

Non-abelian Reciprocity Laws on a Riemann Surface

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Abstract

On a Riemann surface there are relations among the periods of holomorphic differential forms, called Riemann's relations. If one looks carefully in Riemann's proof, one notices that he uses iterated integrals. What I have done in this paper is to generalize these relations to relations among generating series of iterated integrals. Since it is formulated in terms of generating series, it gives infinitely many relations - one for each coefficient of the generating series. The degree one term gives that the sum of the residues of a differential form is zero. The degree 2 term gives Riemann's relations. The new result is reciprocity for the higher degree terms, which give non-trivial relations among iterated integrals on a Riemann surface. We give explicit formulas for degree 3 terms. Also we refine the definition of Manin's non-commutative modular symbols in order to include Eisenstein series.

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0 Introduction

There are several known reciprocity laws on a Riemann surface. For example Weil reciprocity, Riemann relations and reciprocity of two differential forms of third kind. One can think of them as abelian reciprocity laws. In this paper we construct certain type of non-abelian reciprocity laws. They are written in terms of iterated integrals of differential forms of third kind. We use iterated integrals over a path, invented by Chen [Ch].

If one looks carefully in Riemann’s proof of Riemann’s relations, one notices that he uses iterated integrals. What I have done in this paper is to generalize these relations to relations among generating series of iterated integrals. Since it is formulated in terms of generating series, it gives infinitely many relations - one for each coefficient of the generating series. The degree one term gives that the sum of the residues of a differential form is zero. The degree 2 term gives Riemann’s relations. The new result is reciprocity for the higher degree terms, which give non-trivial relations among iterated integrals on a Riemann surface. After constructing the relation among the generating series of iterated integrals on a Riemann surface, we examine in details relations, coming from terms of degree 3 in the generating series.

We describe basic properties of iterated integrals over a path that we need for simplification of iterated integrals and for an analogue of a residue of such integrals. (One can find other analytic consideration of iterated integrals in Goncharov’s paper [G].) We show that the iterated integrals over a path that we consider are invariant under homotopic deformation of the path of integration. Then we construct the generating series of iterated integrals as a solution of an ordinary differential equation, following an idea of Manin [M]. For each closed loop on a Riemann surface, we consider the generating series of the iterated integrals. On a Riemann surface there is ”good” choice of generators of the fundamental group, with only one relation. For each of the generators of the fundamental group we consider the generating series of iterated integrals. Composition of paths corresponds to a composition of generating series of iterated integrals. The relation among the generators in the fundamental group gives us a relation among the generating series of iterated integrals. We call this relation a reciprocity law. This is the key idea in the construction of the reciprocity law.

Before we formulate the final statement of the reciprocity law, we consider several simplifications. The first one is for a simple loop around a pole of a differential form. We present constructively the process of taking residues of iterating integrals, which we generalize to taking residues of generating series of iterated integrals. Another simplification that we make is for commutator of α_i and β_i loops on a Riemann surface. When all this is done we formulate the reciprocity law.

In the last section we apply the developed ideas about non-commutative reciprocity law to Manin’s non-commutative modular symbol. We are able to extend his construction

so that we can include Eisenstein series.

Where is the non-abelian group in the reciprocity law? An iterated integral over a loop depends on the loop. Since iterated integrals are homotopy invariant, we have that an iterated integral over a loop depends only on the element of the fundamental group that this loop represents. How much do iterated integrals distinguish elements of the fundamental group? For the Riemann sphere without a divisor iterated integrals capture the pro-unipotent completion of the fundamental group (see [DG]). I conjecture that iterated integrals of differential forms of third kind on a Riemann surface capture precisely the pro-unipotent part of the fundamental group. For some progress and intuition in this direction one might look at the following papers [G], [Q], and [DG]. By unipotent group we mean that the lower central series of the group stabilizes to the trivial group. For a group G . Let $G_1 = G$ and let $G_{n+1} = [G, G_n]$. Then the series $\{G_n\}$ is called lower central series. The groups G/G_{n+1} is the (n -th) unipotent group. And inverse limit with respect to n

$$\varprojlim G/G_n$$

is called the pro-unipotent completion of G . Note that G/G_2 is the maximal abelian quotient of G . If $G_3 \neq G_2$ then G/G_n is non-abelian for $n \geq 3$. This is always the case for G being a non-abelian fundamental group of a Riemann surface without a divisor.

Denote by $\bar{\pi}_1(X - D, P)$ the quotient of the fundamental group of a Riemann surface X without a divisor D , which is the largest quotient of $\pi_1(X - D, P)$, whose elements can be distinguished by an iterated integral. Where does the non-abelian group $\bar{\pi}_1(X - D, P)$ act? Since the elements of the group $\bar{\pi}_1(X - D, P)$ can be distinguished by an iterated integral, we have that this group is the differential Galois group of the differential equation whose solution is a generating series of iterated integrals.

I would like to make one remark about the reciprocity laws in this paper. Instead of working on a Riemann surface, one can consider a smooth algebraic curve over the algebraic closure of the rational numbers. Using the construction in [G] and [DG] we can conclude that the iterated integrals of algebraic differential forms with logarithmic poles over a path give periods in the sense of algebraic geometry. So the reciprocity laws that we describe in this paper are relations among periods.

We are going to use explicit integrals. However, in a more theoretical approach, one can consider framed mixed Hodge structures associated to the integrals that we consider. The process of taking residues of iterated integrals corresponds to taking co-product of the corresponding framed mixed Hodge structure.

Acknowledgments: I would like to thank Yuri Manin for his inspiring talk on non-commutative modular symbol. I would like to thank also Alexander Goncharov for teaching me iterated integrals. And finally, I would like to thank Glen Stevens for raising the question about non-commutative modular symbols involving Eisenstein series.

1 Background on iterated integrals

1.1 Definition of iterated integrals

Definition 1.1 Let $\omega_1, \dots, \omega_n$ be holomorphic 1-forms on a simply connected open subset U of the complex plane \mathbb{C} . Let

$$\gamma : [0, 1] \rightarrow U$$

be a path. We define an iterated integral of the forms $\omega_1, \dots, \omega_n$ over the path γ to be

$$\int_{\gamma} \omega_1 \circ \dots \circ \omega_n = \int \dots \int_{0 \leq t_1 \leq \dots \leq t_n \leq 1} \gamma^* \omega_1(t_1) \wedge \dots \wedge \gamma^* \omega_n(t_n).$$

It is called iterated because it can be defined inductively by

$$\int_{\gamma} \omega_1 \circ \dots \circ \omega_n = \int_0^1 \left(\int_{\gamma|_{[0,t]}} \omega_1 \circ \dots \circ \omega_{n-1} \right) \gamma^* \omega_n(t).$$

1.2 Homotopy invariance of iterated integrals

Theorem 1.2 *Let $\omega_1, \dots, \omega_n$ be holomorphic 1-forms on a simply connected open subset U of the complex plane \mathbb{C} . Let*

$$H : [0, 1] \times [0, 1] \rightarrow U$$

be a homotopy, fixing the end points, of paths

$$\gamma_s : [0, 1] \rightarrow U$$

such that $\gamma_s(t) = H(s, t)$, and for fixed s , we have a path $\gamma_s : [0, 1] \rightarrow U$. Then

$$\int_{\gamma_s} \omega_1 \circ \dots \circ \omega_n$$

is independent of s .

Proof. We will prove it by induction on n . For $n = 1$ it is obvious. Suppose it is true for $n - 1$. We are going to verify it for n .

Since U is simply connected open subset of \mathbb{C} , it is contractible. Therefore, every 1-form ω_i is exact. Let $\omega_i = df_i$, where f_i is a holomorphic function on U . We have

$$\int_{\gamma_s} \omega_1 \circ \dots \circ \omega_n = \int_{0 < t_1 < \dots < t_n < 1} \gamma_s^* df_1(t_1) \wedge \dots \wedge \gamma_s^* df_n(t_n)$$

Note that

$$df_1(z_1) \wedge \dots \wedge df_n(z_n)$$

is an exact form on U .

