \mathbb{Q})$  by which we mean that  $H_*(M; \mathbb{Q})$  admits a filtration by  $\pi_1 B$  submodules so that the action of  $\pi_1 B$  on the successive subquotients is trivial.

**Corollary 4.5.** *Suppose that  $E \rightarrow B$  is a compact smooth manifold bundle over a connected space  $B$  so that  $\pi_1 B$  acts unipotently on the rational homology of the fiber  $M$ . Then we have a mapping*

$$B \rightarrow |\mathcal{W}h_{\bullet}^h(\mathbb{Q}, 1)|$$

*which is well-defined up to homotopy and we can pull back universal higher torsion invariants to obtain well-defined cohomology classes*

$$\tau_{2k}(E) \in H^{4k}(B; \mathbb{R})$$

*which are trivial if the bundle is diffeomorphic to a product bundle.*

It has been known for many years (by [FH78] using the stability theorem [Igu88]) that there are smooth bundles which are homeomorphic but not diffeomorphic to product bundles and that these exotic smooth structures are detected by algebraic K-theory. Therefore, when the higher FR torsion was successfully defined, it had already been known to be nonzero in these cases.

However, in these exotic examples the fiber  $M$  is either odd dimensional or even dimensional with boundary. We now have the complete calculation of the higher torsion in the case of closed oriented even dimensional fibers.

**Theorem 4.6** (6.6 in [Igu05]). *Suppose that  $M^{2n}$  is a closed oriented even dimensional manifold and  $M \rightarrow E \rightarrow B$  is a smooth bundle so that  $\pi_1 B$  acts unipotently on the rational homology of  $M$ . Then the higher FR torsion invariants  $\tau_{2k}(E)$  are well-defined and given by*

$$\tau_{2k}(E) = \frac{1}{2}(-1)^k \zeta(2k+1) \frac{1}{(2k)!} T_{2k}(E) \in H^{4k}(B; \mathbb{R})$$

*where  $\zeta(s) = \sum \frac{1}{n^s}$  is the Riemann zeta function and*

$$T_{2k}(E) = \text{tr}_B^E \left( \frac{(2k)!}{2} \text{ch}_{2k}(T^v E \otimes \mathbb{C}) \right) \in H^{4k}(B; \mathbb{Z})$$

*with  $T^v E$  being the vertical tangent bundle of  $E$  and*

$$\text{tr}_B^E : H^n(E; \mathbb{Z}) \rightarrow H^n(B; \mathbb{Z})$$

*is the transfer (with  $n = 4k$ ).*

**Remark 4.7.** Note that  $T_{2k}(E)$  is a tangential fiber homotopy invariant. It follows that there are rationally no stable exotic smooth structures on bundles with closed oriented even dimensional fibers. (*Stable* means stable under product with large dimensional disks  $D^N$ . The exotic smooth structure on disk bundles and odd dimensional sphere bundles of [FH78] and the explicit examples given by Hatcher ([Igu02],[Goe01]) are stable.) This will be explain in a future joint paper with Sebastian Goette.

In the special case when  $n = 1$ ,  $M$  is an oriented surface and the bundle  $E$  is classified by a map of  $B$  into the classifying space  $BT_g$  of the Torelli group  $T_g$  where  $g$  is the genus of  $M$ . The tangential homotopy invariant  $T_{2k}$  is equal to the Miller-Morita-Mumford class in this case ([Mum83], [Mor84], [Mil86]). It is still

unknown whether or not any of these classes (tautological classes in degree  $4k$ ) is rationally nontrivial on the Torelli group.

Bismut and Lott [BL95] have defined higher analytic torsion invariants which have been computed in many cases ([BL97], [BG01], [Bun], [Ma97], [Goe01]). In the case of closed oriented even dimensional fibers, the analytic torsion is always zero and Goette has now shown [Goe08] that the expression in Theorem 4.6 gives the difference between the two torsions in all cases.

Higher analytic torsion is defined using flat  $\mathbb{Z}$ -graded superconnections. It was observed by Goette [Goe01] that these are infinitesimal twisting cochains or, as he puts it, that twisting cochains are combinatorial superconnections. We will explain this comment.

## 5. FLAT SUPERCONNECTIONS

When we review the definition of a flat  $\mathbb{Z}$ -graded superconnection we will see that it is the same as an infinitesimal twisting cochain. More precisely, the superconnection is the boundary map of the infinitesimal twisted tensor product. This gives one explanation of the supercommutator rules.

