

COMPARISON OF VIEWS ON FINITISTIC PROJECTIVE DIMENSION

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ABSTRACT. The finitistic dimension conjecture says that the projective dimension of finitely generated modules over an Artin algebra is bounded when finite. The conjecture is known for algebras of representation dimension 3, for modules of Loewy length 2 and for stratifying systems with at most 2 indecomposable modules of infinite projective dimension (Huard, Lanzilotta, Mendoza [4]). We would like to increase these numbers by one.

We will look at stratifying systems of size 3 and examine the corresponding subquivers, Koszul dual systems for the representation dimension 4 case and the advantages of reduction to the case of finite fields. In some cases the corresponding statements are easier to solve and I will show how this works for the self-dual zero relation case. This is joint work with Gordana Todorov.

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1. INTRODUCTION

These are lecture notes for the talk I gave at the XXI meeting on representation theory of algebras at the University of Sherbrooke, Oct 2, 2009 and later at Montevideo, Uruguay on March 5, 2010. This version corrects some of the errors in the previous version, in particular the proof of Lemma 4.1.

Suppose that Λ is a finite dimensional algebra over an algebraically closed field K and $\text{mod-}\Lambda$ is the category of finitely generated right Λ -modules. Then the *finitistic dimension conjecture* says that $\text{pfd}(\Lambda) <$

∞ where

$$\text{pfd}(\Lambda) = \text{pfd}(\text{mod-}\Lambda) = \sup\{\text{pd}_\Lambda(M) \mid \text{pd}_\Lambda(M) < \infty, M \in \text{mod-}\Lambda\}$$

To make this difficult conjecture more accessible, we restrict it to subcategories \mathcal{M} of $\text{mod-}\Lambda$ and ask if $\text{pfd}(\mathcal{M}) < \infty$ where $\text{pfd}(\mathcal{M})$ is defined as above with $\text{mod-}\Lambda$ replaced by \mathcal{M} .

Our starting point is the following.

Theorem 1.1 ([5]). *Let $X \in \text{mod-}\Lambda$ and let $\mathcal{R}_2(X)$ denote the full subcategory of $\text{mod-}\Lambda$ consisting of modules M which admit 2-stage resolution by objects in $\text{add } X$:*

$$0 \rightarrow A_1 \rightarrow A_0 \rightarrow M \rightarrow 0$$

Then $\text{pfd}(\mathcal{R}_2(X)) < \infty$.

The next step, which is still unknown, is to consider modules which admit 3-stage resolutions by objects in a finite category. It will be convenient to take three different finite categories. Suppose that X_0, X_1, X_2 are Λ modules. Then let $\mathcal{R}_3(X_*) \subseteq \text{mod-}\Lambda$ be the category of all modules M for which there is an exact sequence

$$0 \rightarrow A_2 \rightarrow A_1 \rightarrow A_0 \rightarrow M \rightarrow 0$$

Where A_i are in $\text{add } X_i$. Then we want to prove that

$$\text{pfd}(\mathcal{R}_3(X_*)) < \infty$$

Some examples of classes of modules M which fall into this category (i.e. are subcategories of $\mathcal{R}_3(X_*)$ for some X_*) are the following.

- (1) $M = \Omega Y$ for any module Y where Λ has Loevey length 4.
- (2) $M = \Omega^2 Z$ for any module Z where Λ has representation dimension 4.
- (3) $M \in \mathcal{F}(\theta)$ where θ is a stratifying system of size 3.

In the first case we can take $X_2 = \Lambda/r\Lambda$, $X_1 = \Lambda/r^2\Lambda$, $X_0 = \Lambda/r^3\Lambda$. It is known that $\text{pfd}(\Lambda) < \infty$ for artin algebras Λ with Loevey length ≤ 3 , so Loevey length 4 is the next case.

In the second case there exists, by definition of representation dimension, a generator-cogenerator X so that $\text{End}_\Lambda(X)$ has global dimension 4. Then $\Omega^2 Z$ is an object of $\mathcal{R}_3(X)$ for all modules Z . It follows from the theorem above that $\text{pfd}(\Lambda) < \infty$ when the representation dimension of Λ is ≤ 3 . So, $\text{rep dim } \Lambda = 4$ is the next case.