Using Stokes theorem, we make the following computation

$$\begin{aligned}
0 &= \int_{0 < s < 1, 0 < t_1 < \dots < t_n < 1} d(\gamma_s^* df_1(t_1) \wedge \dots \wedge \gamma_s^* df_n(t_n)) = \\
&= \int_{0 < t_1 < \dots < t_n < 1} \gamma_1^* df_1(t_1) \wedge \dots \wedge \gamma_1^* df_n(t_n) - \\
&\quad - \int_{0 < t_1 < \dots < t_n < 1} \gamma_0^* df_1(t_1) \wedge \dots \wedge \gamma_0^* df_n(t_n) + \\
&\quad + \int_{0 < s < 1, 0 = t_1 < t_2 < \dots < t_n < 1} d(\gamma_s^* df_2(t_2) \wedge \dots \wedge \gamma_s^* df_n(t_n)) + \\
&\quad + \sum_{i=1}^{n-1} (-1)^i \int_{0 < s < 1, 0 < t_1 < \dots < t_i = t_{i+1} < \dots < t_n < 1} d(\gamma_s^* df_1(t_1) \wedge \dots \\
&\quad \dots \wedge \gamma_s^*(f_i(z_i) df_{i+1}(z_i) - f_{i+1}(z_i) df_i(z_i)) \wedge \dots \wedge \gamma_s^* df_n(t_n)) + \\
&\quad + (-1)^n \int_{0 < s < 1, 0 < t_1 < \dots < t_{n-1} < t_n = 1} d(\gamma_s^* df_1(t_1) \wedge \dots \wedge \gamma_s^* df_n(t_{n-1})) = \\
&= \int_{0 < t_1 < \dots < t_n < 1} \gamma_1^* df_1(t_1) \wedge \dots \wedge \gamma_1^* df_n(t_n) - \\
&\quad - \int_{0 < t_1 < \dots < t_n < 1} \gamma_0^* df_1(t_1) \wedge \dots \wedge \gamma_0^* df_n(t_n).
\end{aligned}$$

The first equality follows from the fact that

$$df_1(z_1) \wedge \dots \wedge df_n(z_n)$$

is an exact form on U^n . For the second equality, consider γ as a two dimensional membrane, sending (s, t) to $\gamma_s(t)$.

$$\gamma^* df_1(s, t_1) \wedge \dots \wedge \gamma^* df_n(s, t_n)$$

restricted to

$$\{(s, t_1, \dots, t_n) | 0 < s < 1, 0 < t_1 < \dots < t_i = t_{i+1} < \dots < t_n < 1\}$$

gives

$$\gamma^* df_1(s, t_1) \wedge \dots \wedge \gamma^*(f_i df_{i+1} - f_{i+1} df_i) \wedge \dots \wedge \gamma^* df_n(s, t_n),$$

where

$$\gamma^*(f_i df_{i+1} - f_{i+1} df_i)$$

depends on s and t_i . And the third equality uses the assumption that iteration of $n - 1$ 1-forms is homotopy invariant.

1.3 Differential equation

When we consider an iterated integral, we can let the end point vary in a small neighborhood. Then the iterated integral becomes an analytic function.

Let $\omega_1, \dots, \omega_n$ be differentials of 3rd kind on a Riemann surface X . Consider the differential equation

$$dF = F \sum_{i=1}^n A_i \omega_i,$$

where A_1, \dots, A_n are non-commuting formal variables. Let P be a point of X such that none of the differential forms has a pole at P . It is easy to check that the function

$$F(z) = 1 + \sum_i (A_i \int_P^z \omega_i) + \sum_{i,j} A_i A_j \int_P^z \omega_i \circ \omega_j + \sum_{i,j,k} A_i A_j A_k \int_P^z \omega_i \circ \omega_j \circ \omega_k + \dots$$

is a solution to the differential equation with initial condition at P , $F(P) = 1$. The summation continues so that every iterated integral in the given n 1-forms is present in the summation. Note that

$$d \int_P^z \omega_i \circ \dots \circ \omega_j \circ \omega_k = \left(\int_P^z \omega_i \circ \dots \circ \omega_j \right) \omega_k.$$

1.4 Multiplication formulas

We can take a path γ from P to z . We denote the solution of the differential equation by F_γ . If γ_1 is a path that ends at Q and γ_2 is a path that starts at Q we can compose them. Denote the composition by $\gamma_1 \gamma_2$.

Theorem 1.3 (*Composition of paths*) *With the above notation, we have*

$$F_{\gamma_1} F_{\gamma_2} = F_{\gamma_1 \gamma_2}.$$

Proof. This property holds for any ordinary differential equation of first order. In terms of iterated integrals, we are going to prove a multiplication formula, which leads to the above formula for solutions of the differential equation.

Corollary 1.4 (*Composition of paths*) *Let $\omega_1, \dots, \omega_n$ be differential forms, some of them could repeat. Let also γ_1 be a path that ends at Q and γ_2 be a path that starts at Q . We can compose them. Denote the composition by $\gamma_1 \gamma_2$. Then*

$$\int_{\gamma_1 \gamma_2} \omega_1 \circ \dots \circ \omega_n = \sum_{i=0}^n \int_{\gamma_1} \omega_1 \circ \dots \circ \omega_i \int_{\gamma_2} \omega_{i+1} \circ \dots \circ \omega_n,$$

where for $i = 0$ we define $\int_{\gamma_1} \omega_1 \circ \dots \circ \omega_i = 1$, and similarly, for $i = n$ we define $\int_{\gamma_2} \omega_{i+1} \circ \dots \circ \omega_n = 1$.

Proof. It follows by comparing the coefficients next to a monomial on the left hand side and on the right hand side in the equation in theorem 1.3.

An alternative proof can be done combinatorially, involving the simplices, over which we integrate.

1.5 Shuffle relations

Definition 1.5 Denote by $Sh(m, n)$ the shuffles, which are permutations τ of the set $\{1, \dots, m, m+1, \dots, m+n\}$ such that

$$\tau(1) < \tau(2) < \dots < \tau(m)$$

and

$$\tau(m+1) < \tau(m+2) < \dots < \tau(m+n).$$

Theorem 1.6 (*Shuffle relation*) Let $\omega_1, \dots, \omega_m, \omega_{m+1}, \dots, \omega_{m+n}$ be differential 1-forms, some of them could repeat. Let also γ be a path that does not pass through any of the poles of the given differential forms. Then

$$\int_{\gamma} \omega_1 \circ \dots \circ \omega_m \int_{\gamma} \omega_{m+1} \circ \dots \circ \omega_{m+n} = \sum_{\tau \in Sh(m,n)} \int_{\gamma} \omega_{\tau(1)} \circ \omega_{\tau(2)} \circ \dots \circ \omega_{\tau(m+n)}.$$

Proof. From the definition of an iterated integral, we know that an m -fold iteration can be expressed as an integral over an m dimensional simplex, where the coordinates are ordered in increasing order. If we multiply one such integral with an iterated integral defined by n differential forms then the product is an integral over the domain defined by the product of the two simplices. In order to obtain the product as a sum of iterated integrals, we insert inequalities between the coordinates of the product of the two simplices. By inserting inequalities between the coordinates, we obtain an iterated integral. In order of the equality to hold, we have to take a sum over all possible ways of inserting inequalities between the coordinates.

1.6 Reversing the path

Lemma 1.7 (*Reversing the path*) Let γ be a path. Let γ^{-1} be the same path but going to the opposite direction. Then

$$\int_{\gamma^{-1}} \omega_1 \circ \omega_2 \circ \dots \circ \omega_n = (-1)^n \int_{\gamma} \omega_n \circ \omega_{n-1} \circ \dots \circ \omega_1.$$

Proof. Change the parameter t on $[0, 1]$ of γ to $1 - s$. Then s gives a parametrization of γ^{-1} . Using the definition of an iterated integral as an integral over a simplex of an n -form, we see that changing t to $1 - s$, gives a multiple of $(-1)^n$ times the iterated integral defined using the variable s . That iterated integral is precisely the integrals defined by γ^{-1} .

2 Non-abelian reciprocity law on a Riemann surface

This section is the heart of the article. It continuous with the topic of iterated integrals. However, now this is done in the direction of building the reciprocity law. It is more technical than the previous section. And it ends with the statement of the non-abelian reciprocity law on Riemann surfaces.

2.1 Iterated integrals over a loop around a pole

Let U be an open simply connected subset of the complex plane. We can assume that 0 belongs to U . Let $f_1 dz, \dots, f_n dz$ be holomorphic differentials on U . We are going to iterate these differential forms together with the form dz/z . Let P be a point in U different from 0. And let σ be a simple loop in U that starts and ends at P and goes around 0 once in a counterclockwise direction. We can assume that σ does not intersect itself. Let γ_{ϵ} be a path that starts at P and ends at ϵ for a point $\epsilon \neq 0$ in U very close to 0. For convenience we take ϵ to be a positive real number. Let γ be a path starting from P and ending at 0, which is the limit of γ_{ϵ} , when ϵ tends to zero. We define also σ_{ϵ} to be

a loop that starts and ends at ϵ and goes around 0 once in a counterclockwise direction along a circle of radius ϵ . We can deform σ homotopic in $U - \{0\}$ to $\gamma_\epsilon \sigma_\epsilon \gamma_\epsilon^{-1}$. Note that iterated integrals are invariant with respect to homotopic deformation of the path of integration (see theorem 1.2). We are going to use corollary 1.4 for the composition of the paths $\gamma_\epsilon \sigma_\epsilon \gamma_\epsilon^{-1}$.