Instead of defining superconnections and showing their relationship to twisting cochains we will take the opposite approach. We ask the question: What is the natural definition of an “infinitesimal twisting cochain?” This question will lead us to the definition of a flat superconnection and we will see that the “superconnection complex”  $(\Omega(B, V), D)$  is dual to a twisted tensor product.

Suppose that  $B$  is a smooth manifold and  $C = \bigoplus_{n \geq 0} C_n$  is a nonnegatively graded complex vector bundle over  $B$ . We need a graded flat connection  $\nabla$  on  $C$  making each  $C_n$  into a locally constant coefficient sheaf for the twisting cochain that we want. The example that we keep in mind is when  $C$  is the fiberwise homology of a smooth manifold bundle  $F \rightarrow E \rightarrow B$ . This makes the dual bundle

$$C^* := \bigoplus_{n \geq 0} \text{Hom}(C_n, \mathbb{C})$$

the fiberwise cohomology bundle of  $E$ .

Now, imagine that  $B$  is subdivided into tiny simplices and we have a twisting cochain on  $B$  with coefficients in  $(C, \nabla)$ . Then, at each vertex  $v$  we have a degree  $-1$  endomorphism  $\psi_0(v)$  of  $C(v)$ . This gives a degree 1 endomorphism  $A_0 = \psi_0^*$  of the dual  $C^*(v)$ . We should extend this to a smooth family of such maps

$$A_0 \in \Omega^0(B, \text{End}(C^*)) = \Omega^0(B, \text{End}(C)^{op})$$

so that  $A_0(x)$  has degree 1 and square zero ( $A_0(x)^2 = 0$ ) at all  $x \in B$ .

Next, we take the edges of  $B$ . If an edge  $e$  goes from  $v_0$  to  $v_1$  we get a degree 0 map

$$C(v_0) \xleftarrow{\psi_1(e)} C(v_1)$$

so that  $\psi_1(e)$  together with the map (parallel transport) given by the connection  $\nabla$  is a chain map. If we dualize we get (an approximate) cochain map  $A_1(\Delta v) : C^*(v_0) \rightarrow C^*(v_1)$ . To obtain the smooth version we need to take local coordinates for  $C$  so that parallel transport of  $\nabla$  is constant, i.e., so that, on  $C^*$ ,  $\nabla^* = d$ . Then  $A_1$  should be a matrix 1-form on  $B$

$$A_1 \in \Omega^1(B, \text{End}(C^*))$$

so that parallel transport by the new connection  $\nabla_1 = d - A_1$  on  $C^*$  keeps  $A_0$  invariant. (The change in sign comes from the fact that parallel transport by  $d - A_1$  is given infinitesimally by  $I + A_1(\Delta v)$  where  $A_1(\Delta v)$  is evaluation of the matrix 1-form  $A_1$  on the vector  $\Delta v$ .) This means that

$$\begin{aligned} [\nabla_1, A_0] &= [d - A_1, A_0] = 0 \\ dA_0 &= [A_1, A_0] = A_1A_0 + A_0A_1. \end{aligned}$$

We interpret this as an approximately commutative diagram:

$$\begin{array}{ccc} C^*(v_0) & \xrightarrow{I+A_1(\Delta v)} & C^*(v_1) \\ A_0 \uparrow & & \uparrow_{A_0+\Delta A_0} \\ C^*(v_0) & \xrightarrow{I+A_1(\Delta v)} & C^*(v_1) \end{array}$$

Higher order terms are need to make the diagram actually commute. The linear terms give the following approximate equation:

$$\Delta A_0 \cong A_1(\Delta v)A_0 - A_0A_1(\Delta v)$$

Since  $A_0$  is odd, we get two changes of signs:

$$\Delta A_0 \cong -dA_0(\Delta v) \quad A_1(\Delta v)A_0 = -(A_1A_0)(\Delta v)$$

As  $\Delta v \rightarrow 0$  we get the equation  $dA_0 = A_1A_0 + A_0A_1$  as claimed.