2. STRATIFYING SYSTEMS

Stratifying systems are a generalization of stratified algebras which were introduced by Dlab and Ringel [2]. We use the following definition which is due to Marcos, Mendoza and Sáenz.

Definition 2.1 ([6]). A *stratifying system* of size t is a collection of indecomposable modules $\theta(1), \dots, \theta(t)$ so that

- (1) $\text{Hom}_\Lambda(\theta(j), \theta(i)) = 0$ for all $j > i$
- (2) $\text{Ext}_\Lambda^1(\theta(j), \theta(i)) = 0$ for $j \geq i$.

If θ is a stratifying system then $\mathcal{F}(\theta)$ is defined to be the full subcategory of $\text{mod-}\Lambda$ of all modules M having filtrations with quotients in $\text{add}(\coprod \theta(i))$.

Since the finitistic dimension conjecture seems too hard to tackle directly, we are interested in the following weaker conjecture of Huard, Lanzillota and Mendoza.

Conjecture 2.2 ([4]). $\text{pfd}(\mathcal{F}(\theta)) < \infty$.

We know a special case of this:

Theorem 2.3 ([4]). $\text{pfd}(\mathcal{F}(\theta)) < \infty$ if at most 2 of the $\theta(i)$ have finite projective dimension.

So, the next case is when 2 is increased to 3. We will consider the case $t = 3$ when $\theta(1), \theta(2), \theta(3)$ have infinite projective dimension. In that case we are reduced to modules having 3 stage finite resolutions by the following theorem.

Theorem 2.4 ([6]). For any stratifying system θ of size t , there exist $Q(i) \in \mathcal{F}(\theta)$ so that every $M \in \mathcal{F}(\theta)$ admits a t -stage resolution:

$$0 \rightarrow Q_{t-1}(M) \rightarrow \dots \rightarrow Q_1(M) \rightarrow Q_0(M) \rightarrow M \rightarrow 0$$

where

$$Q_i(M) \in \text{add} \left(\coprod_{j>i} Q(j) \right)$$

This theorem reduces the study of stratified modules to modules having finite resolutions in a finite category. We want to go the other way. This works to some extent in the case $t = 3$ using Koszul duality.

3. KOSZUL DUALITY

The first step is to construct the Koszul algebra which governs these examples. This will be a graded basic algebra $E = \coprod E_j$ which is a subalgebra of the endomorphism algebra of $X = \coprod X_i$. E will be

generated in degrees 0 and 1 and will have only quadratic relations. In other words, it will be a Koszul algebra. [1, 3]

Definition 3.1. Let $\Sigma = \Sigma_0 \amalg \Sigma_1 \amalg \Sigma_2$ be the graded subalgebra of $\text{End}_\Lambda(X)$ given as follows.

- (1) $\Sigma_0 = K^{n_0+n_1+n_2} = K^{n_0} \times K^{n_1} \times K^{n_2}$ where n_i is the number of indecomposable components of X_i and $K^{n_i} \subseteq \text{End}_\Lambda(X_i)$ is the subalgebra of “trivial morphisms” given by multiplication by a scalar on each component. (Fixed decompositions of each X_i must be chosen.)
- (2) $\Sigma_1 = \text{Hom}_\Lambda(X_2, X_1) \amalg \text{Hom}_\Lambda(X_1, X_0)$
- (3) Σ_2 is the image of the composition map

$$\circ : \text{Hom}_\Lambda(X_1, X_0) \otimes_{K^{n_1}} \text{Hom}_\Lambda(X_2, X_1) \rightarrow \text{Hom}_\Lambda(X_2, X_0)$$

If we let R be the kernel of this composition map we get a short exact sequence of K^{n_0} - K^{n_2} -bimodules:

$$(3.1) \quad 0 \rightarrow R \rightarrow \text{Hom}_\Lambda(X_1, X_0) \otimes_{K^{n_1}} \text{Hom}_\Lambda(X_2, X_1) \xrightarrow{\circ} \Sigma_2 \rightarrow 0$$

Note that Σ is, by construction, a basic algebra which can be expressed as the quotient of a path algebra by the homogeneous relation ideal R . There are $n = n_0 + n_1 + n_2$ vertices in the quiver which lie in degree 0,1,2. These vertices correspond to the components of X_0, X_1, X_2 which we take to be fixed.

