

Cohomology of $GL_4(\mathbb{Z})$ with Non-trivial Coefficients

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Abstract

In this paper we compute the cohomology groups of $GL_4(\mathbb{Z})$ with coefficients in symmetric powers of the standard representation twisted by the determinant. This problem arises in Goncharov's approach to the study of motivic multiple zeta values of depth 4. The techniques that we use include Kostant's formula for cohomology groups of nilpotent Lie subalgebras of a reductive Lie algebra, Borel-Serre compactification, a result of Harder on Eisenstein cohomology. Finally, we need to show that the ghost class, which is present in the cohomology of the boundary of the Borel-Serre compactification, disappears in the Eisenstein cohomology of $GL_4(\mathbb{Z})$. For this we use a computationally effective version for the homological Euler characteristic of $GL_4(\mathbb{Z})$ with non-trivial coefficients.

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1 Introduction

1.1 Main result and applications

The main goal of this paper is to present a computation of cohomology groups

$$H^i(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes det),$$

where $S^{n-4}V_4$ is the $(n-4)$ -th symmetric power of the standard representation V_4 and det is the determinant representation.

The above cohomology groups describe certain spaces of motivic multiple zeta values. This relation was revealed by Goncharov who suggested to me the problem of computing the cohomology groups of $GL_4(\mathbb{Z})$.

Recall the definition of multiple zeta values

$$\zeta(k_1, \dots, k_m) = \sum_{0 < n_1 < \dots < n_m} \frac{1}{n_1^{k_1} \dots n_m^{k_m}},$$

where $k_1 + \dots + k_m$ is called *weight* and m is called *depth*.

Goncharov has described the cases of *depth*=2 [G2] and of *depth*=3 [G3]. He relates the space of motivic multiple zeta values of *depth*=2 and *weight*= n to the cohomology groups of $GL_2(\mathbb{Z})$ with coefficients in the $(n-2)$ -symmetric power of the standard representation V_2 , namely, to

$$H^i(GL_2(\mathbb{Z}), S^{n-2}V_2).$$

He calls this a mysterious relation between the multiple zeta values of *depth*= m and the "modular variety"

$$GL_m(\mathbb{Z}) \backslash GL_m(\mathbb{R}) / SO_m(\mathbb{R}) \times \mathbb{R}_{>0}^\times.$$

In the paper [G3], he relates the spaces of motivic multiple zeta values of *depth*=3 and *weight*= n to the cohomology of $GL_3(\mathbb{Z})$ with coefficients in the $(n-3)$ -symmetric power of the standard representation V_3 , namely,

$$H^i(GL_3(\mathbb{Z}), S^{n-3}V_3).$$

Goncharov has also related the case of multiple zeta values of *depth*=4 and *weight*= n to the computation of the cohomology of $GL_4(\mathbb{Z})$ with coefficients in the

$(n - 4)$ -symmetric power of the standard representation V_4 twisted by the determinant (private communications). That is, in order to compute the spaces of motivic multiple zeta values of *depth*=4 and *weight*= n one has to know

$$H^i(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes \det).$$

The main result of this paper is the following.

Theorem 1.1 *The dimensions of the cohomology groups of $GL_4(\mathbb{Z})$ with coefficients the symmetric powers of the standard representation twisted by the determinant are given by*

$$H^i(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes \det) = \begin{cases} \mathbb{Q} \oplus H_{cusp}^1(GL_2(\mathbb{Z}), S^{n-2}V_2 \otimes \det) & \text{for } i = 3, \\ 0 & \text{for } i \neq 3. \end{cases}$$

More explicitly,

$$\dim(H^3(GL_4(\mathbb{Z}), S^{12n-4+k}V_4 \otimes \det)) = \begin{cases} n + 1 & \text{for } k = 0, 4, 6, 8, 10, \\ n & \text{for } k = 2, \\ 0 & \text{for } k \text{ odd.} \end{cases}$$

1.2 Computational methods and notation

All representations that we consider are finite dimensional representations of $GL(\mathbb{Q})$ defined over \mathbb{Q} . However, we shall consider them as representations of the arithmetic subgroups via inclusion. We assume that the reader is familiar with group cohomology. For a good introduction to this subject and to various Euler characteristics of group see [Br].