Lemma 2.1 *With the above notation*

$$\int_{\sigma_\epsilon} \frac{dz}{z} \circ \dots \circ \frac{dz}{z} = \frac{(2\pi i)^r}{r!},$$

where we iterate the form dz/dz with itself r -times

Proof. We can take the following parametrization of z along σ_ϵ . Let $z = \epsilon e^{2\pi i t}$ for $0 \leq t \leq 1$. Then $dz/z = 2\pi i dt$. Therefore,

$$\int_{\sigma_\epsilon} \frac{dz}{z} \circ \dots \circ \frac{dz}{z} = (2\pi i)^r \int_0^1 dt \circ \dots \circ dt = \frac{(2\pi i)^r}{r!}.$$

Corollary 2.2 *If $\omega_1, \dots, \omega_r$ are holomorphic forms on $U - \{0\}$ with simple poles at 0 then*

$$\int_{\sigma_\epsilon} \omega_1 \circ \dots \circ \omega_r = \frac{1}{r!} \prod_{i=1}^r \text{Res}_0 \omega_i.$$

Lemma 2.3 *Consider an iterated integral over σ_ϵ of the differential forms*

$$\frac{dz}{z}, f_1 dz, \dots, f_n dz$$

in various order with possible repetition, so that not all of them need to appear in the integral. If at least one of the holomorphic differentials $f_1 dz, \dots, f_n dz$ on U is present in the integral then the limit of the iterated integral, as ϵ goes to 0, is 0.

Proof. Expand the holomorphic differential forms around $z = 0$. Consider the following parametrization of the variable z along the path σ_ϵ : $z = \epsilon e^{2\pi i t}$ for $0 \leq t \leq 1$. It is enough to prove the statement of the lemma for $f_i(z) = z^{n_i}$. (The general statement will follow since we have a uniform convergence of the power series of f_i on a compact subset of U .) Note that $dz/z = 2\pi i dt$ and $z^n dz = 2\pi i \epsilon^{n+1} e^{2\pi i(n+1)t} dt$ for some $n \geq 0$. Consider the parametrization in terms of t for $0 \leq t \leq s$, where s is close to 1. Then the iterated integral over $0 \leq t \leq s$ becomes

$$C \epsilon^N e^{2\pi i N s} g(s),$$

where C is a constant $N > 0$ and $g(t)$ is a polynomial in t . So the limit as s approaches 1 will be $C \epsilon^N g(1)$. So the iterated integral has value $C \epsilon^N g(1)$. Finally, this value approaches zero as ϵ tends to zero, which proves the lemma.

Lemma 2.4 *Let $f_1 dz, \dots, f_n dz$ be holomorphic forms on $U - \{0\}$. Let f_i and f_{i+1} be also holomorphic at 0. Denote by ω^{or} r -fold iteration of ω . Then*

$$\lim_{\epsilon \rightarrow 0} \sum_{j=0}^r \int_{\gamma_\epsilon} f_1 dz \circ \dots \circ f_i dz \circ \left(\frac{dz}{z}\right)^{\circ j} \int_{\gamma_\epsilon^{-1}} \left(\frac{dz}{z}\right)^{\circ(r-j)} \circ f_{i+1} dz \circ \dots \circ f_n dz = 0.$$

The limit is still zero in the cases when the set $\{f_1 dz, \dots, f_i dz\}$ is empty and/or the set $\{f_{i+1} dz, \dots, f_n dz\}$ is empty.

Proof. Consider the integral

$$\int_{\gamma_\epsilon} f_1 dz \circ \dots \circ f_i dz$$

as a function of ϵ . It is a sum of terms of the type a constant times $\epsilon^k \log^l(\epsilon)$. Since f_i is holomorphic, we have that if $l \geq 0$ then $k \geq 0$. Also if $l > 0$ then $k > 0$, because the integral is convergent for $\epsilon = 0$. When ϵ tends to zero $\epsilon^k \log^l(\epsilon)$ tend to zero. Now, consider the integral

$$\int_{\gamma_\epsilon} f_1 dz \circ \dots \circ f_i dz \circ \left(\frac{dz}{z}\right)^{\circ j}.$$

As a function of ϵ it is a sum of terms of the type constant times $\epsilon^k \log^{l+j}(\epsilon)$ for $k \geq 0$ and $l \geq 0$. From the previous considerations if $l > 0$ then $k > 0$ and the term tends to zero as ϵ approaches zero. Thus, the only term that does not tend to zero as ϵ approaches zero is the constant term of

$$\int_{\gamma_\epsilon} f_1 dz \circ \dots \circ f_i dz$$

times $\log^j(\epsilon)$. Then

$$\lim_{\epsilon \rightarrow 0} \int_{\gamma_\epsilon} f_1 dz \circ \dots \circ f_i dz \circ \left(\frac{dz}{z}\right)^{\circ j} - \frac{1}{j!} \log^j(\epsilon) \int_{\gamma} f_1 dz \circ \dots \circ f_i dz = 0.$$

For the other integral in this lemma we use

$$\int_{\gamma_\epsilon^{-1}} \left(\frac{dz}{z}\right)^{\circ(r-j)} \circ f_{i+1} dz \circ \dots \circ f_n dz = (-1)^{n-i+r-j} \int_{\gamma_\epsilon} f_n dz \circ \dots \circ f_{i+1} dz \circ \left(\frac{dz}{z}\right)^{\circ(r-j)}.$$

Using the same arguments as in the beginning of the proof we find that

$$\lim_{\epsilon \rightarrow 0} \int_{\gamma_\epsilon} f_n dz \circ \dots \circ f_{i+1} dz \circ \left(\frac{dz}{z}\right)^{\circ(r-j)} - \frac{1}{(r-j)!} \log^{r-j}(\epsilon) \int_{\gamma} f_n dz \circ \dots \circ f_{i+1} dz = 0.$$

Note also that

$$(-1)^{n-i+r-j} \int_{\gamma} f_n dz \circ \dots \circ f_{i+1} dz = (-1)^{r-j} \int_{\gamma^{-1}} f_{i+1} dz \circ \dots \circ f_n dz.$$

Therefore,

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \sum_{j=0}^r \int_{\gamma_\epsilon} f_1 dz \circ \dots \circ f_i dz \circ \left(\frac{dz}{z}\right)^{\circ j} \int_{\gamma_\epsilon^{-1}} \left(\frac{dz}{z}\right)^{\circ(r-j)} \circ f_{i+1} dz \circ \dots \circ f_n dz = \\ & = \int_{\gamma} f_1 dz \circ \dots \circ f_i dz \int_{\gamma^{-1}} f_{i+1} dz \circ \dots \circ f_n dz \sum_{j=0}^r \lim_{\epsilon \rightarrow 0} \frac{1}{j!} (-1)^{r-j} \frac{1}{(r-j)!} \log^r(\epsilon). \end{aligned}$$

Finally, we have

$$\sum_{j=0}^r \frac{1}{j!} \log^j(\epsilon) (-1)^{r-j} \frac{1}{(r-j)!} \log^{r-j}(\epsilon) = 0,$$

using binomial coefficients, after multiplying by $r!$. Thus, the limit in the lemma is zero.

Now we are ready to give the general algorithm for expressing an iterated integral over a loop σ around 0 in terms of integrals over a path γ starting at P and ending at 0 and residues at $z = 0$.

Theorem 2.5 Let $\omega_1, \dots, \omega_n$ be holomorphic forms on U . Let i_1, \dots, i_m be integers such that

$$0 \leq i_1 < i_2 < \dots < i_m \leq n.$$

Let j_1, \dots, j_m be positive integers. Then

$$\begin{aligned} & \int_{\sigma} \omega_1 \circ \dots \circ \omega_{i_1} \circ \left(\frac{dz}{z}\right)^{\circ j_1} \circ \omega_{i_1+1} \circ \dots \circ \omega_{i_2} \circ \left(\frac{dz}{z}\right)^{\circ j_2} \circ \omega_{i_2+1} \circ \dots \\ & \dots \circ \omega_{i_m} \circ \left(\frac{dz}{z}\right)^{\circ j_m} \circ \omega_{i_m+1} \circ \dots \circ \omega_n = \\ = & \sum_{k=1}^m \frac{(2\pi i)^{j_k}}{j_k!} \int_{\gamma} \omega_1 \circ \dots \circ \omega_{i_1} \circ \dots \circ \left(\frac{dz}{z}\right)^{\circ j_{k-1}} \circ \omega_{i_{k-1}+1} \circ \dots \circ \omega_{i_k} \times \\ & \times \int_{\gamma^{-1}} \omega_{i_k+1} \circ \dots \circ \omega_{i_{k+1}} \circ \left(\frac{dz}{z}\right)^{\circ j_{k+1}} \circ \omega_{i_{k+1}+1} \circ \dots \circ \omega_{i_m} \circ \left(\frac{dz}{z}\right)^{\circ j_m} \circ \omega_{i_m+1} \dots \circ \omega_n. \end{aligned}$$

Proof. First we use that σ is homotopic to $\gamma_{\epsilon} \sigma_{\epsilon} \gamma_{\epsilon}^{-1}$. Then we use the formula for composition of paths in corollary 1.4, expressing the integral over σ in terms of a sum of products of an integral over γ_{ϵ} , an integral over σ_{ϵ} and an integral over γ_{ϵ}^{-1} . Consider the portion of this sum, where there are no differential forms integrated over σ_{ϵ} . The sum of all such terms is zero because by corollary 1.4 this is the same as the integral over $\gamma_{\epsilon} \gamma_{\epsilon}^{-1}$, which is homotopic to the constant loop at P .