At the next step, we take two small triangles in  $B$  forming a rectangle. The following diagram which commutes up to homotopy by  $\psi_2^* = A_2$  indicates what is happening. Here  $A_1 = A_1^x dx + A_1^y dy$  where  $A_1^x, A_1^y$  are (even) matrix 0-forms and  $A_1^x \Delta x$  indicates multiplication by the scalar quantity  $\Delta x$ .

$$\begin{array}{ccc} C^*(v'_1) & \xrightarrow{I+A_1^x \Delta x + \Delta A_1^x \Delta x} & C^*(v_2) \\ I+A_1^y \Delta y \uparrow & & \uparrow_{I+A_1^y \Delta y + \Delta A_1^y \Delta y} \\ C^*(v_0) & \xrightarrow{I+A_1^x \Delta x} & C^*(v_1) \end{array}$$

This gives the following approximate equation where  $\Delta x, \Delta y$  are scalar quantities and  $\Delta v_x, \Delta v_y$  are the corresponding vector quantities giving our rectangle in  $B$ .

$$\begin{aligned} (A_0A_2 + A_2A_0)(\Delta v_x, \Delta v_y) &\cong A_1^y \Delta y + \Delta A_1^y \Delta y A_1^x \Delta x - A_1^x \Delta x + \Delta A_1^x \Delta x A_1^y \Delta y \\ &\cong A_1^y A_1^x \Delta x \Delta y + \frac{\partial A_1^y}{\partial x} \Delta x \Delta y - A_1^x A_1^y \Delta x \Delta y - \frac{\partial A_1^x}{\partial y} \Delta x \Delta y \end{aligned}$$

Since  $A_1^x, A_1^y$  are even, the right hand side can be written as

$$(dA_1 - A_1^2)(\Delta v_x, \Delta v_y)$$

In other words, we have

$$A_2 \in \Omega^2(B, \text{End}(C^*))$$

satisfying the equation

$$dA_1 = A_0A_2 + A_1^2 + A_2A_0.$$

In general we should have

$$dA_{n-1} = \sum_{p+q=n} A_p A_q.$$

(See [Igu09] for a full explanation.) This leads to the following definition.

**Definition 5.1.** An *infinitesimal twisting cochain* on  $B$  with coefficients in a graded vector bundle  $C^*$  with graded flat connection  $\nabla$  ( $\nabla = \sum \nabla_k$  where  $(-1)^k \nabla_k$  is a flat connection on  $C^k$ ) is equal to a sequence of  $\text{End}(C^*)$ -valued forms

$$A_p \in \Omega^p(B, \text{End}_{1-p}(C^*)) = \Omega^0(B, \text{End}_{1-p}(C^*)) \otimes_{\Omega^0(B)} \Omega^p(B)$$

of total degree 1 so that

$$(7) \quad \nabla A_{n-1} = \sum_{p+q=n} A_p A_q.$$

In order to get a superconnection we need to pass to the superalgebra of operators on  $\Omega(B, C^*)$  in which the supercommutator rules apply.

## 6. SUPERCOMMUTATOR RULES

If  $A \in \Omega(B, \text{End}(C^*))$  is written as  $A = \sum \phi_i \otimes \alpha_i$  with fixed total degree  $|A| = |\phi_i| + |\alpha_i|$ , let  $\tilde{A}$  be the linear operator on

$$\Omega(B, C^*) = \Omega^0(B, C^*) \otimes_{\Omega^0(B)} \Omega(B)$$

given by

$$(8) \quad \tilde{A}(c \otimes \gamma) := \sum_i (-1)^{|c| \cdot |\alpha_i|} \phi_i(c) \otimes \alpha_i \wedge \gamma$$

**Proposition 6.1** (Prop. 1 in [Qui85]). *If  $\omega \in \Omega^k(B)$  then*

$$\tilde{A} \circ \omega = (-1)^{k|A|} \omega \circ \tilde{A}.$$

*Conversely, any linear operator on  $\Omega(B, C^*)$  of fixed total degree having this property is equal to  $\tilde{A}$  for a unique  $A \in \Omega(B, \text{End}(C^*))$ .*

*Proof.* Since  $\tilde{A}$  acts only on the first tensor factor we get

$$\tilde{A} \circ \omega = \tilde{A}\omega = (-1)^{k|A|} \omega \tilde{A} = (-1)^{k|A|} \omega \circ \tilde{A}$$

as required. Conversely, any linear operator which is  $\Omega(B)$ -linear in this sense must be “local” and thus we may restrict to a coordinate chart  $U$  over which  $C^*$  has a basis of sections. This makes  $\Omega(U, C^*|U)$  into a free module over  $\Omega(U)$ . Thus any  $\Omega(U)$ -linear operator is uniquely given by  $\tilde{A}$  where  $A \in \Omega(U, \text{End}(C^*|U))$  is given by the value of the operator on the basis of sections of  $C^*|U$ . Patch these together using a partition of unity.  $\square$

Here is another straightforward calculation.