We are going to describe briefly various types of cohomology groups of arithmetic groups, namely, boundary cohomology, cohomology at the infinity, Eisenstein cohomology, interior cohomology and cusp cohomology. All of them are based on a compactification of certain space, called Borel-Serre compactification. The reader who is not familiar with these constructions should not be discouraged. We have tried to present a piece of "Calculus" for cohomology of arithmetic groups. That is, we give the definitions intuitively rather than strictly, and describe the computational tools which we are going to use. The constructions and the proofs of the basic tools could be found in the cited literature. What we do in the main part of this paper is to present the desired computation based on these tools.

We start with the Borel-Serre compactification [BoSe]. Let Γ be a subgroup of $GL_m(\mathbb{Q})$ which is commensurable to $GL_m(\mathbb{Z})$. That is, the intersection $\Gamma \cap GL_m(\mathbb{Z})$ is of finite index both in Γ and in $GL_m(\mathbb{Z})$. Let

$$X = GL_m(\mathbb{R})/SO_m(\mathbb{R}) \times \mathbb{R}_{>0}^{\times}.$$

Then X is a contractable topological space on which Γ acts on the left. And let

$$Y_{\Gamma} = \Gamma \backslash X.$$

Then the Borel-Serre compactification of Y_{Γ} , denoted by \overline{Y}_{Γ} , is a compact space, containing Y_{Γ} . Moreover, it is of the same homotopy type as Y_{Γ} . If V is a representation of Γ and V^{\sim} is the corresponding sheaf then

$$H_{top}^i(\overline{Y}_{\Gamma}, V^{\sim}) = H_{group}^i(\Gamma, V).$$

The space \overline{Y}_Γ can be split into strata, where each stratum corresponds to a parabolic subgroup P of GL_m/\mathbb{Q} and the maximal stratum is Y_Γ . Also the closure of a stratum corresponding to a parabolic subgroup P consists of all strata corresponding to parabolic subgroups Q so that $Q \subset P$. Let $Y_{\Gamma,P}$ be the stratum corresponding to a parabolic subgroup P . Let

$$P(\mathbb{Z}) = P(\mathbb{Q}) \cap \Gamma.$$

Then the topological cohomology of \overline{Y}_P coincides with the group cohomology of $P(\mathbb{Z})$. More precisely,

$$H_{top}^i(\overline{Y}_P, j_P^* V^\sim) = H_{group}^i(P(\mathbb{Z}), V),$$

where V is a representation over the rational numbers and V^\sim the corresponding sheaf on \overline{Y}_Γ and $j_P^* V^\sim$ is its restriction on $\overline{Y}_{\Gamma,P}$.

The boundary of the Borel-Serre compactification is

$$\partial \overline{Y}_\Gamma = \overline{Y}_\Gamma - Y_\Gamma = \cup_P \overline{Y}_{\Gamma,P}.$$

The inclusion

$$j : \partial \overline{Y}_\Gamma \subset \overline{Y}_\Gamma$$

induces

$$j^\# : H_{top}^i(\overline{Y}_\Gamma, V^\sim) \rightarrow H_{top}^i(\partial \overline{Y}_\Gamma, j^* V^\sim).$$

We call the range of the last map $j^\#$ cohomology of the boundary. We use the notation

$$H_{\partial}^i(\Gamma, V) := H_{top}^i(\partial \overline{Y}_\Gamma, j^* V^\sim).$$

We warn the reader that it is not a standard notation.

The image of the map $j^\#$ is called cohomology at the infinity of Γ . We use the notation

$$H_{inf}^i(\Gamma, V) := \text{Im}(j^\#).$$

And the kernel of the map $j^\#$ is called interior cohomology of Γ . We use the notation

$$H_{\dagger}^i(\Gamma, V) := \text{Ker}(j^\#).$$

For the representations that we will consider we have that the cohomology at infinity coincides with the Eisenstein cohomology. This is used for describing certain maps between cohomology groups. Also the interior cohomology coincides with the cusp cohomology. In the representations which we will consider we are going to use that fact in order to show that the interior cohomology vanishes.