Next, we examine what possible iterated integrals over σ_{ϵ} we can have. If we have a holomorphic differential form at zero in the iterated integral over σ_{ϵ} then, by lemma 1.9, the value of the integral tends to zero as ϵ approaches zero. Therefore, it is enough to consider only iterations of dz/z over σ_{ϵ} .

Consider iterated integrals of dz/z over σ_{ϵ} . If the corresponding iterated integral over γ ends with dz/z and/or the corresponding iterated integral over γ_{ϵ}^{-1} starts with dz/z then by lemma 2.3 the sum of all such integrals tends to zero as ϵ approaches zero.

Therefore, the only terms we have to consider are the ones in the theorem.

2.2 Generating series of iterated integrals over a loop around a pole

Consider n differential 1-forms with simple poles $\omega_1, \dots, \omega_n$, defined on an open and simply connected set $U \subset \mathbb{C}$. Let

$$dF = F \sum_{i=1}^n A_i \omega_i$$

be a type of differential equation that we considered in the beginning, where A_1, \dots, A_n are non-commuting formal variables. Let $Q \in U$ be a point where at least one of the differential forms has a pole. Let $P \in U$ be a point, which is not a pole for any of the differential forms. Consider a simple loop σ with the following properties: it starts at P ; it does not self-intersect; it bounds a region V homeomorphic to a disk; and the only pole of the differential forms that lies inside V is Q .

Let us simplify the solution F_{σ} , using theorem 2.5. Define γ to be the path, starting at P and ending at Q , that sits in the region V , bounded by σ . Using theorem 2.5, we can decompose F_{σ} . Let F_{γ}^{reg} be regularization of F_{γ} , which contains only the iterated integrals over the path γ , whose iteration does not end with a differential form that has

a pole at P . The series F_γ contains the summand 1 and also all the iterated integrals mentioned above times the corresponding non-commuting variables. Similarly, $F_{\gamma^{-1}}^{reg}$ is regularization of $F_{\gamma^{-1}}$, which contains only the iterated integrals over the path γ , whose iteration does not start with a differential form that has a pole at Q , where γ^{-1} is the reversed path of γ .

Lemma 2.6 *With the notation in this subsection, we have*

$$(F_\gamma^{reg})^{-1} = F_{\gamma^{-1}}^{reg}.$$

Proof. It follows from lemma 1.7.

We call F_σ^{Res} the residual part of F_σ , defined by the portion of F_σ that contains only the iterated integrals of differential forms that have a pole at Q . In particular, F_σ^{Res} does contain the constant 1. Let

$$F_\sigma^{Res+} = -1 + F_\sigma^{Res}$$

be the residual part without the constant term 1. Recall, Q is a point, where at least one of the differential forms has a pole. Then we have the following version of theorem 2.5 in terms of generating series.

Theorem 2.7 *With the notation from this subsection, we have*

$$F_\sigma = 1 + F_\gamma^{reg} F_\sigma^{Res+} F_{\gamma^{-1}}^{reg}.$$

Proof. From theorem 1.12 we have

$$F_\sigma = F_\gamma^{reg} F_\sigma^{Res} F_{\gamma^{-1}}^{reg}.$$

We simplify the right hand side, using lemma 2.6

$$F_\gamma^{reg} F_{\gamma^{-1}}^{reg} = 1.$$

$$F_\gamma^{reg} F_\sigma^{Res} F_{\gamma^{-1}}^{reg} = F_\sigma = F_\gamma^{reg} (1 + F_\sigma^{Res+}) F_{\gamma^{-1}}^{reg} = 1 + F_\gamma^{reg} F_\sigma^{Res+} F_{\gamma^{-1}}^{reg}.$$

2.3 Generating series over α and β cycles of a Riemann surface

Let X be a Riemann surface of genus g . Let $\omega_1, \dots, \omega_n$ be differential forms of third kind on X . Let also $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ be loops on X starting at P , which do not pass through a pole of any of the differential forms, such that they generate the fundamental group $\pi_1(X, P)$ with only one relation

$$[\alpha_1, \beta_1] \dots [\alpha_g, \beta_g] = 1.$$

We used the notation $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$.

We are going to simplify

$$F_{[\alpha, \beta]} = F_\alpha F_\beta F_{\alpha^{-1}} F_{\beta^{-1}}.$$

Note that if we take the constant term 1 from F_β then we will have

$$F_\alpha F_{\alpha^{-1}} F_{\beta^{-1}} = F_{\beta^{-1}},$$

which follows from the homotopy invariance of $F_{\alpha\alpha^{-1}}$ (theorem 1.2). In order to capture such a cancelation, we define for a path γ

$$F_\gamma^+ = F_\gamma - 1.$$

Lemma 2.8 *With the above notation*

$$F_{[\alpha,\beta]} = 1 + F_\beta^+ F_{\alpha^{-1}}^+ - F_\alpha^+ F_{\beta^{-1}}^+ + F_\alpha^+ F_\beta^+ F_{\alpha^{-1}}^+ + F_\beta^+ F_{\alpha^{-1}}^+ F_{\beta^{-1}}^+ + F_\alpha^+ F_\beta^+ F_{\alpha^{-1}}^+ F_{\beta^{-1}}^+.$$

Proof. We are going to use many times $F_\gamma^+ = F_\gamma - 1$. We have

$$\begin{aligned} F_{[\alpha,\beta]} &= F_\alpha F_\beta F_{\alpha^{-1}} F_{\beta^{-1}} = F_\alpha F_\beta^+ F_{\alpha^{-1}} F_{\beta^{-1}} + F_\alpha F_{\alpha^{-1}} F_{\beta^{-1}} = \\ &= F_\alpha F_\beta^+ F_{\alpha^{-1}} F_{\beta^{-1}} + F_{\beta^{-1}} = F_\alpha F_\beta^+ F_{\alpha^{-1}}^+ F_{\beta^{-1}} + F_\alpha F_\beta^+ F_{\beta^{-1}} + F_{\beta^{-1}} = \\ &= F_\alpha F_\beta^+ F_{\alpha^{-1}}^+ F_{\beta^{-1}} + F_\alpha F_\beta F_{\beta^{-1}} - F_\alpha F_{\beta^{-1}} + F_{\beta^{-1}} = \\ &= F_\alpha F_\beta^+ F_{\alpha^{-1}}^+ F_{\beta^{-1}} + (F_\alpha - F_\alpha F_{\beta^{-1}} + F_{\beta^{-1}} - 1) + 1 = \\ &= F_\alpha F_\beta^+ F_{\alpha^{-1}}^+ F_{\beta^{-1}} - F_\alpha^+ F_{\beta^{-1}}^+ + 1. \end{aligned}$$

Also

$$\begin{aligned} F_\alpha F_\beta^+ F_{\alpha^{-1}}^+ F_{\beta^{-1}} &= F_\alpha^+ F_\beta^+ F_{\alpha^{-1}}^+ F_{\beta^{-1}} + F_\beta^+ F_{\alpha^{-1}}^+ F_{\beta^{-1}} = \\ &= F_\alpha^+ F_\beta^+ F_{\alpha^{-1}}^+ F_{\beta^{-1}} + F_\alpha^+ F_\beta^+ F_{\alpha^{-1}}^+ + F_\beta^+ F_{\alpha^{-1}}^+ F_{\beta^{-1}} + F_\beta^+ F_{\alpha^{-1}}^+. \end{aligned}$$

From these two sequences of equalities the lemma follows.

2.4 Non-abelian reciprocity law

Let $\omega_1 \dots \omega_n$ be differential forms of third kind on a Riemann surface X . Let Y be the open subset of X obtained by removing the poles of $\omega_1 \dots \omega_n$. Consider the differential equation on Y

$$dF = F \left(\sum_{i=1}^n A_i \omega_i \right),$$

where A_i for $i = 1, \dots, n$ are non-commuting formal variables. Fix a point P in Y . Let

$$\gamma : [0, 1] \rightarrow Y$$

be a piecewise smooth path that starts at P and ends at z . Then the solution of the differential equation with initial condition $F(P) = 1$ is

$$F_\gamma = 1 + \sum_{i=1}^n A_i \int_0^1 \gamma^* \omega_i + \sum_{i,j=1}^n A_i A_j \int_0^1 \gamma^* \omega_i \circ \gamma^* \omega_j + \dots$$

If γ_1 and γ_2 are two paths such that the end point of γ_1 is the beginning point of γ_2 . Let, $\gamma_1 \gamma_2$ be the composition of the two paths. Then, $F_{\gamma_1 \gamma_2} = F_{\gamma_1} F_{\gamma_2}$ (theorem 1.2).