Choose a basis for $\text{Hom}_\Lambda(X_2, X_1)$ consisting of homomorphisms f_j going from one component of X_2 to one component of X_1 . Choose a similar basis of homomorphisms $g_k : X_1 \rightarrow X_0$. Each f_j and g_k corresponds to an arrow of degree -1. Paths are composed right to left and the longest paths have length 2 and have the form $g_k f_j$. R is the set of all linear combinations of these length 2 paths which are zero in the algebra Σ . Σ act on $X = \amalg X_i$ on the left since it is a subalgebra of $\text{End}_\Lambda(X)$.

Next we construct the dual of Σ using the duality functor $D = \text{Hom}_K(-, K)$. Note that if V is an R - S -bimodule then DV is an S - R -bimodule. Also, $DK^n = K^n$.

Definition 3.2. Let $E = E(\Sigma)$ be the graded algebra $E = E_0 \amalg E_1 \amalg E_2$ given as follows.

- (1) $E_0 = D\Sigma_0 = K^{n_0+n_1+n_2} = K^{n_0} \times K^{n_1} \times K^{n_2}$
- (2) $E_1 = D\Sigma_1 = D \text{Hom}_\Lambda(X_2, X_1) \amalg D \text{Hom}_\Lambda(X_1, X_0)$
- (3) $E_2 = DR$ with multiplication $E_1 \otimes E_1 \rightarrow E_2$ given by the dual of the sequence (3.1):

$$0 \leftarrow DR \xleftarrow{\text{mult in } E} D \text{Hom}_\Lambda(X_2, X_1) \otimes_{K^{n_1}} D \text{Hom}_\Lambda(X_1, X_0) \leftarrow D\Sigma_2 \leftarrow 0$$

Although E does not act on X , we want to consider right E -modules and compose the paths in the quiver of E from left to right. The basis $\{f_j, g_k\}$ for Σ_1 gives a dual basis $\{f_j^*, g_k^*\}$ for $E_1 = D\Sigma_1$ so the length 2 paths in E_2 have the form $f_j^* g_k^*$.

The following is a simple example of basic Koszul duality. This has been vastly generalized in all possible directions by many authors and is the underlying principle behind Kontsevich graph cohomology.

Proposition 3.3. *There is an exact equivalence of K -categories:*

$$\Phi : \text{mod-}E \cong \mathcal{C}$$

where \mathcal{C} is a subcategory of the category of chain complexes $A_2 \xrightarrow{d} A_1 \xrightarrow{d} A_0$ with $d^2 = 0$ in $\text{mod-}\Lambda$ where $A_i \in \text{add } X_i$.

Proof. We will construct an exact embedding of $\text{mod-}E$ into the category of chain complexes $A_2 \xrightarrow{d} A_1 \xrightarrow{d} A_0$ with $d^2 = 0$ in $\text{mod-}\Lambda$ and examine its image to determine the precise definition of \mathcal{C} .

Let V_* be an E -module. As a vector space $V_* = V_0 \amalg V_1 \amalg V_2$ where V_i is a K^{n_i} -module and the E -module structure is given by a linear map $V_* \otimes E_1 \rightarrow V_*$ which is the direct sum of two maps:

$$\alpha : V_2 \otimes_{K^{n_2}} D \text{Hom}_\Lambda(X_2, X_1) \rightarrow V_1$$

$$\beta : V_1 \otimes_{K^{n_1}} D \text{Hom}_\Lambda(X_1, X_0) \rightarrow V_0$$

The chain complex $\Phi(V_*)$ is given by taking the adjoints of these two maps: The adjoint of α is

$$\hat{\alpha} : V_2 \rightarrow V_1 \otimes_{K^{n_1}} \text{Hom}_\Lambda(X_2, X_1)$$

with $\hat{\beta}$ defined similarly. Taking the adjoint again we get:

$$V_2 \otimes_{K^{n_2}} X_2 \xrightarrow{\hat{\alpha}} V_1 \otimes_{K^{n_1}} X_1 \xrightarrow{\hat{\beta}} V_0 \otimes_{K^{n_0}} X_0$$