In our problem we have

$$H_{cusp}^i(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes \det) = 0,$$

where V_m is the standard m -dimensional representation of $GL_m(\mathbb{Q})$. And S^n is the n -th symmetric power. The last equality holds for $n > 4$ because the representation

$$S^{n-4}V_4 \otimes \det$$

is not self-dual. For $n = 4$ it is true because

$$H_{cusp}^i(SL_4(\mathbb{Z}), \mathbb{Q}) = 0.$$

Thus, we need to compute only the Eisenstein cohomology.

The highest weight representation will be denoted by $L[a_1, \dots, a_m]$, where the weight $[a_1, \dots, a_m]$ sends $diag[H_1, \dots, H_m]$ to $a_1(H_1) + \dots + a_m(H_m)$. Sometimes we shall denote the weight simply by λ . At a later stage there will be a number of cohomologies to consider. In order to make the answer more observable, sometimes we abbreviate. For example:

$$H^i(L[a_1, \dots, a_d]) := H^i(GL_d(\mathbb{Z}), L[a_1, \dots, a_d]).$$

For further abbreviation we set

$$\begin{aligned} (a_1, a_2|a_3|a_4) &:= H^1(L[a_1, a_2]) \otimes H^0(L[a_3]) \otimes H^0(L[a_4]) \\ (a_1|a_2, a_3|a_4) &:= H^0(L[a_1]) \otimes H^1(L[a_2, a_3]) \otimes H^0(L[a_4]) \\ (a_1|a_2|a_3, a_4) &:= H^0(L[a_1]) \otimes H^0(L[a_2]) \otimes H^1(L[a_3, a_4]) \\ (a_1, a_2|a_3, a_4) &:= H^1(L[a_1, a_2]) \otimes H^1(L[a_3, a_4]) \\ (a_1|a_2|a_3|a_4) &:= H^0(L[a_1]) \otimes H^0(L[a_2]) \otimes H^0(L[a_3]) \otimes H^0(L[a_4]) \end{aligned}$$

We also will use the abbreviation

$$(\overline{a_1, a_2}|a_3|a_4) := H_{cusp}^1(L[a_1, a_2]) \otimes H^0(L[a_3]) \otimes H^0(L[a_4]).$$

We consider the parabolic subgroups of GL_4 that contain a fixed Borel subgroup. We shall consider the standard representation of GL_4 with the choice of the Borel subgroup B being the upper triangular matrices. Then the parabolic subgroups can be listed in the following way: P_{ij} is the smallest parabolic subgroup containing a non-zero a_{ji} -entry. And $P_{12,34}$ is the smallest parabolic subgroup containing $a_{21} \neq 0$ and $a_{43} \neq 0$. More precisely: All parabolic subgroups contain B which is upper triangular. Also, P_{12} has a quotient $GL_2 \times GL_1 \times GL_1$, P_{23} has a quotient $GL_1 \times GL_2 \times GL_1$, P_{34} has a quotient $GL_1 \times GL_1 \times GL_2$, P_{13} has a quotient $GL_3 \times GL_1$, P_{24} has a quotient $GL_1 \times GL_3$, and $P_{12,34}$ has a quotient $GL_2 \times GL_2$.

We are going to use the Kostant's theorem [K] in order to obtain information about the parabolic subgroups. To do that we need to examine carefully the action of the Weyl group, W on the root system of gl_n . Also we need the Weyl group, W_P associated to the algebra P . In order to use the Kostant theorem, we need to examine the action of the Weyl group W on the root system of gl_n up to permutation of the root system of P . That is, we need to consider representatives of the quotient $W_P \backslash W$. We state Kostant's theorem [K].

Theorem 1.2 *Let V be a representation of highest weight λ . Let N_P be nilpotent radical of a parabolic group P , and let ρ be half of the sum of the positive roots. Then*

$$H^i(N_P, V) = \bigoplus_{\omega} L_{\omega(\lambda+\rho)-\rho},$$

where the sum is taken over the representatives of the quotient $W_P \backslash W$ with minimal length such that their length is exactly i . In the above notation L_{λ} means representation of N_P with highest weight λ .