Consider a simple loop σ_i in Y with the following properties: it starts at P ; it does not self-intersect; it bounds a open region V_i homeomorphic to a punctured disk; and the only pole of the differential forms that lies inside the closure of V_i is P_i . Let γ_i be a path with the following properties: γ_i lies in the closure of V_i ; γ_i starts at P and ends

at P_i . We can choose the loops σ_i so that in the counterclockwise we start with loops around points of the poles of ω_1 . Then it continues with loops around the points of the poles of ω_2 , which are not poles of ω_1 , then it continues with loops around the poles of ω_3 , which are not poles of ω_1 or ω_2 and so on. Call these loops $\sigma_1, \dots, \sigma_N$. We assume that σ_i bounds a disk - the closure of V_i , containing only one point P_i from the poles of the differential forms. Let also α_i and β_i be loops on Y for $i = 1, \dots, g$, where g is the genus of Y . We can choose such that α_i and β_i so that α_i and β_i for $i = 1, \dots, g$ and $\sigma_1, \dots, \sigma_N$ generate $\pi_1(Y)$ and the only relation between them is

$$\sigma_1 \dots \sigma_N [\alpha_1, \beta_1] \dots [a_g, b_g] = 1$$

We define a pro-unipotent tame symbol to be F_σ where σ is one of the loops that we have defined above. Note that we have made some choices of loops. If we multiply all the loops going in counterclockwise direction, then we will obtain a loop homotopic to the zero loop at P . This gives the global reciprocity law for pro-unipotent tame symbols after simplification that uses section 1. We can write this reciprocity law in the following way. Let $\sigma_1, \dots, \sigma_N$ in this order be the above loops counted in counterclockwise direction. Consider F_{σ_i} as an element of the formal power series $\mathbb{C} \ll A_1, \dots, A_n \gg$.

In the formulation of the pro-unipotent reciprocity law, we are going to use the notation from this section and from subsections 2.2 and 2.3.

Theorem 2.9 *The non-abelian reciprocity law on a Riemann surface is*

$$\begin{aligned} & \left(\prod_{i=1}^N 1 + F_{\gamma_i}^{reg} F_{\sigma_i}^{Res} F_{\gamma_i^{-1}}^{reg} \right) \times \\ & \times \left(\prod_{j=1}^g 1 + F_{\beta_j}^+ F_{\alpha_j}^+ - F_{\alpha_j}^+ F_{\beta_j}^+ + F_{\alpha_j}^+ F_{\beta_j}^+ F_{\alpha_j}^+ + F_{\beta_j}^+ F_{\alpha_j}^+ F_{\beta_j}^+ + F_{\alpha_j}^+ F_{\beta_j}^+ F_{\alpha_j}^+ F_{\beta_j}^+ \right) = 1. \end{aligned}$$

Proof. For the solutions of this differential equation we have that for two loops σ and τ starting at the same point P we have that $F_\sigma F_\tau = F_{\sigma\tau}$ (see theorem 1.3). The composition

$$\sigma_1 \dots \sigma_N [\alpha_1, \beta_1] \dots [\alpha_g, \beta_g]$$

is homotopic to the trivial loop at P . Therefore,

$$F_{\sigma_1} \dots F_{\sigma_N} F_{[\alpha_1, \beta_1]} \dots F_{[\alpha_g, \beta_g]} = 1.$$

Using lemma 2.6 in subsection 2.2, we have

$$F_{\sigma_i} = 1 + F_{\gamma_i}^{reg} F_{\sigma_i}^{Res} F_{\gamma_i^{-1}}^{reg}.$$

From theorem 2.7 from subsection 2.3, we have

$$F_{[\alpha_j, \beta_j]} = 1 + F_{\beta_j}^+ F_{\alpha_j}^+ - F_{\alpha_j}^+ F_{\beta_j}^+ + F_{\alpha_j}^+ F_{\beta_j}^+ F_{\alpha_j}^+ + F_{\beta_j}^+ F_{\alpha_j}^+ F_{\beta_j}^+ + F_{\alpha_j}^+ F_{\beta_j}^+ F_{\alpha_j}^+ F_{\beta_j}^+.$$

The following two lemmas are useful for explicit computations, which we are going to consider in section 4.

Lemma 2.10 *The coefficient next to the linear terms in the formal variables in*

$$\prod_{i=1}^N F_{\sigma_i}$$

is zero.

Proof. It is enough to prove the lemma for the coefficient A_1 . The coefficient next to A_1 is

$$\sum_{i=1}^N \int_{\sigma_i} \omega_1.$$

The sum is zero because it is equal to the sum of the residues of ω_1 .

Lemma 2.11 *The coefficient next to A , B or C in*

$$F_{[\alpha_l, \beta_l]}$$

is zero for every $l = 1, \dots, g$.

Proof. From lemma 2.8 we see that there is no term in degree 1 in the formal variables.

3 Reciprocity laws for differentials of third kind

Let ω_1 and ω_2 be differential forms of third kind on a Riemann surface X of genus at least $g \geq 1$. Consider the differential equation

$$dF = F(A\omega_1 + B\omega_2),$$

where A and B are non-commuting formal variables. Let P be point on X , which is not a pole for ω_1 and ω_2 . Assume that there are no common poles between ω_1 and ω_2 .

Let γ is a path starting at P and ending at Q . Let $F_\gamma = F(Q)$ be the solution of the differential equation with initial conditions $F(P) = 1$ solved along the path γ and $F(Q)$ is the evaluation of that solution at the point Q .

Let P_1, \dots, P_p be the poles of ω_1 . Let Q_1, \dots, Q_q be the poles of ω_2 . Let $Y = X - \{P_1, \dots, P_p, Q_1, \dots, Q_q\}$. Consider a simple loop σ_{P_i} in Y with the following properties: it starts at P ; it does not self-intersect; it bounds a open region V_{P_i} homeomorphic to a punctured disk; and the only pole of the differential forms that lies inside the closure of V_{P_i} is P_i . We can choose these loops so that they do not intersect each other except at the point P . Call these loops $\sigma_{P_1}, \dots, \sigma_{P_p}, \sigma_{Q_1}, \dots, \sigma_{Q_q}$. We can choose the loops so that in the counterclockwise we start with loops $\sigma_{P_1}, \dots, \sigma_{P_p}$ around the poles of ω_1 in this order, followed by the loops $\sigma_{Q_1}, \dots, \sigma_{Q_q}$ around the poles of ω_2 . Let also α_l and β_l be loops on Y for $l = 1, \dots, g$, where g is the genus of Y . We can choose α_l and β_l so that α_l and β_l for $l = 1, \dots, g$ and $\sigma_{P_1}, \dots, \sigma_{P_p}, \sigma_{Q_1}, \dots, \sigma_{Q_q}$ generate $\pi_1(Y, P)$ and the only relation between them is

$$\sigma_{P_1} \dots \sigma_{P_p} \sigma_{Q_1} \dots \sigma_{Q_q} [\alpha_1, \beta_1] \dots [\alpha_g, \beta_g] = 1.$$

Let γ_{P_i} be a path from P to P_i that lies inside the disk bound by σ_{P_i} . Similarly, let γ_{Q_j} be a path from P to Q_j that lies inside the disk bound by σ_{Q_j} .

Let σ be any of the above loops σ_{P_i} , σ_{Q_j} , α_k or b_k . Consider the tame symbol F_σ .

Consider the coefficients contributing to AB in the reciprocity law associated to the differential equation

$$dF = F(A\omega_1 + B\omega_2).$$

Theorem 3.1 *Let ω_1 and ω_2 be two differential forms of third kind without common poles. Then with the above notation we have a reciprocity law for differential forms of third kind*

$$\begin{aligned} & \sum_{i=1}^p \text{Res}_{P_i} \omega_1 \int_{\gamma_{P_i}^{-1}} \omega_2 + \sum_{j=1}^q \text{Res}_{Q_j} \omega_2 \int_{\gamma_{Q_j}} \omega_1 + \\ & + \sum_{k=1}^g (\int_{\alpha_k} \omega_1 \int_{\beta_k} \omega_2 - \int_{\beta_k} \omega_1 \int_{\alpha_k} \omega_2) = \\ & = 0. \end{aligned}$$

Proof. We have that

$$F_{\sigma_{P_1} \dots \sigma_{P_p} \sigma_{Q_1} \dots \sigma_{Q_q} [\alpha_1, \beta_1] \dots [\alpha_g, \beta_g]} = 1.$$

From lemma 2.10 and lemma 2.11 we have that there is no linear terms in A or B in $F_{\sigma_{P_1} \dots \sigma_{P_p} \sigma_{Q_1} \dots \sigma_{Q_q}}$ and in $F_{[\alpha_1, \beta_1] \dots [\alpha_g, \beta_g]}$. Using theorem 2.9, lemma 2.8 and lemma 2.7, we obtain that the coefficient next to AB in $F_{\sigma_{P_i}}$ is

$$\text{Res}_{P_i} \omega_1 \int_{\gamma_{P_i}^{-1}} \omega_2.$$

Similarly, the coefficient next to AB in $F_{\sigma_{Q_j}}$ is

$$\text{Res}_{Q_j} \omega_2 \int_{\gamma_{Q_j}} \omega_1.$$

And finally, the coefficient next to AB in $F_{[\alpha_k, \beta_k]}$ is

$$\int_{\alpha_k} \omega_1 \int_{\beta_k} \omega_2 - \int_{\beta_k} \omega_1 \int_{\alpha_k} \omega_2.$$

When the two forms do not have a common pole we obtain a reciprocity law for differential forms of third kind (see [GH]).