**Proposition 6.2.**  $[d, \tilde{A}] = d \circ \tilde{A} - (-1)^{|A|} \tilde{A} \circ d = \tilde{d}\tilde{A}$ .

If  $A'$  is another  $\text{End}(C^*)$  valued form on  $B$  then  $\tilde{A}\tilde{A}' = \tilde{A} \circ \tilde{A}'$ . So

$$[d, \tilde{A}_{n-1}] = \tilde{d}\tilde{A}_{n-1} = \sum_{p+q=n} \tilde{A}_p \circ \tilde{A}_q$$

which, in coordinate free notation, is

$$[\nabla, \tilde{A}] = \tilde{A} \circ \tilde{A}$$

Since  $|A| = 1$  and  $\nabla^2 = 0$ , we get

$$(\nabla - \tilde{A})^2 = (\nabla - \tilde{A}) \circ (\nabla - \tilde{A}) = 0.$$

This leads to the following definition due to Bismut and Lott [BL95]. (A similar definition appeared in [Che75].)

**Definition 6.3.** Let  $V = \bigoplus_{n \geq 0} V^n$  be a graded complex vector bundle over a smooth manifold  $B$ . Then a *superconnection* on  $V$  is defined to be a linear operator  $D$  on  $\Omega(B, V)$  of total degree 1 so that

$$D\alpha = d\alpha + (-1)^{|\alpha|}\alpha D$$

for all  $\alpha \in \Omega(B)$ . The superconnection  $D$  is called *flat* if

$$D^2 = 0.$$

If  $D$  is flat then  $(\Omega(B, V), D)$  is a chain complex which we call the *superconnection complex*. We will see later that it is homotopy equivalent to the dual of a twisted tensor product.

A flat superconnections on  $V$  corresponds to a contravariant  $A_\infty$  functor on  $B$ . To get a twisting cochain we need an ordinary graded flat connection  $\nabla$  on  $V$ . Then  $D - \nabla$  gives a twisting cochain by reversing the above process.

The first step is to get out of the superalgebra framework by writing  $D$  as a sum

$$D = \nabla - \tilde{A} = \nabla - \tilde{A}_0 - \tilde{A}_1 - \tilde{A}_2 - \dots$$

where  $A_p \in \Omega^p(B, \text{End}_{1-p}(V))$  corresponds to  $\tilde{A}_p$  by (8) and satisfies (7).

Next, we obtain a contravariant twisting cochain on the category of smooth simplices in  $B$  with coefficients in the category of cochain complexes by iterated integration of  $A_*$ . Then we dualize, relying on Proposition 1.2 to recover the original twisting cochain.

## 7. CHEN'S ITERATED INTEGRALS

This section gives a very short discussion and proof of the first two steps in the process of integrating a flat superconnection to obtain a twisting cochain. Details are fully explained in [Igu09] although the original idea is contained in Chen's work [Che73], [Che75], [Che77].

Since  $\nabla$  is a flat connection, we can choose local coordinates so that  $V$  is a trivial bundle and  $\nabla = d$ . Starting with  $p = 0$  we note that  $A_0(x)$  is a degree 1 endomorphism of  $V_x$  with  $A_0(x)^2 = 0$  making  $C(x) = (V_x, A_0(x))$  into a cochain complex for all  $x \in B$ . Putting  $n = 2$  in (7) we see that the curvature  $(d - A_1)^2$  of the connection  $d - A_1$  is null homotopic. Also we will see that parallel transport of this connection is a cochain map.

