Hatcher handles and the rigidity conjecture

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Basic definition

- Given a smooth manifold bundle $F \to M \to B$ where F, M, B are smooth (C^∞) oriented closed manifolds,
- we want to find all smooth bundles $F \to M' \to B$ which are *fiberwise tangentially homeomorphic* to $p : M \to B$.
- $T^vM = ker(Tp : TM \to TB)$ is the *vertical tangent bundle* of $M \to B$.
- A fiberwise tangential homeomorphism is a homeomorphism $M' \to M$ over B together with an isomorphism of vector bundles $T^vM' \to T^vM$.
- $\mathcal{S}_B(M) =$ space of all such $M' \to B$.

Smoothing theory

Classical smoothing theory

- A smooth structure on a topological manifold is equivalent to a vector bundle structure on the “tangent microbundle.”
- M', M tangentially homeomorphic $\Rightarrow$ diffeomorphic.
- But M', M are not *fiberwise diffeomorphic* over B.

Fiberwise smoothing

- $\mathcal{S}_B(M) =$ space of all such $M' \to B$ fiberwise tangentially homeomorphic to $M \to B$
- $\mathcal{S}_B^s(M) = colim\, \mathcal{S}_B(M \times D^N) =$ space of stable smooth structures on $M \to B$.

Dwyer-Weiss-Williams smoothing theory

Proposition

$\mathcal{S}_B^s(M)$ is an infinite loop space $\Rightarrow \pi_0\mathcal{S}_B^s(M)$ is an abelian group.

Theorem

$\pi_0\mathcal{S}_B^s(M) \otimes \mathbb{R} \cong \bigoplus_{k>0} H_{q-4k}(M)$ (coef in $\mathbb{R}$) where $q = \dim B$.

Definition

Let Θ_M be the composition

$$\Theta : \pi_0\mathcal{S}_B(M) \to \pi_0\mathcal{S}_B^s(M) \to \bigoplus_{k>0} H_{q-4k}(M)$$

We call $\Theta_M(M')$ the relative smooth structure class of M'

Rigidity conjecture

Conjecture

If the fiber of $M \to B$ is a closed even dimensional manifold then $\Theta_M = 0$. In other words, no rational stable smooth structures can be realized.

Theorem (Bismut-Lott)

The higher analytic torsion of these bundles is always zero when defined.

Realization theorem

Sebastian Goette and I used "positive and negative Hatcher handles" to construct virtually all elements of $\pi_0\mathcal{S}_B^s(M)$. We call our construction the *Arc de Triomph* (AdT) construction.

Theorem (Goette-I)

Given any smooth bundle $M \to B$ whose fiber dimension is odd and $\geq 2q+3$, the AdT construction produces smooth bundles $M' \to B$ which are fiberwise tangentially homeomorphic to M and whose relative smooth structure classes span $\bigoplus_{k>0} H_{q-4k}(M)$. In fact, the images of these M' in $\pi_0\mathcal{S}_B^s(M)$ form a subgroup of finite index.

Hatcher's construction

- Take a map $\xi : B \to G/O$.
- This is an n-plane bundle over B together with a fiberwise homotopy equivalence:

$$S^{n-1}(\xi) \simeq S^{n-1} \times B$$

- Take the fiberwise mapping cone and thicken it up to form a smooth disk bundle $E(\xi) \to B$ where

$$E(\xi) = D^n(\xi) \times D^m \cup S^{n-1} \times D^{m+1} \times B$$

Theorem

The set of all $E(\xi)$ forms a subgroup of $\mathcal{S}^s_B(B)$ of finite index.

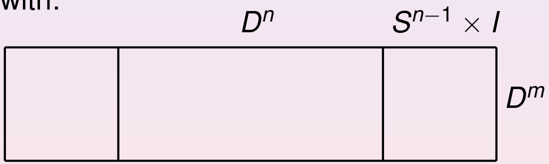
Hatcher's disk bundle

The fibers of $E(\xi)$ looks like this (drawing for $n = 1$)

Hatcher's disk bundle

The fibers of $E(\xi)$ looks like this (drawing for $n = 1$)

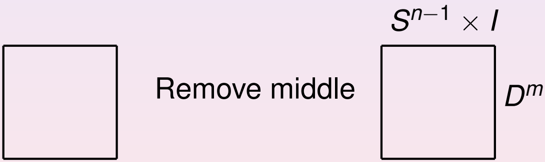
Start with:



$D^n \times D^m$

Hatcher's disk bundle

The fibers of $E(\xi)$ looks like this (drawing for $n = 1$)