This also implies Weil reciprocity law.

Corollary 3.2 *(Weil reciprocity) With the above notation, consider $\omega_1 = df/f$ and $\omega_2 = dg/g$, where f and g are meromorphic functions on the Riemann surface with disjoint divisors. Let the corresponding divisors (f) and (g) be*

$$(f) = \sum_{i=1}^p a_i P_i$$

and

$$(g) = \sum_{j=1}^q b_j Q_j.$$

Then

$$\prod_{i=1}^p g(P_i)^{-a_i} \prod_{j=1}^q f(Q_j)^{b_j} = 1.$$

Proof. Consider the above theorem with $\omega_1 = df/f$ and $\omega_2 = dg/g$. Use the simplification up to a summand of $2\pi i$ of

$$\int_{\gamma_{P_i}^{-1}} \frac{dg}{g} = \log(g(P)) - \log(g(P_i)).$$

Similarly, up to a summand of $2\pi i$, we have

$$\int_{\gamma_{Q_j}} \frac{df}{f} = \log(f(Q_j)) - \log(f(P)).$$

When we divide by $2\pi i$ and exponentiate we obtain

$$\prod_{i=1}^p g(P_i)^{-a_i} \prod_{j=1}^q f(Q_j)^{b_j}.$$

It is enough to show that the remaining terms contribute a multiple of $(2\pi i)^2$. Also,

$$\int_{\alpha_k} \frac{df}{f}$$

gives a multiple of $2\pi i$. Indeed, consider f as a function

$$f : X \rightarrow \mathbb{C}P^1.$$

Then

$$\int_{\alpha_k} \frac{df}{f} = \int_{\alpha_k} f^* \frac{dz}{z} = \int_{f_* \alpha_k} \frac{dz}{z},$$

which is a multiple of $2\pi i$.

4 Non-abelian reciprocity laws involving three differential forms

Consider the differential equation

$$dF = F(A\omega_1 + B\omega_2 + C\omega_3),$$

where A , B and C are non-commuting formal variables. Assume ω_1 , ω_2 and ω_3 have distinct poles. Let P be point on X , which is not a pole for ω_1 , ω_2 or ω_3 .

Let γ is a path starting at P and ending at Q . Let $F_\gamma = F(Q)$ be the solution of the differential equation with initial conditions $F(P) = 1$ solved along the path γ and $F(Q)$ is the evaluation of that solution at the point Q .

Let P_1, \dots, P_p be the poles of ω_1 , Q_1, \dots, Q_q be the poles of ω_2 and R_1, \dots, R_r be the poles of ω_3 . Let $Y = X - \{P_1, \dots, P_p, Q_1, \dots, Q_q, R_1, \dots, R_r\}$. Consider a simple loop σ_{P_i} in Y with the following properties: it starts at P ; it does not self-intersect; it bounds a open region V_{P_i} homeomorphic to a punctured disk; and the only pole of the differential forms that lies inside the closure of V_{P_i} is P_i . We can choose these loops so that they do not intersect each other except at the point P . Call these loops

$\sigma_{P_1}, \dots, \sigma_{P_p}, \sigma_{Q_1}, \dots, \sigma_{Q_q}, \sigma_{R_1}, \dots, \sigma_{R_r}$. We can choose the loops so that in the counter-clockwise we start with loops $\sigma_{P_1}, \dots, \sigma_{P_p}$ around the poles of ω_1 in this order, followed by the loops $\sigma_{Q_1}, \dots, \sigma_{Q_q}$ around the poles of ω_2 , followed by the loops $\sigma_{R_1}, \dots, \sigma_{R_r}$ around the poles of ω_3 . Let also α_l and β_l be loops on Y for $l = 1, \dots, g$, where g is the genus of Y . We can choose α_l and β_l so that α_l and β_l for $l = 1, \dots, g$ and $\sigma_{P_1}, \dots, \sigma_{P_p}, \sigma_{Q_1}, \dots, \sigma_{Q_q}, \sigma_{R_1}, \dots, \sigma_{R_r}$ generate $\pi_1(Y, P)$ and the only relation between them is

$$\sigma_{P_1} \dots \sigma_{P_p} \sigma_{Q_1} \dots \sigma_{Q_q} \sigma_{R_1} \dots \sigma_{R_r} [\alpha_1, \beta_1] \dots [\alpha_g, \beta_g] = 1$$

Let γ_{P_i} be a path from P to P_i that lies inside the punctured disk V_{P_i} bound by σ_{P_i} . Similarly, let γ_{Q_j} be a path from P to Q_j that lies inside the punctured disk V_{Q_j} bound by σ_{Q_j} and let γ_{R_k} be a path from P to R_k that lies inside the punctured disk V_{R_k} bound by σ_{R_k} .

Let σ be any of the above loops $\sigma_{P_i}, \sigma_{Q_j}, \sigma_{R_k}, \alpha_l$ or β_l . Consider the tame symbol F_σ . For them we have the following reciprocity.

Theorem 4.1 *Using the above notation, for three differential forms of third kind, with disjoint poles, we have*

$$\begin{aligned} & \sum_{i=1}^p \text{Res}_{P_i} \omega_1 \int_{\gamma_{P_i}^{-1}} \omega_2 \circ \omega_3 + \sum_{j=1}^q \text{Res}_{Q_j} \omega_2 \int_{\gamma_{Q_j}^{-1}} \omega_1 \int_{\gamma_{Q_j}} \omega_3 + \\ & \sum_{k=1}^r \text{Res}_{R_k} \omega_3 \int_{\gamma_{R_k}} \omega_1 \circ \omega_2 + \\ & + \sum_{l=1}^g (\int_{\alpha_l} \omega_1 \circ \omega_2 \int_{\beta_l} \omega_3 - \int_{\beta_l} \omega_1 \circ \omega_2 \int_{\alpha_l} \omega_3 + \\ & + \int_{\alpha_l} \omega_3 \circ \omega_2 \int_{\beta_l} \omega_1 - \int_{\beta_l} \omega_3 \circ \omega_2 \int_{\alpha_l} \omega_1 - \\ & - \int_{\alpha_l} \omega_1 \int_{\beta_l} \omega_2 \int_{\alpha_l} \omega_3 + \int_{\beta_l} \omega_1 \int_{\alpha_l} \omega_2 \int_{\beta_l} \omega_3) = 0. \end{aligned}$$

Proof.

Consider only the coefficients of the tame symbol F_σ next to the A, B, C, AB, BC and ABC terms in the non-commuting power series in A, B and C .

The coefficient next to ABC in $F_{\sigma_{P_i}}$ is

$$\int_{\sigma_{P_i}} \omega_1 \circ \omega_2 \circ \omega_3 = \text{Res}_{P_i} \omega_1 \int_{\gamma_{P_i}^{-1}} \omega_2 \circ \omega_3.$$

Let $\sigma_1 = \sigma_{P_1} \sigma_{P_2} \dots \sigma_{P_p}$ be a product of loops that go around the poles of ω_1 . Then the coefficient next to ABC in $F_{\sigma_1} = F_{\sigma_{P_1}} \dots F_{\sigma_{P_p}}$ is

$$\int_{\sigma_1} \omega_1 \circ \omega_2 \circ \omega_3 = \sum_{i=1}^p \text{Res}_{P_i} \omega_1 \int_{\gamma_{P_i}^{-1}} \omega_2 \circ \omega_3.$$

Also the coefficient next to A in F_{σ_1} is equal to the sum of the residues of ω_1 , which is zero. Also the coefficient next to B and C in F_{σ_1} is zero because the disks that σ_1 bounds do not contain any of the poles of ω_2 or ω_3 .

Let $\sigma_3 = \sigma_{R_1} \sigma_{R_2} \dots \sigma_{R_r}$ be the product of the loops around the poles of ω_3 . Consider the coefficients next to A, B and C in F_{σ_3} . The coefficient next to A and B is zero because the loop does not bound any poles of ω_1 or ω_2 . The coefficient next to B is

zero because the sum of the residues of ω_3 is zero. Similarly to the commutation in the previous paragraph the coefficient next to ABC in F_{σ_3} is

$$\int_{\sigma_3} \omega_1 \circ \omega_2 \circ \omega_3 = \sum_{k=1}^r \text{Res}_{R_k} \omega_3 \int_{\gamma_{R_k}} \omega_1 \circ \omega_2.$$

Let $\sigma_2 = \sigma_{Q_1} \sigma_{Q_2} \dots \sigma_{Q_q}$ be the product of the loops around the poles of ω_2 . Consider the coefficients next to A , B and C in F_{σ_2} . The coefficient next to A and C is zero because the loop does not bound any poles of ω_1 or ω_3 . The coefficient next to B is zero because the sum of the residues of ω_2 is zero.