It is well-known that the parallel transport  $\Phi_1$  associated to the connection  $d - A_1$  on  $V$  is given by an iterated integral of the matrix 1-form  $A_1$ . Given any piecewise smooth path  $\gamma : [0, 1] \rightarrow B$ , parallel transport is the family of degree zero homomorphisms  $\Phi_1(t, s) : C(\gamma(s)) \rightarrow C(\gamma(t))$  so that  $\Phi_1(s, s) = I = id_V$  and  $d - A_1 = 0$ , i.e.,

$$\begin{aligned} \frac{\partial}{\partial t} \Phi_1(t, s) &= A_1/t \Phi_1(t, s) \\ \frac{\partial}{\partial s} \Phi_1(t, s) &= -\Phi_1(t, s) A_1/s \end{aligned}$$

where  $A_1/t = A_1(\gamma(t))(\gamma') \in \text{End}(C(\gamma(t)))$ . The solution is given by Chen's iterated integral [Che77]:

$$\begin{aligned} \Phi_1(s_0, s_1) = I &+ \int_{s_0 \geq t \geq s_1} dt_1 A_1/t + \int_{s_0 \geq t_1 \geq t_2 \geq s_1} dt_1 dt_2 (A_1/t_1)(A_1/t_2) \\ &+ \int_{s_0 \geq t_1 \geq t_2 \geq t_3 \geq s_1} dt_1 dt_2 dt_3 (A_1/t_1)(A_1/t_2)(A_1/t_3) + \cdots \end{aligned}$$

which we abbreviate as:

$$\Phi_1(s_0, s_1) = I + \int_{\gamma} A_1 + \int_{\gamma} (A_1)^2 + \int_{\gamma} (A_1)^3 + \cdots .$$

This can also be written as the limit of a product

$$\Phi_1 = \lim_{\Delta t \rightarrow 0} \prod (I + (A_1/t_i) \Delta t)$$

In the case when  $A_1$  is constant, parallel transport  $C(\gamma(0)) \rightarrow C(\gamma(1))$  is given by  $e^{A_1}$ . The inverse is given by  $\Phi_1(0, 1) = e^{-A_1}$ .

**Proposition 7.1.**  $A_0(\gamma(t))\Phi_1(t, s) = \Phi_1(t, s)A_0(\gamma(s))$ , i.e.,  $\Phi_1(t, s)$  gives a cochain map

$$C(\gamma(t)) \leftarrow C(\gamma(s)).$$

*Proof.* By (7) we have:

$$-\frac{d}{dt} A_0(\gamma(t)) = dA_0(\gamma(t))(\gamma') = A_0(\gamma(t))A_1/t - (A_1/t)A_0(\gamma(t))$$

where both negative signs come from the fact that  $A_0$  is odd. So,  $X(t) = A_0(\gamma(t))\Phi_1(t, s)$  is the unique solution of the differential equation

$$\frac{\partial}{\partial t} X(t) = (A_1/t)X(t)$$

with initial condition  $X(s) = A_0(\gamma(s))$ . So,  $X(t)$  must also be equal to the other solution of this differential equation which is  $X(t) = \Phi_1(t, s)A_0(\gamma(s))$ .  $\square$

Let

$$\Delta^2 = \{(x, y) \in \mathbb{R}^2 \mid 1 \geq x \geq y \geq 0\}$$

and suppose that  $\sigma : \Delta^2 \rightarrow B$  is a smooth simplex with vertices  $v_0 = \sigma(0, 0)$ ,  $v_1 = \sigma(1, 0)$ ,  $v_2 = \sigma(1, 1) \in B$ . Then a chain homotopy

$$\Phi_1(v_0, v_2) \simeq \Phi_1(v_0, v_1)\Phi_1(v_1, v_2)$$

can be obtained by an iterated integral of the form

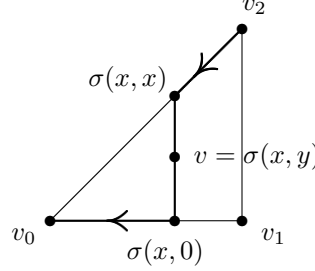
$$\psi_2(\sigma) = \int_{\sigma} A_2 + \int_{\sigma} A_2 A_1 + \int_{\sigma} A_1 A_2 + \int_{\sigma} A_2 A_1 A_1 + \int_{\sigma} A_1 A_2 A_1 + \cdots .$$