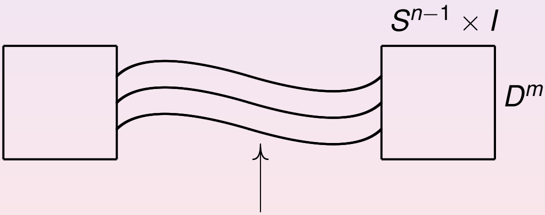


$$S^{n-1} \times I \times D^m$$

Hatcher's disk bundle

The fibers of $E(\xi)$ looks like this (drawing for $n = 1$)

$E(\xi)$:



Insert: $D^n(\xi) \times D^m$

positive Hatcher handle

A positive Hatcher handle is

$$B(\xi) = E(\xi) \times I$$

positive Hatcher handle

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$E(\xi)$

positive Hatcher handle

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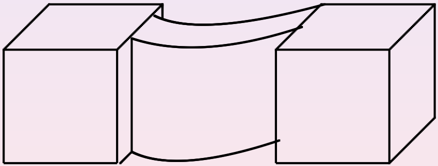
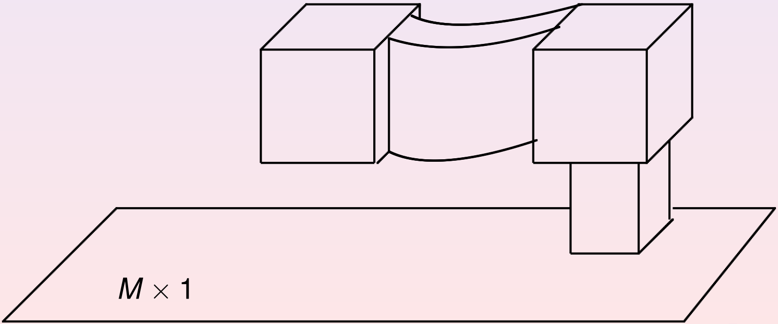


positive Hatcher handle

A positive Hatcher handle is

$$B(\xi) = E(\xi) \times I$$

attached on the top $M \times 1$ of a thickened bundle $M \times I$



negative Hatcher handle

This is $A(\xi) = E(\xi) \times [0,1] \cup D^n \times D^m \times [1,2]$ attached on the top $M \times 1$ of $M \times I$ along an embedding $E(\xi) \times 0 \to M \times 1$.

negative Hatcher handle

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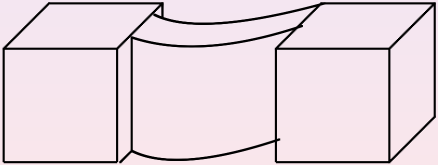
$E(\xi) \times [0,1]$



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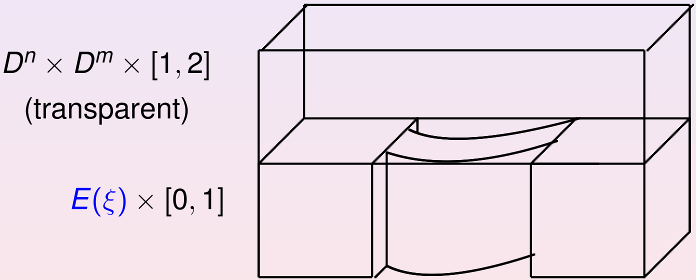

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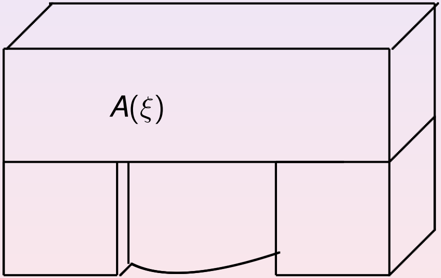

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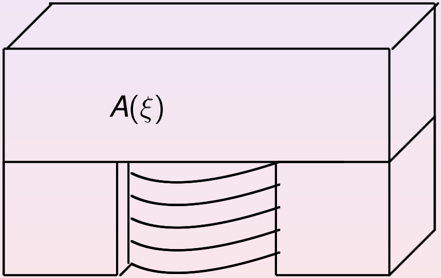