The composition is zero since it is the adjoint of $\beta\alpha : V_2 \otimes DR \rightarrow V_0$ and R goes to 0 in $\text{Hom}_\Lambda(X_2, X_0)$.

Conversely, suppose that $A_2 \rightarrow_d A_1 \rightarrow_d A_0$ is a chain complex with $A_i \in \text{add } X_i$. Then we can expand A_i in the form $A_i = V_i \otimes_{K^{n_i}} X_i$ and the boundary maps are adjoint of maps α, β as above. Furthermore, the composition

$$\beta(\alpha \otimes id) : \alpha : V_2 \otimes_{K^{n_2}} D \text{Hom}_\Lambda(X_2, X_1) \otimes_{K^{n_1}} D \text{Hom}_\Lambda(X_1, X_0) \rightarrow V_0$$

factors through $V_2 \otimes_{K^{n_2}} DR$ since $d^2 = 0$. \square

From the above proof we see that the category \mathcal{C} can be described as follows.

Definition 3.4. The objects of \mathcal{C} are 3-term chain complexes $A_2 \rightarrow_d A_1 \rightarrow_d A_0$ of Λ -modules (with $d^2 = 0$) together with an explicit decomposition of each A_i in the form $A_i = V_i \otimes_{K^{n_i}} X_i$. The morphisms $A_* \rightarrow B_*$ are the chain maps $\{h_i : A_i \rightarrow B_i\}$ induced by K^{n_i} -linear maps $V_i \rightarrow W_i$ if $B_i = W_i \otimes_{K^{n_i}} X_i$.

We envision that this duality principle can be used in the following way. Suppose that we have a 3-stage resolution of a module M in $\text{mod-}\Lambda$:

$$0 \rightarrow A_2 \rightarrow A_1 \rightarrow A_0 \rightarrow M \rightarrow 0$$

where $A_i \in \text{add } X_i$. Then $A_* = (A_2 \rightarrow A_1 \rightarrow A_0) \in \mathcal{C}$. Suppose that A_* has a 2-stage resolution by a finite subcategory $\text{add } Y_*$ in \mathcal{C} (equivalently, the corresponding E -module has a 2-stage finite resolution in $\text{mod-}E$):

$$0 \rightarrow C_* \rightarrow B_* \rightarrow A_* \rightarrow 0$$

where $C_*, B_* \in \text{add } Y_*$ for some $Y_* \in \mathcal{C}$. Then the long exact sequence of this short exact sequence of chain complexes ends with:

$$\underbrace{H_1(A_*)}_{=0} \rightarrow \underbrace{H_0(C_*) \rightarrow H_0(B_*)}_{\in \text{add } H_0(Y_*)} \rightarrow \underbrace{H_0(A_*)}_{=M} \rightarrow 0$$

which gives a 2-stage resolution of M by the finite subcategory $\text{add } H_0(Y_*)$ in $\text{mod-}\Lambda$. This would imply that $\text{pd}_\Lambda M$ is bounded if finite. Thus we are reduced to the following problem:

Show that every right $E\Sigma$ -module W has a 2-stage resolution in a finite category.

Lemma 3.5. *Every right $E\Sigma$ -module has a 2-stage resolution in a finite category if and only if every right Σ -module has a 2-stage resolution in a finite category.*

Proof. Since $EE\Sigma = \Sigma$ it suffices to prove the implication in one direction. So suppose that W is an $E\Sigma$ -module. The quiver of $E\Sigma$ have vertices in degree 2,1,0 and all arrow of degree -1. Let $M \subseteq W$ be the submodule generated by $W_2 = WK^{n_2}$, the restriction of W to the vertices of degree 2. Then the quotient W/M has support at the vertices of degree 1,0 and is thus a module of projective dimension 1. Thus W has a 2-stage resolution in a finite category if and only if M has such a resolution.