The integral over  $\sigma$  is the double integral of the pull-back to  $\Delta^2$ . The factors of  $A_1$  will just give the parallel transport  $\Phi_1$  along paths connecting  $v_0$  and  $v_2$  to the point  $v = \sigma(x, y)$

$$\psi_2(\sigma) = \int_{1 \geq x \geq y \geq 0} \sigma^*(\Phi_1(v_0, v)A_2(v)\Phi_1(v, v_2)) \in \text{Hom}(C(v_2), C(v_0))$$

where  $\Phi_1(v_0, v)$ ,  $\Phi_1(v, v_2)$  are given by parallel transport along paths given by two straight lines each as shown in the Figure.

$$\Phi_1(v, v_2) = \Phi_1(v, \sigma(x, x))\Phi_1(\sigma(x, x), v_2)$$

FIGURE 1.  $\Phi_1(v_0, v)$ ,  $\Phi_1(v, v_2)$  are parallel transport along dark lines.


$$\Phi_1(v_0, v) = \Phi_1(v_0, \sigma(x, 0))\Phi_1(\sigma(x, 0), v).$$

**Proposition 7.2.**  $A_0(v_0)\psi_2(\sigma) + \psi_2(\sigma)A_0(v_2) = \Phi_1(v_0, v_2) - \Phi_1(v_0, v_1)\Phi_1(v_1, v_2)$ .

*Proof.* For  $x \in [0, 1]$  let  $\Phi(x)$  be the parallel transport of  $d - A_1$  along the three segment path:

$$\Phi(x) = \Phi_1(v_0, \sigma(x, 0))\Phi_1(\sigma(x, 0), \sigma(x, x))\Phi_1(\sigma(x, x), v_2).$$

Then

$$\Phi(0) = \Phi_1(v_0, v_2)$$

$$\Phi(1) = \Phi_1(v_0, v_1)\Phi_1(v_1, v_2).$$

So the right hand side of the formula we are proving is

$$(9) \quad \Phi(0) - \Phi(1) = - \int_0^1 d\Phi(x).$$

By Proposition 7.1 the left hand side is equal to

$$\int_{\sigma} \Phi_1(v_0, v)[A_0(v)A_2(v) + A_2(v)A_0(v)]\Phi_1(v, v_2)$$

(The sign in front of  $A_2(v)A_0(v)$  is  $(-1)^2 = +1$  since the form degree of  $A_2$  is 2.) By (7) this is equal to

$$(10) \quad = \int_{\sigma} \Phi_1(v_0, v)[-A_1(v)A_1(v) + dA_1(v)]\Phi_1(v, v_2)$$

In (9), we have

$$-\frac{d\Phi(x)}{dx} = \Phi_1(v_0, \sigma(x, 0))X(x)\Phi_1(\sigma(x, x), v_2)$$

where

$$X(x) = A_1^x\Phi_1(\sigma(x, 0), \sigma(x, x)) - \frac{d}{dx}\Phi_1(\sigma(x, 0), \sigma(x, x)) - \Phi_1(\sigma(x, 0), \sigma(x, x))A_1^x$$

using the notation  $\sigma^*(A_1) = A_1^x dx + A_1^y dy$ . (The term  $\Phi_1(\sigma(x, 0), \sigma(x, x))A_1^y$  which occurred with positive sign in the second term and negative sign in the third term was cancelled.) Comparing this to (10) we are reduced to showing that  $X(x) = Y(x)$  where

$$Y(x) = \int_{0 \leq y \leq z} dy \Phi_1(\sigma(x, 0), v) \left( -A_1^x A_1^y + A_1^y A_1^x + \frac{\partial A_1^y}{\partial x} - \frac{\partial A_1^x}{\partial y} \right) \Phi_1(v, \sigma(x, x)).$$

Expressing  $\Phi_1(\sigma(x, 0), \sigma(x, x))$  as an iterated integral of  $A_1^y dy$  we see that the second term of  $X(x)$  is equal to the third term of  $Y(x)$  (with  $\frac{\partial A_1^y}{\partial x}$ ). The negative sign comes from the fact that we are going backwards along the  $y$  direction ( $dt = -dy$ ). The other three terms of  $Y(x)$  form the commutator of  $A_1^x$  with each factor  $A_1/dt = -A_1^y$  in the iterated integral representation of  $\Phi_1(\sigma(x, 0), \sigma(x, x))$ . This can be more easily seen in the product limit form:

$$\Phi_1(\sigma(x, 0), \sigma(x, x)) = \lim_{\substack{n \rightarrow \infty \\ \Delta y = x/n}} \prod_1^n (I - A_1^y \Delta y)$$