The coefficient next to ABC in F_{σ_2} is

$$\int_{\sigma_2} \omega_1 \circ \omega_2 \circ \omega_3 = \sum_{j=1}^q \text{Res}_{Q_j} \omega_2 \int_{\gamma_{Q_j}^{-1}} \omega_1 \int_{\gamma_{Q_j}} \omega_3.$$

Therefore, the coefficient next to ABC in $F_{\sigma_1 \sigma_2 \sigma_3}$ is

$$\sum_{i=1}^p \text{Res}_{P_i} \omega_1 \int_{\gamma_{P_i}^{-1}} \omega_2 \circ \omega_3 + \sum_{j=1}^q \text{Res}_{Q_j} \omega_2 \int_{\gamma_{Q_j}^{-1}} \omega_1 \int_{\gamma_{Q_j}} \omega_3 + \sum_{k=1}^r \text{Res}_{R_k} \omega_3 \int_{\gamma_{R_k}} \omega_1 \circ \omega_2.$$

Consider the coefficients next A , B , C and ABC of $F_{[\alpha_i, \beta_i]}$. Using lemma 2.8, we notice that the coefficient next to A , B or C in $F_{[\alpha_i, \beta_i]}$ is zero. Also, the linear coefficients next to F_{σ_1} , F_{σ_2} and F_{σ_3} are zero. Therefore, there will be any contribution from the quadratic terms in A , B and C in the series $F_{[\alpha_i, \beta_i]}$, since it has to be multiplies by a linear term, which is zero. From the reciprocity law, theorem 2.9, we have consider only the coefficient next to ABC of. We only need to consider the sum of the coefficients next to ABC in $F_{[\alpha_i, \beta_i]}$, F_{σ_1} , F_{σ_2} and F_{σ_3} , because the linear terms are zero. By lemma 1.5 (see also theorem 2.9), we have that the coefficient next to ABC in $F_{[\alpha_i, \beta_i]}$ is

$$\begin{aligned} & \int_{\beta_i} \omega_1 \int_{\alpha_i^{-1}} \omega_2 \circ \omega_3 + \int_{\beta_i} \omega_1 \circ \omega_2 \int_{\alpha_i^{-1}} \omega_3 - \\ & - \int_{\alpha_i} \omega_1 \int_{\beta_i^{-1}} \omega_2 \circ \omega_3 - \int_{\alpha_i} \omega_1 \circ \omega_2 \int_{\beta_i^{-1}} \omega_3 + \\ & + \int_{\alpha_i} \omega_1 \int_{\beta_i^{-1}} \omega_2 \int_{\alpha_i^{-1}} \omega_3 + \int_{\beta_i} \omega_1 \int_{\alpha_i^{-1}} \omega_2 \int_{\beta_i^{-1}} \omega_3 \end{aligned}$$

Using lemma 1.7 for reversing a path of the above iterated integrals, we finish the proof of the theorem.

Consider the differential equation

$$F = F(A\omega_1 + B\omega_2),$$

where A and B are non-commuting formal variables. Assume ω_1 and ω_2 have distinct poles. Let P be point on X , which is not a pole for ω_1 or ω_2 .

Let γ is a path starting at P and ending at Q . Let $F_\gamma = F(Q)$ be the solution of the differential equation with initial conditions $F(P) = 1$ solved along the path γ and $F(Q)$ is the evaluation of that solution at the point Q .

Let P_1, \dots, P_p be the poles of ω_1 and Q_1, \dots, Q_q be the poles of ω_2 . Let $Y = X - \{P_1, \dots, P_p, Q_1, \dots, Q_q\}$. Consider a simple loop σ_{P_i} in Y with the following properties: it starts at P ; it does not self-intersect; it bounds a open region V_{P_i} homeomorphic to

a punctured disk; and the only pole of the differential forms that lies inside the closure of V_{P_i} is P_i . We can choose these loops so that they do not intersect each other except at the point P . Call these loops $\sigma_{P_1}, \dots, \sigma_{P_p}, \sigma_{Q_1}, \dots, \sigma_{Q_q}$. We can choose the loops so that in the counterclockwise we start with loops $\sigma_{P_1}, \dots, \sigma_{P_p}$ around the poles of ω_1 in this order, followed by the loops $\sigma_{Q_1}, \dots, \sigma_{Q_q}$ around the poles of ω_2 . Let also α_l and β_l be loops on Y for $l = 1, \dots, g$, where g is the genus of Y . We can choose α_l and β_l so that α_l and β_l for $l = 1, \dots, g$ and $\sigma_{P_1}, \dots, \sigma_{P_p}, \sigma_{Q_1}, \dots, \sigma_{Q_q}$ generate $\pi_1(Y, P)$ and the only relation between them is

$$\sigma_{P_1} \dots \sigma_{P_p} \sigma_{Q_1} \dots \sigma_{Q_q} [\alpha_1, \beta_1] \dots [\alpha_g, \beta_g] = 1$$

Let γ_{P_i} be a path from P to P_i that lies inside the disk bound by σ_{P_i} . Similarly, let γ_{Q_j} be a path from P to Q_j that lies inside the disk bound by σ_{Q_j} .

Let σ be any of the above loops $\sigma_{P_i}, \sigma_{Q_j}, \sigma_{R_k}, \alpha_l$ or β_l . Consider the tame symbol F_σ . For them we have the following reciprocities

Theorem 4.2 *Using the above notation, for two differential forms of third kind, with disjoint poles, we have*

$$\begin{aligned} & -\frac{1}{2} \sum_{i=1}^p \text{Res}_{P_i}^2 \omega_1 \int_{\gamma_{P_i}} \omega_2 + \frac{1}{2} \sum_{j=1}^q \text{Res}_{Q_j} \omega_2 (\int_{\gamma_{Q_j}} \omega_1)^2 + \\ & + \sum_{l=1}^g (\int_{\alpha_l} \omega_1 \int_{\beta_l} \omega_1 \circ \omega_2 - \int_{\beta_l} \omega_1 \int_{\alpha_l} \omega_1 \circ \omega_2 - \\ & - \int_{\alpha_l} \omega_1 \circ \omega_1 \int_{\beta_l} \omega_2 - \int_{\beta_l} \omega_1 \circ \omega_1 \int_{\alpha_l} \omega_2 = 0. \end{aligned}$$

Theorem 4.3 *Using the above notation, for two differential forms of third kind, with disjoint poles, we have*

$$\begin{aligned} & \sum_{i=1}^p 2 \text{Res}_{P_i} \omega_1 \int_{\gamma_{P_i}} \omega_1 \circ \omega_2 - \sum_{j=1}^q \text{Res}_{Q_j} \omega_2 (\int_{\gamma_{Q_j}} \omega_1)^2 + \\ & + 2 \sum_{l=1}^g (\int_{\alpha_l} \omega_1 \circ \omega_2 \int_{\beta_l} \omega_1 - \int_{\beta_l} \omega_1 \circ \omega_2 \int_{\alpha_l} \omega_1 = 0. \end{aligned}$$

The proofs of theorems 4.2 and 4.3 are similar to the proof of theorem 4.1. For theorem 4.2 one examines the contributions to the coefficient in front of A^2B . And for theorem 4.3 one examines the contributions to the coefficient next to ABA .

Remark 4.4 Theorems 4.1, 4.2 and 4.3 can be considered as a consequence of a more general reciprocity law. One can consider three differential forms of third kind ω_1, ω_2 and ω_3 allowing poles of different differential forms to coincide. Then examine the differential equation

$$dF = F(A\omega_1 + B\omega_2 + C\omega_3).$$

And consider the contribution to the coefficient next to ABC . That will lead to theorem 4.1 in the case when the poles of ω_1, ω_2 and ω_3 are disjoint. It will lead to theorem 4.2 in the case when $\omega_1 = \omega_2$. And finally it will lead to theorem 4.3 in the case when $\omega_1 = \omega_3$. The reason for not writing the more general reciprocity law is that it has too many terms.

5 Manin's non-commutative modular symbol, involving Eisenstein series

In this section we give an application of our reciprocity law to the non-commutative modular symbol that Manin has defined (see [M]). Before we can apply the reciprocity law, we make a generalization of Manin's non-commutative modular symbol. We generalize it so that we can consider modular symbols involving Eisenstein series.