Let $[a, b, c, d]$ denote an element of the root lattice (inside h^*) whose value on the diagonal entry $[H_{11}, H_{22}, H_{33}, H_{44}]$ in h is $aH_{11} + bH_{22} + cH_{33} + dH_{44}$. The Weyl group acts on the weight lattice by permuting the entries of $[a, b, c, d]$. It is well known that the Weyl group is generated by reflections perpendicular to the primitive roots. We can choose positivity so that the primitive roots correspond to the permutation (12), (23) and (34), (having sl_4 in mind; (12) sends $[a, b, c, d]$ to $[b, a, c, d]$.) Then the length of an element of the Weyl group is precisely the (minimal) number of successive transpositions, or equivalently, the (minimal) number of reflections w.r.t. the primitive roots. In this setting the right quotient $W_P \backslash W$ can be interpreted as shuffles in the following way: Take for example the parabolic subalgebra P_{23} . Its Levi quotient $M_P = M_{P_{23}}$ is $gl_1 \times gl_2 \times gl_1$. Thus, W_P is generated by (23). Among the representatives of the quotient $W_P \backslash W$ we can consider the ones that preserve the order of the subset $\{23\}$ inside $\{1234\}$. Thus, we can consider all shuffles of $\{1|23|4\}$. Similarly, if we take the parabolic subalgebra $P_{12,34}$, we need to consider the shuffles of $\{12|34\}$ so that the order of $\{12\}$ and the order of $\{34\}$ is preserved. And for the subalgebras P_{13} we consider the shuffles of the set $\{123|4\}$, which means permutations of $\{1234\}$ such that the order $\{123\}$ is preserved.

In order to apply Kostant's theorem, we need to examine the length of each element ω in the Weyl group W , and also the resulting weight $\omega(\lambda + \rho) - \rho$, where $\lambda = [a, b, c, d]$ is the weight of V and ρ is half of the sum of the positive roots.

After we obtain the cohomology of the parabolic groups we have to consider a spectral sequence involving these cohomologies in order to obtain the cohomology of the boundary of the Borel-Serre compactification. Then we use homological Euler characteristics in order to compute the cohomology groups of $GL_m(\mathbb{Z})$ for $m = 2, 3, 4$.

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2 Homological Euler characteristics of $GL_m(\mathbb{Z})$

We call *homological Euler characteristic of a group* Γ the alternating sum of the dimension of the cohomology of the group. We denote it by $\chi_h(\Gamma, V)$, where V is a finite dimensional representation of Γ . More precisely,

$$\chi_h(\Gamma, V) = \sum_i (-1)^i \dim H^i(\Gamma, V).$$

In this section we compute the homological Euler characteristics of $GL_m(\mathbb{Z})$ for $m = 2, 3, 4$ with representations which later will occur in the Kostant's formula applied to $GL_4(\mathbb{Z})$ with coefficients in the representation $(n-4)$ -th symmetric power of the standard representation twisted by the determinant which is $L[n-3, 1, 1, 1]$

The material in this section is in the spirit of the papers [Ho2] and [Ho1]. Most of the formulas and notations are taken from there. The only exception is the computation of $\chi_h(GL_3(\mathbb{Z}), L[n-3, 1, 0])$, done here in details.

We start with $GL_2(\mathbb{Z})$.

Theorem 2.1 *Let $S^n V_2$ be the n -th symmetric power of the standard representation of GL_2 . Then*

$$\chi_h(GL_2(\mathbb{Z}), S^{12n+k} V_2) = \begin{cases} -n+1 & k=0 \\ -n & k=2, 4, 6, 8 \\ -n-1 & k=10 \\ 0 & k=\text{odd}, \end{cases}$$

and

$$\chi_h(GL_2(\mathbb{Z}), S^{12n+k} V_2 \otimes \det) = \begin{cases} -n & k=0 \\ -n-1 & k=2, 4, 6, 8 \\ -n-2 & k=10 \\ 0 & k=\text{odd}. \end{cases}$$