The commutator of  $A_1^x$  with  $I - A_1^y \Delta y$  is:

$$A_1^x(I - A_1^y \Delta y) - (I - A_1^y \Delta y)(A_1^x + \Delta_y A_1^x) = -A_1^x A_1^y \Delta y + A_1^y \Delta y A_1^x - \Delta_y A_1^x + o(\Delta y)$$

So, the commutator of  $A_1^x$  with  $\Phi_1(\sigma(x, 0), \sigma(x, x))$  is

$$A_1^x \Phi_1 - \Phi_1 A_1^x = \lim_{\Delta y \rightarrow 0} \sum_i^{i-1} \prod_1^{i-1} (I - A_1^y \Delta y) [-A_1^x A_1^y \Delta y + A_1^y \Delta y A_1^x - \Delta_y A_1^x] \prod_{i+1}^n (I - A_1^y \Delta y)$$

So, the sum of the remaining two terms of  $X(x)$  is equal to the sum of the remaining three terms of  $Y(x)$  and we conclude that  $X(x) = Y(x)$  proving the proposition.  $\square$

The higher steps in this process are explained in detail in [Igu09]. Here we will show how the entire process can be done in an easier way.

## 8. ANOTHER METHOD

There is another method for constructing a simplicial twisting cochain from a flat connection. We assume that  $B$  is compact and we choose a finite ‘‘good cover’’ for  $B$ . (See [BT82].) This is a covering of  $B$  by contractible open sets  $U$  so that all nonempty intersections

$$U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_n}$$

are also contractible.

**Lemma 8.1.** *If  $U$  is a contractible open subset of  $B$  and  $D$  is a flat superconnection on a graded vector bundle  $V$  over  $B$  then the cohomology of the superconnection complex over  $U$  is isomorphic to the cohomology of  $V$  using  $A_0$  as differential:*

$$H^n(\Omega(U, V|U), D) \cong H^n(V, A_0)$$

where the isomorphism is given by restriction to any point in  $U$ .

*Proof.* The spectral sequence collapses since its  $E_2$ -term is  $H^p(U; H^q(V))$ .  $\square$

This lemma implies that

$$F : U \mapsto (\Omega(U, V|U), D)$$

is a functor from the nerve  $\mathcal{N}\mathcal{U}$  of the good cover  $\mathcal{U}$  of  $B$  to the category of cochain complexes over  $\mathbb{C}$  and cochain homotopy equivalences. Applying the  $A_\infty$  cohomology functor we get a contravariant  $A_\infty$  functor  $H^*F$  on  $\mathcal{N}\mathcal{U}$ . By Proposition 1.2 we can dualize to get a covariant  $A_\infty$  functor  $\Phi_1 + \psi$  on  $\mathcal{N}\mathcal{U}$  with coefficients in  $(\Phi, \Phi_1) = (V^*, \nabla^*)$ . Subtracting  $\Phi_1 = \nabla^*$  we get the twisting cochain  $\psi$  satisfying the following.

**Theorem 8.2.** *The twisted tensor product  $C_*(\mathcal{N}\mathcal{U}) \otimes_\psi V^*$  is homotopy equivalent to the dual of the superconnection complex. I.e.,*

$$(\Omega(B, V), D) \simeq \text{Hom}(C_*(\mathcal{N}\mathcal{U}) \otimes_\psi V^*, \mathbb{C}).$$

*Proof.* This holds by induction on the number of open sets in the finite good covering  $\mathcal{U}$ . When the number is 1 we use Lemma 8.1. To increase the number we use Mayer-Vietoris.  $\square$

**Remark 8.3.** By Ed Brown's Theorem 2.3 this implies that, if the superconnection is constructed correctly, the superconnection complex gives the cohomology of the total space of a smooth manifold bundle. This construction also allows us to compare flat superconnections with twisting cochains, giving a K-theory difference class with a well defined higher torsion. But, this is the subject of another paper.