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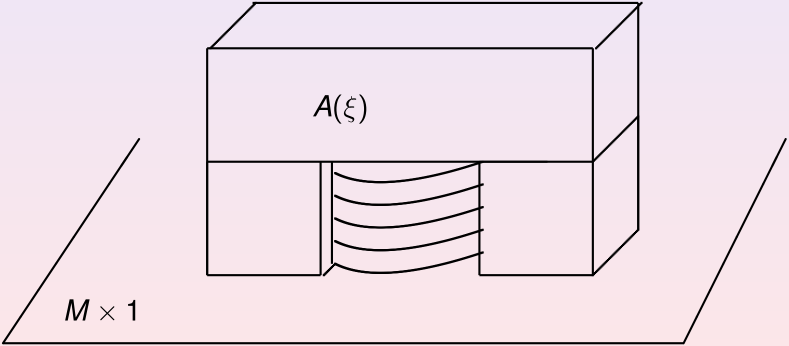
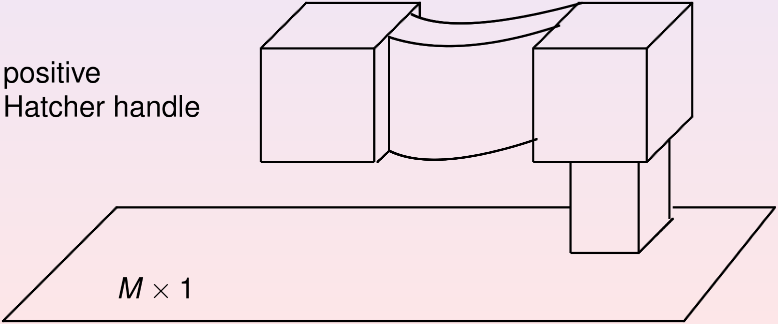