Let P_2 be the projective cover of M . Then $Y = \Omega M = \ker(P_2 \rightarrow M)$ is the direct sum $Y = Z \amalg S$ where S is a projective semisimple module in degree 0 and Z is generated in degree 1. Again, M has a 2-stage

resolution in a finite category if and only if P_2/Z has such a resolution. Therefore, we may assume that $S = 0$. So, we have 3-stage resolution

$$0 \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow M \rightarrow 0$$

But the category of complexes $P_0 \rightarrow P_1 \rightarrow P_2$ is exactly equivalent to $\text{mod-}\Sigma$. \square

4. THE MONOMIAL CASE

We restrict to the special case when Σ is monomial. In that case, the exact sequence (3.1) is:

$$0 \rightarrow \underbrace{R}_{\text{zero paths}} \rightarrow K(\text{paths of length 2}) \rightarrow \underbrace{\Sigma_2}_{\text{nonzero paths}} \rightarrow 0$$

Koszul duality then just switched the zero paths with the nonzero paths. So, $E\Sigma$ is also monomial.

Being monomial implies that all projective modules are images of projective modules on rooted tree quivers with all arrows pointing away from the root. On the tree quivers, the projective modules have only a finite number of quotients with supports on subtrees of the tree. This gives a finite number of the quotients of projective $E\Sigma$ -modules which we call “tree modules”. Note that the radicals of the longer projective $E\Sigma$ -modules are direct sums of tree modules.

There are n_2 tree quivers T_2^k whose roots map to the vertices of degree 2. These give projective $E\Sigma$ -modules with Loevey length ≤ 3 . There are n_1 tree quivers T_1^j whose roots map to the vertices of degree 1 corresponding to the components X_1^j of X_1 . The number of leaves in the tree T_1^j is $d_j = \dim_K(\text{Hom}_\Lambda(X_1^j, X_0))$. Thus each leaf of the quiver T_1^j is an arrow in the quiver of $E\Sigma$ starting at vertex j . These arrows might have the same target.

Let Q be the direct sum of all indecomposable projective $E\Sigma$ -modules and all tree modules with tops in degree 1 as described above. For any $E\Sigma$ -module M , let $p : A \twoheadrightarrow M$ be the right Q -approximation. This is onto since Q is a generator.

Lemma 4.1. *The components of the kernel of $p : A \twoheadrightarrow M$ are either simple modules or isomorphic images of representations of the T_1^j 's as described above. In particular, if $d_j \leq 3$ for all j , M has a 2-stage resolution in a finite category.*

Proof. The kernel of $p : A \rightarrow M$ is equal to the kernel of the restriction of p to the submodules B, W of A, M giving by restricting the support to the vertices of degree 1,0.

For each j let $\widetilde{B}_j, \widetilde{M}_j = \widetilde{W}_j$ be the representation of the quiver T_1^j which maps onto the support of T_1^j in B, M, W and is universal with this property. For example, the top of \widetilde{W}_j is the representation W_j of W at vertex j and the value of \widetilde{W}_j at each other vertex α is equal to the image of W_j under the mapping $\alpha : W_j \rightarrow W_{i\alpha}$. This construction is functorial and the homomorphism $B \rightarrow W$ induces a mapping $\widetilde{B}_j \rightarrow \widetilde{W}_j$ for each j . Furthermore, since B is in *add* Q , $B = (\coprod \widetilde{B}_j) \coprod B_0$ where B_0 is the sum of the simple projective components of B which lie in degree 0. Let W_0 is defined similarly. Then the epimorphism $B \rightarrow W$ lifts to $\widetilde{W} = (\coprod \widetilde{W}_j) \coprod W_0$ by construction and the kernel of $q : \widetilde{W} \rightarrow W$ is a semi-simple projective module S in degree 0.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker p & \longrightarrow & B & \xrightarrow{p} & W \longrightarrow 0 \\ & & \downarrow & & \downarrow h & & \downarrow = \\ 0 & \longrightarrow & S & \longrightarrow & \widetilde{W} & \xrightarrow{q} & W \longrightarrow 0 \end{array}$$