#### REFERENCES

- [BG01] Jean-Michel Bismut and Sebastian Goette, *Families torsion and Morse functions*, Astérisque (2001), no. 275, x+293.
- [BL95] Jean-Michel Bismut and John Lott, *Flat vector bundles, direct images and higher real analytic torsion*, J. Amer. Math. Soc. **8** (1995), no. 2, 291–363.
- [BL97] ———, *Torus bundles and the group cohomology of  $\text{gl}(N, \mathbf{Z})$* , J. Differential Geom. **47** (1997), no. 2, 196–236.
- [Bor74] A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. ENS **7** (1974), 235–272.
- [Bro59] Edgar H. Brown, Jr., *Twisted tensor products. I*, Ann. of Math. (2) **69** (1959), 223–246.
- [BT82] R. Bott and L. Tu, *Differential forms in algebraic topology*, Graduate Texts in Math., vol. 82, Springer-Verlag, New York, 1982.
- [Bun] U. Bunke, *Higher analytic torsion of sphere bundles and continuous cohomology of  $\text{Diff}(S^{2n-1})$* , math.DG/9802100.
- [Che73] Kuo-tsai Chen, *Iterated integrals of differential forms and loop space homology*, Ann. of Math. (2) **97** (1973), 217–246.
- [Che75] Kuo Tsai Chen, *Connections, holonomy and path space homology*, Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 1, Stanford Univ., Stanford, Calif., 1973), Amer. Math. Soc., Providence, R. I., 1975, pp. 39–52.
- [Che77] K.-T. Chen, *Iterated path integrals*, Bull. Amer. Math. Soc. **83** (1977), 831–879.
- [EM53] Samuel Eilenberg and Saunders MacLane, *On the groups of  $h(\pi, n)$ . I.*, Ann. of Math. (2) **58** (1953), 55–106.
- [FH78] F. T. Farrell and W. C. Hsiang, *On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds*, Algebraic and Geometric Topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1, Amer. Math. Soc., Providence, R.I., 1978, pp. 325–337.
- [Fuk93] Kenji Fukaya, *Morse homotopy,  $A^\infty$ -category, and Floer homologies*, Proceedings of GARC Workshop on Geometry and Topology '93 (Seoul, 1993) (Seoul), Lecture Notes Ser., vol. 18, Seoul Nat. Univ., 1993, pp. 1–102.
- [Goe01] Sebastian Goette, *Morse theory and higher torsion invariants I*, math.DG/0111222.
- [Goe08] Sebastian Goette, *Torsion invariants for families*, arXiv:math/0804.3020.
- [Igu87] Kiyoshi Igusa, *The space of framed functions*, Trans. Amer. Math. Soc. **301** (1987), no. 2, 431–477.
- [Igu88] ———, *The stability theorem for smooth pseudoisotopies*, K-Theory **2** (1988), no. 1–2, vi+355.
- [Igu02] ———, *Higher Franz-Reidemeister Torsion*, AMS/IP Studies in Advance Mathematics, vol. 31, International Press, 2002.
- [Igu05] ———, *Higher complex torsion and the framing principle*, Mem. Amer. Math. Soc. **177** (2005), no. 835, xiv+94.
- [Igu09] ———, *Iterated integrals of superconnections*, preprint.
- [Kad80] T.V. Kadeishvili, *On the homology theory of fiber spaces*, Uspekhi Mat.Nauk **35** (1980), 183–188, Translated in Russ Math. Surv. **35** (1980), 231–238.

- [Ma97] Xiaonan Ma, *Formes de torsion analytique et familles de submersions*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 2, 205–210.
- [Mil86] Edward Y. Miller, *The homology of the mapping class group*, J. Differential Geom. **24** (1986), no. 1, 1–14.
- [Mor84] Shigeyuki Morita, *Characteristic classes of surface bundles*, Bull. Amer. Math. Soc. (N.S.) **11** (1984), no. 2, 386–388.
- [Mum83] David Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and geometry, Vol. II, Birkhäuser Boston, Boston, MA, 1983, pp. 271–328.
- [Qui85] Daniel Quillen, *Superconnections and the Chern character*, Topology **24** (1985), 89–95.
- [Sta63] J.D. Stasheff, *Homotopy associativity of H-spaces I,II*, Trans. AMS **108** (1963), 275–292, 293–312.
- [Sus81] A. A. Suslin, *On the equivalence of K-theories*, Comm. Algebra **9** (1981), no. 15, 1559–1566.
- [Vol71] I. A. Volodin, *Algebraic K-theory as an extraordinary homology theory of the category of associative rings with unit*, Math. USSR Izv. **5** (1971), 859–887.