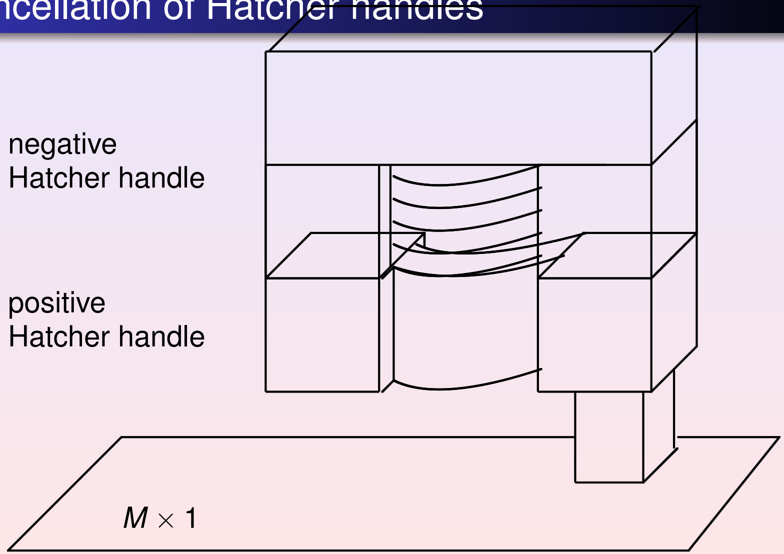
cancellation of Hatcher handles

cancellation of Hatcher handles

positive
Hatcher handle



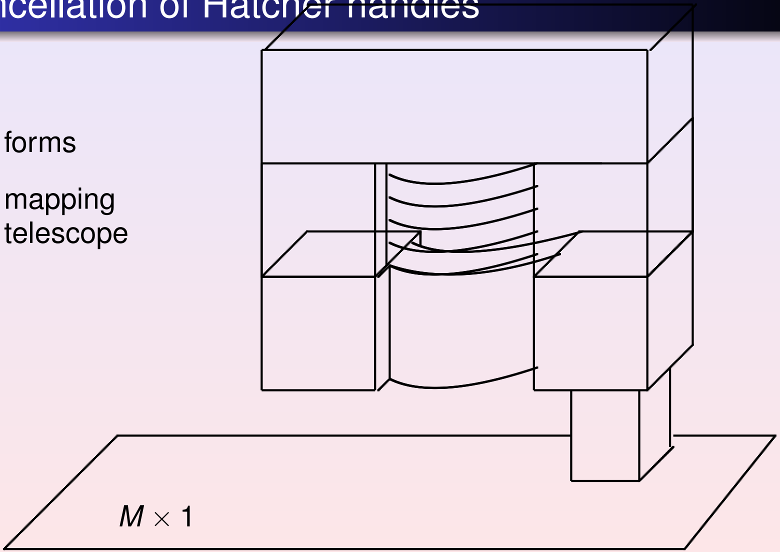
cancellation of Hatcher handles



cancellation of Hatcher handles

forms

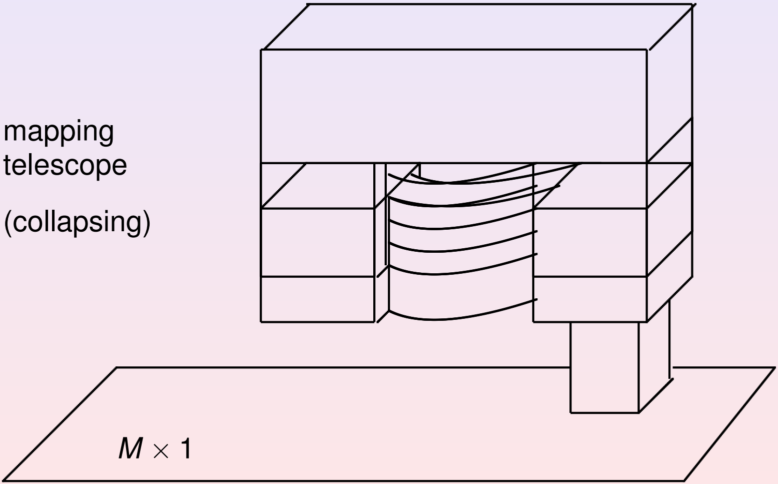
mapping telescope



cancellation of Hatcher handles

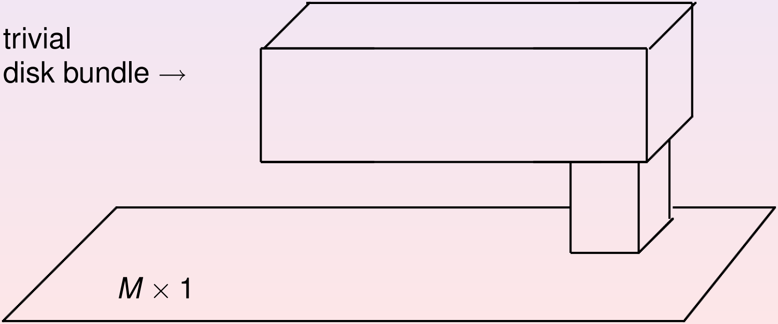
mapping
telescope

(collapsing)



cancellation of Hatcher handles

trivial
disk bundle →



$M \times 1$

cancellation of Hatcher handles

trivial
disk bundle
collapses

cancellation of Hatcher handles

$$M \times I \cup B(\xi) \cup A(\xi) \;\cong\; M \times I$$

Proof of realization theorem

Proof.

Place positive and negative Hatcher handles on the top $M \times 1$ of $M \times I$ is a suitable way. The new top boundary is fiberwise tangentially homeomorphic to M. When the fiber dimension is odd, this process constructs virtually all smooth structures on M. □

Remark

When the fiber dimension is even, the same procedure does not give any rationally nontrivial smooth structures. (We get only 2-torsion elements of $\pi_0 \mathcal{S}_B^s(M)$).

What is higher torsion?

- Under certain conditions (B simply connected is sufficient), a smooth bundle $M \to B$ gives a real cohomology class

$$\tau(M) \in \bigoplus_{k>0} H^{4k}(B)$$

- If $\tau(M) \neq \tau(M')$ then M, M' are not fiberwise diffeomorphic.
- For Hatcher's disk bundle we get the chern character of ξ

$$\tau(E(\xi)) = \sum_{k>0} (-1)^k \zeta(2k+1) \frac{1}{2} ch_{4k}(\xi \otimes \mathbb{C})$$

Definitions for higher torsion

There are three different definitions of higher Reidemeister torsion:

- Bismut-Lott higher analytic torsion
- I-Klein higher torsion defined using Morse theory
- Dwyer-Weiss-Williams higher torsion defined using homotopy theory

Theorem (Badzioch-Dorabiala-Klein-Williams)

DWW-torsion = IK-torsion.

Theorem (S.Goette)

BL-torsion = IK-torsion.

formula for higher torsion

Theorem

The relative higher Reidemeister torsion

$$\tau(M', M) \in \bigoplus_{k>0} H^{4k}(B)$$

is always defined ($\tau(M', M) = \tau(M') - \tau(M)$ *when RHS is defined) and equal to the Poicaré dual of the image of the smooth structure class:*

$$\tau(M', M) = p_*\left(\Theta_M(M')\right) \in \bigoplus_{k>0} H_{q-4k}(B)$$

even dimensional fibers

Theorem

When $p : M \to B$ has closed even dimensional fibers,

$$\tau(M', M) = 0$$

Corollary

The following composition is trivial in the even case:

$$\pi_0 \mathcal{S}_B(M) \to \pi_0 \mathcal{S}_B^s(M) \otimes \mathbb{R} \cong \bigoplus_{k>0} H_{q-4k}(M) \xrightarrow{p_*} \bigoplus_{k>0} H_{q-4k}(B)$$

$\Rightarrow$ *counterexamples to rigidity conjecture lie in ker p_*.*