For $GL_m(\mathbb{Z})$ $m=3$ and 4 we need to consider the representations

$$L[n-3, 1, 0] = \text{Ker}(S^{n-3} V_3 \otimes V_3 \rightarrow S^{n-2} V_3),$$

$$L[n-2, 1, 1] = S^{n-3} V_3 \otimes \det,$$

$$L[n-2, 2, 2] = S^{n-4} V_3,$$

$$L[n-3, 1, 1, 1] = S^{n-4} V_4 \otimes \det.$$

Theorem 2.2 *The homological Euler characteristics of $GL_3(\mathbb{Z})$ and $GL_4(\mathbb{Z})$ with coefficients in the above representation are given by*

$$(a) \chi_h(GL_3(\mathbb{Z}), L[n-3, 1, 0]) = \chi_h(GL_2(\mathbb{Z}), S^{n-4} V_2) - \chi_h(GL_2(\mathbb{Z}), S^{n-2} V_2),$$

$$(b) \chi_h(GL_3(\mathbb{Z}), L[n-2, 1, 1]) = \chi_h(GL_2(\mathbb{Z}), S^{n-2} V_2 \otimes \det),$$

$$(c) \chi_h(GL_3(\mathbb{Z}), L[n-2, 2, 2]) = \chi_h(GL_2(\mathbb{Z}), S^{n-4} V_2),$$

$$(d) \chi_h(GL_4(\mathbb{Z}), L[n-3, 1, 1, 1]) = \chi_h(GL_2(\mathbb{Z}), S^{n-2} V_2 \otimes \det).$$

The technique that we are going to use involves a substantial simplification of the trace formula which works when $\Gamma = GL_m(\mathbb{Z})$ or a group co-mensurable to $GL_m(\mathbb{Z})$. The simplification of the trace formula for $GL_m(\mathbb{Z})$ was developed in [Ho1, Ho2]. Besides the simplification we are going to use some computation which were done in the above two papers.

Now we present the simplification of the trace formula in the case of $GL_m(\mathbb{Z})$. An arithmetic group Γ has also an orbifold Euler characteristic. We denote it by $\chi(\Gamma)$, without subscript. It is in fact an Euler characteristic of a certain orbifold. There is a more algebraic description. If an arithmetic group Γ has no torsion then the orbifold Euler characteristic coincides with the homological Euler characteristic with coefficients in the trivial representation.

$$\chi(\Gamma) = \chi_h(\Gamma, \mathbb{Q}).$$

If Γ has torsion choose a torsion free finite index subgroup Γ_0 . Then

$$\chi(\Gamma) = \frac{\chi(\Gamma_0)}{[\Gamma : \Gamma_0]}.$$

Let $C(A)$ denote the centralizer of the element A inside Γ . Then the classical trace formula is

$$\chi_h(\Gamma, V) = \sum_A \chi(C(A)) \text{Tr}(A|V),$$

where the sum is taken over all torsion elements considered up to conjugation. And $C(A)$ denotes the centralizer of the element A inside Γ . We remark that in this formula the identity element is also considered as a torsion element.

For the simplification of the trace formula we need the following definition. Let A be an element in $GL_m(\mathbb{Z})$. Consider it as an $m \times m$ matrix. Let f be its characteristic polynomial. Let

$$f = f_1^{a_1} \dots f_l^{a_l}$$

be the factorization of f into irreducible over \mathbb{Q} polynomials. Denote by

$$R(g, h) = \prod_{i,j} (\alpha_i - \beta_j)$$

the resultant of the polynomials

$$g = \prod_i (x - \alpha_i) \text{ and } h = \prod_j (x - \beta_j).$$

Denote by

$$R(A) = \prod_{i < j} R(f_i^{a_i}, f_j^{a_j})$$

Theorem 2.3 *Let V be a finite dimensional representation of $GL_m(\mathbb{Q})$. Then the homological Euler characteristic of $GL_m(\mathbb{Z})$ with coefficients in V is given by*

$$\chi_h(GL_m(\mathbb{Z}), V) = \sum_A |R(A)| \chi(C(A)) \text{Tr}(A|V),$$

where the sum is taken over torsion matrices A consisting of square blocks A_{11}, \dots, A_{ll} on the block-diagonal and zero blocks off the diagonal. Also the matrices A_{ii} are non-conjugate to each other. And they are chosen from the set $\{+1, +I_2, -1, -I_2, T_3, T_4, T_6\}$, where

$$T_3 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, T_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, T_6 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

The blocks on the diagonal are chosen up to permutation. And the characteristic polynomial f_i of A_{ii} is a power of an irreducible polynomial, and f_i and f_j are relatively prime.