The kernel of $h : B \rightarrow \widetilde{W}$ is the direct sum of the kernels of the maps $\widetilde{B}_j \rightarrow \widetilde{W}_j$ plus a semisimple projective module. Since \widetilde{B}_j and \widetilde{W}_j are representations of the Dynkin quiver T_1^j , this kernel lies in a finite category. Since S is semi-simple projective, $\ker p$ is a direct sum of $\ker h$ and a submodule of S and therefore also lies in a finite category. \square

This implies that $\text{pfd}(\mathcal{F}(\theta))$ is finite if θ is a stratifying system of size 3 which is monomial with the dimension restrictions given in the lemma. One of the correct ways to state this is the following.

Definition 4.2. We define a linear mapping $c : V \otimes W \rightarrow U$ to be *monomial* if there exists bases $\{v_k\}, \{w_j\}$ for V, W so that the kernel of c is generated by a subset of the set of monomials $\{v_i \otimes w_j\}$. We define a stratifying system θ of size 3 to be *monomial* if $\text{Ext}_\Lambda^1(\theta(1), \theta(3)) = 0$ and the Yoneda product

$$c : \text{Ext}_\Lambda^1(\theta(2), \theta(3)) \otimes \text{Ext}_\Lambda^1(\theta(1), \theta(2)) \rightarrow \text{Ext}_\Lambda^2(\theta(1), \theta(3))$$

is monomial.

Following a procedure similar to the discussion above we have the following construction.

Definition 4.3. Given any stratifying system θ of size 3, let $F = F(\theta)$ be the graded algebra $F = F_0 \coprod F_1 \coprod F_2$ defined as follows.

- (1) $F_0 = K^3 \subseteq \text{End}_\Lambda(\theta)$
- (2) $F_1 = D \text{Ext}_\Lambda^1(\theta(1), \theta(2)) \coprod D \text{Ext}_\Lambda^1(\theta(2), \theta(3))$

(3) $F_2 = DR$ where $R = \ker c$ is the kernel of the Yoneda product.

If $\text{Ext}_\Lambda^1(\theta(1), \theta(3)) = 0$ then an $F(\theta)$ -module V_* is equivalent to an object M of $\mathcal{F}(\theta)$ together with an explicit filtration

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq M_3 = 0$$

and isomorphisms of the quotients $M_{i-1}/M_i \cong V_i \otimes \theta(i)$. The modules $Q(i)$ of [6] correspond to projective modules over $F(\theta)$ and we get the following corollary.

Corollary 4.4. *If θ is a monomial stratifying system of size 3 and either $\text{Ext}_\Lambda^1(\theta(1), \theta(2))$ or $\text{Ext}_\Lambda^1(\theta(2), \theta(3))$ has dimension ≤ 3 then $\text{pfd}(\mathcal{F}(\theta)) < \infty$.*

5. FINITE FIELDS

“Reduction to the case of finite fields” which was in the abstract but not in the lecture refers to the following easy theorem.

Theorem 5.1. *For a fixed prime p and positive integer m , $\text{pfd}(\Lambda) < \infty$ for all finite dimensional algebras Λ of Loewey length $\leq m$ over $\overline{\mathbb{F}_p}$, the algebraic closure of the prime field of characteristic p , if and only if $\text{pfd}(A) < \infty$ for all basic algebras A of Loewey length $\leq m$ over all finite fields of characteristic p .*

This follows easily from the fact that

$$\text{pfd}(\Lambda) = \text{pfd}(\Lambda \otimes_K L)$$

for any field extension L of K . This in turn is very easy except in the inseparable case which is also not difficult.

If we assume that the ground field is finite, then indecomposable projective modules have only a finite number of quotient modules and we can hope to imitate the monomial relation case where we used the fact that there are only finitely many tree modules.

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