5.1 Construction and properties of the symbols

Let Γ be an arithmetically defined, torsion free, discrete group, acting on the upper half plane \mathcal{H} , whose quotient is a Riemann surface X without a divisor D . (We allow compact quotients. That is $D = 0$.) We will consider loops on the modular curve $X - D$. Let $f_1 dz, \dots, f_n dz$ be cusp or Eisenstein modular forms on \mathcal{H} with respect to Γ . Consider the differential equation

$$dJ(z) = J(z) \sum_{i=1}^n A_i f_i dz,$$

where A_i 's are non-commuting formal variables. Let P be the image of $i = \sqrt{-1}$ in the projection

$$pr : \mathcal{H} \rightarrow X - D.$$

That is,

$$P = pr(i).$$

Usually, by a modular form $f(z)$ people denote a holomorphic *function* on the upper half plane that has 'good' transformation properties under the action of the arithmetic group Γ . We are going to use a different trivialization of the modular form. A modular form will be a holomorphic *1-form* on the upper half plane that has a 'good' transformation properties under the action of the group Γ .

We are going to identify the upper half plane \mathcal{H} with $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ by identifying i in the upper half plane \mathcal{H} with the projection of the identity element I in $SL_2(\mathbb{R})$. Let $P \in X - D$ be the projection of i from the upper half plane. Then

$$\pi_1(X - D, P) = \Gamma.$$

Let Q_1, \dots, Q_N be the points of the divisor D . They are called cusp points. Consider the punctured Riemann surface X . We can slice it to $(N - 1 + 4g)$ -gon, so that the points of the divisor D are vertices of this polygon.

Denote by γ_j a path, which is a piece of one of the sides of this polygon, that starts at an orbit of P and ends at a vertex with label Q_j . Denote by $\tilde{\gamma}_j^{-1}$ a path, which is a piece of one of the sides of this polygon, that starts at Q_j and ends at at the other orbit of P compared to γ_j .

Let $\omega_1, \dots, \omega_n$ be modular 1-forms. Consider a neighborhood U of a cusp point $Q = Q_j$ on the modular curve X such that the modular 1-forms $\omega_1, \dots, \omega_n$ have trivialization on U . Let Q' be in U and $Q' \neq Q$. As before, let $\sigma = \sigma_j$ be a simple loop on the modular curve $X - D$ that starts at P and bounds a disc, which contains Q . Let Q_ϵ be a point inside U in an ϵ -neighborhood of Q . The loop σ is homotopic to

$$\gamma' \gamma_\epsilon \sigma_\epsilon \gamma_\epsilon^{-1} \gamma'^{-1},$$

where γ' is a path from P to Q' , γ_e a path from Q' to Q_ϵ and σ_e a loop, starting at Q_e around Q in an ϵ -neighborhood of Q . Let $\tilde{\sigma} = \tilde{\sigma}_j$ be a lift of σ on the upper half plane. Define lifts $\tilde{\gamma}', \tilde{\gamma}_\epsilon, \tilde{\sigma}_e, \tilde{\gamma}_\epsilon^{(-1)}, \tilde{\gamma}'^{(-1)}$ from the modular curve $X - D$ to the upper half plane of $\gamma', \gamma_\epsilon, \sigma_e, \gamma_e^{-1}$ and γ'^{-1} , respectively. We also require that the ending point of a lifted path is the starting point of the next path.

Definition 5.1 Given modular 1-forms $(\omega_1, \dots, \omega_n) = \Omega$ and a path γ in the upper half plane \mathcal{H} , define

$$J_\gamma(\Omega)$$

to be the generating series of iterated integrals of $\omega_1, \dots, \omega_n$ over the path γ , if the modular forms do not have poles at the starting point and at the ending point. If the starting point of γ is a pole of any of the modular 1-forms, then we define

$$J_\gamma^{Reg}(\Omega)$$

to be the generating series of iterated integrals whose first 1-form in the iteration has no pole at the starting point of γ . Similarly, if the ending point of γ is a pole of any of the modular 1-forms, then we define

$$J_\gamma^{Res}(\Omega)$$

to be the generating series of iterated integrals whose last 1-form in the iteration has no pole at the ending point of γ . We define

$$J_Q^{Res}(\Omega)$$

where Q is a cusp point, to be exponential of the sum of the residues of the modular 1-forms at the point Q on the modular curve, computed in a small neighborhood of Q . And finally,

$$J_Q^{Res+}(\Omega) = -1 + J_Q^{Res}(\Omega).$$

We have

Theorem 5.2 *With the notation from this subsection, we have*

$$J_{\tilde{\sigma}}(\Omega) = 1 + J_{\tilde{\gamma}}^{reg}(\Omega) J_Q^{Res+}(\Omega) J_{\tilde{\gamma}^{(-1)}}^{reg}(\Omega),$$

where γ is a path starting at P and ending at Q , $\tilde{\gamma}$ and $\tilde{\gamma}^{(-1)}$ are lifts of γ and γ^{-1} , so that $\tilde{\gamma}$ starts at the initial point of $\tilde{\sigma}$ and ends at a cusp corresponding to Q and $\tilde{\gamma}^{(-1)}$ at the ending point of $\tilde{\gamma}$ and ends at the ending point of $\tilde{\sigma}$

Proof. We apply theorem 2.7 to the loop $\gamma_e \sigma_e \gamma_e^{-1}$. Then we use that $J_{\gamma'}(\Omega) J_{\gamma_e}(\Omega) = J_{\gamma' \gamma_e}(\Omega)$. The same equality hold for regularized generating series.

We have a non-abelian reciprocity law on the upper half plane involving holomorphic modular 1-forms, which are either cusp forms or Eisenstein series (as 1-forms). It is similar to the non-abelian reciprocity law on Riemann surface involving differential forms of third kind. See theorem 2.9. In the following theorem we are going to omit Ω from the notation. That is, we are going to write $J_{\tilde{\gamma}}$ instead of $J_{\tilde{\gamma}}(\Omega)$.

Theorem 5.3 *With the above notation, we have the following reciprocity law for Eisenstein and cusp forms*

$$\left(\prod_{j=1}^M 1 + J_{\tilde{\gamma}_j}^{reg} J_{Q_j}^{Res+} J_{\tilde{\gamma}_j^{(-1)}}^{reg} \right) = 1,$$

where the composition of the paths $\tilde{\gamma}_j$ and $\tilde{\gamma}_j^{(-1)}$ for all j , namely,

$$\prod_{j=1}^M \tilde{\gamma}_j \tilde{\gamma}_j^{(-1)}$$

is contractible and $M = N + 4g$.

Definition 5.4 Let P and P' be in the orbit of i in the upper half plane under the action of the arithmetic group Γ . Let Q and Q' be a cusp points. Let also $\tilde{\gamma}$, $\tilde{\gamma}'^{(-1)}$ and $\tilde{\alpha}$ be paths in the upper half plane that connect P with Q , Q' with P and P with P' , respectively. We define non-commutative modular symbols to be

$$J_{\tilde{\gamma}}^{reg}(\Omega), J_{\tilde{\gamma}'^{(-1)}}^{reg}(\Omega), J_{\tilde{\alpha}}(\Omega)$$

and

$$J_Q^{Res}(\Omega).$$

The non-commutative symbol that Manin has defined can be recovered by considering the product

$$J_{\tilde{\gamma}'^{(-1)}}^{reg}(\Omega) J_{\tilde{\gamma}}^{reg}(\Omega),$$

where the regularization is not needed for cusp forms.

5.2 Main Example

A very important example occurs, when we consider generating series of iterated integrals J of 1-forms fdz and dz for a torsion free arithmetic groups Γ commensurable to $SL_2(\mathbb{Z})$, where fdz is a cusp form.

Remark 5.5 Note that fdz is a cusp 1-form but f is not a cusp form. The poles of f cancel with the poles of dz . In terms of algebraic geometry a cusp form is a section of a certain line bundle on the modular curve X . Instead of considering local trivialization in terms of the rational functions \mathcal{O}_X , we consider local trivialization in terms of 1-forms on X , Ω_X . The cusp form F trivialized as a function on the upper half plane is related to the cusp form trivialized as a holomorphic 1-form fdz by

$$F(x) = \int_0^x f dz$$

for a point x in the chosen fundamental domain.

Definition 5.6 We denote by

$$dz^{\text{on}}$$

the n -fold iteration of the form dz .

With this notation we have the following theorem.

Theorem 5.7

- (a) $L(F, n) := \int_{i\infty}^0 F(z)z^{n-1}dz = (n+1)! \int_{i\infty}^0 fdz \circ dz^{\circ n};$
- (b) $L(F, n_1, F, n_2, \dots, F, n_k) := \int_{i\infty}^0 F(z)z^{n_1-1}dz \circ \dots \circ F(z)z^{n_k-1}dz =$
 $= \prod_{j=1}^k (n_j + 1)! \int_{i\infty}^0 fdz \circ dz^{\circ n_1} \circ \dots \circ fdz \circ dz^{\circ n_k};$
- (c) $J_{\tilde{\gamma}}^{reg}(fdz, dz)$ is the generating series of the L-functions in part (b).

Proof. The form dz on the upper half plane corresponds to a differential form of third kind on the modular curve. Also at the point 0 of the upperhalf plane both fdz and dz are holomorphic. Thus, the regularization is well defined.

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