Remark: There is one more simplification that we can make. In the formula in theorem 3.3 for the homological Euler characteristic one can do the summation in the following way. If the $-I_m$ acts on V nontrivially then all the cohomologies of $GL_m(\mathbb{Z})$ with coefficients in V vanish and the homological Euler characteristic vanishes. If $-I_m$ acts trivially on V then $\text{Tr}(-A|V) = \text{Tr}(A|V)$. Also, $C(-A) = C(A)$ and $|R(-A)| = |R(A)|$. If $-A$ is not conjugate to A then in the sum of theorem 3.3 we can compute the invariants for A . And for $-A$ they are the same. Note that $-A$ is conjugate to A if and only if one can obtain $-A$ by permuting the blocks on the diagonal of A .

Proof. (of theorem 3.2) Parts (b), (c) and (d) are computed [?]. We are going to prove part (a). We are going to use the following notation. Given a matrix A whose blocks on the diagonal are A_{11}, \dots, A_{ll} and whose blocks off the diagonal are zero, we write it as

$$A = [A_{11}, \dots, A_{ll}].$$

Using this notation and the notation of theorem 2.3 we quote lemma 4.2 of the paper [Hol]

Lemma 2.4 *For the centralizers and the resultants of the torsion elements in $GL_3(\mathbb{Z})$ we have*

$$(c) |R([I_2, -1])| \chi(C([I_2, -1])) = -\frac{1}{12},$$

$$(e) |R([T_3, 1])| \chi(C([T_3, 1])) = \frac{1}{4},$$

$$(f) |R([T_6, 1])| \chi(C([T_6, 1])) = \frac{1}{12},$$

$$(i) |R([T_4, 1])| \chi(C([T_4, 1])) = \frac{1}{4}.$$

Also, we are going to use lemma 5.3 from the same paper [?].

Lemma 2.5 *The traces of the torsion elements in $GL_3(\mathbb{Z})$ acting on the symmetric power of the standard representation are given by:*

$$(c) \quad \text{Tr}([I_2, -1] | S^{2n+k} V_3) = \begin{cases} n+1 & k=0 \\ n+1 & k=1, \end{cases}$$

$$(e) \quad \text{Tr}([T_3, 1] | S^{3n+k} V_3) = \begin{cases} 1 & k=0 \\ 0 & k=1 \\ 0 & k=2, \end{cases}$$

$$(f) \quad \text{Tr}([T_6, 1] | S^{6n+k} V_3) = \begin{cases} 1 & k=0 \\ 2 & k=1 \\ 2 & k=2 \\ 1 & k=3 \\ 0 & k=4 \\ 0 & k=5, \end{cases}$$

$$(i) \quad \text{Tr}([T_4, 1]|S^{4n+k}V_3) = \begin{cases} 1 & k = 0 \\ 1 & k = 1 \\ 0 & k = 2 \\ 0 & k = 3. \end{cases}$$

In order to compute $\text{Tr}(A|L[w-3, 1, 0])$ for torsion elements A , we are going to use

$$L[w-3, 1, 0] = \text{Ker}(S^{w-3}V_3 \otimes V_3 \rightarrow S^{w-2}V_3).$$

Also, we are going to use that

$$\text{Tr}(A|V \otimes W) = \text{Tr}(A|V)\text{Tr}(A|W).$$

Using the above two equalities together with lemma 3.5, we obtain the following.

Lemma 2.6 *The traces of the torsion elements in $GL_3(\mathbb{Z})$ acting on the $L[w-3, 1, 0]$ are given by:*

$$(c) \quad \text{Tr}([I_2, -1]|L[2n-1+k, 1, 0]) = \begin{cases} -1 & k = 0 \\ 0 & k = 1, \end{cases}$$

$$(e) \quad \text{Tr}([T_3, 1]|L[3n-1+k, 1, 0]) = \begin{cases} -1 & k = 0 \\ 0 & k = 1 \\ 0 & k = 2, \end{cases}$$

$$(f) \quad \text{Tr}([T_6, 1]|L[6n-1+k, 1, 0]) = \begin{cases} -1 & k = 0 \\ 0 & k = 1 \\ 2 & k = 2 \\ 3 & k = 3 \\ 2 & k = 4 \\ 0 & k = 5, \end{cases}$$

$$(i) \quad \text{Tr}([T_4, 1]|L[4n-1+k, 1, 0]) = \begin{cases} -1 & k = 0 \\ 0 & k = 1 \\ 1 & k = 2 \\ 0 & k = 3. \end{cases}$$

For each of the torsion elements A in $GL_3(\mathbb{Z})$ we have that A and $-A$ are not conjugated. When we use theorem 2.3 we can count only four of the torsion elements listed in lemmas 2.4, 2.5 and 2.6 and multiply by two in order to consider the contribution of the negative of these torsion elements. Thus, using theorem 2.3, lemma 2.4 and lemma 2.6 we obtain

$$\chi_h(GL_3(\mathbb{Z}), L[12n - 1, 1, 0]) = 2\left(\frac{1}{12} - \frac{1}{4} - \frac{1}{12} - \frac{1}{4}\right) = -1,$$

$$\chi_h(GL_3(\mathbb{Z}), L[12n + 1, 1, 0]) = 2\left(\frac{1}{12} + 0 + \frac{2}{12} + \frac{1}{4}\right) = 1,$$

$$\chi_h(GL_3(\mathbb{Z}), L[12n + 3, 1, 0]) = 2\left(\frac{1}{12} + 0 + \frac{2}{12} - \frac{1}{4}\right) = 0,$$

$$\chi_h(GL_3(\mathbb{Z}), L[12n + 5, 1, 0]) = 2\left(\frac{1}{12} - \frac{1}{4} - \frac{1}{12} + \frac{1}{4}\right) = 0,$$

$$\chi_h(GL_3(\mathbb{Z}), L[12n + 7, 1, 0]) = 2\left(\frac{1}{12} + 0 + \frac{2}{12} - \frac{1}{4}\right) = 0,$$

$$\chi_h(GL_3(\mathbb{Z}), L[12n + 1, 1, 0]) = 2\left(\frac{1}{12} + 0 + \frac{2}{12} + \frac{1}{4}\right) = 1,$$

Consider the statement of theorem 2.2 part (a). The above computation of homological Euler characteristics gives the left hand side of part(a). The right hand side can be computed directly from theorem 2.1. They do coincide. Thus, part (a) of theorem 2.2 is proven.

3 Cohomology of $GL_2(\mathbb{Z})$.

This section is to show how the computational method works for $GL_2(\mathbb{Z})$. All the results are known, but we need them for the later sections. We are going to compute Eisenstein cohomology and cusp cohomology of $GL_2(\mathbb{Z})$ with coefficients in some representations.

First we are going to compute the cohomology of the boundary using Kostant's theorem. Let $L[a, b]$ be the irreducible representation with highest weight $[a, b]$. The group GL_2 has one parabolic subgroup up to conjugation - the Borel subgroup B . It has a nilpotent radical N and a Levi quotient $GL_1 \times GL_1$. The Weyl group has two elements. Also, the half of the 'sum' of the positive roots is $\rho = [1/2, -1/2]$ Consider the following table:

$\omega \in W$	length	$\omega(\lambda + \rho) - \rho$
12	0	$[a, b]$
21	1	$[b - 1, a + 1]$

From Kostant's theorem we obtain that

$$H^n(N, L[a, b]) = \begin{cases} L[a, b] & n = 0, \\ L[b - 1, a + 1] & n = 1. \end{cases}$$

The integral points of the Levi quotient of B are $GL_1(\mathbb{Z}) \times GL_1(\mathbb{Z})$. Using the Hochschild-Serre spectral sequence we compute $H^n(B, L[a, b])$. If both a and b are even then $H^0(B, L[a, b]) = H^0(GL_1(\mathbb{Z}), L[a]) \otimes H^0(GL_1(\mathbb{Z}), L[b]) = \mathbb{Q}$, and the rest of the cohomology groups are trivial. If both a and b are odd then $H^1(B, L[a, b]) = H^0(GL_1(\mathbb{Z}), L[b-1]) \otimes H^0(GL_1(\mathbb{Z}), L[a+1]) = \mathbb{Q}$. If $a+b$ is odd then $H^n(B, L[a, b]) = 0$ for all n .

There are several cases. If $a + b$ is odd then $-I$ acts non-trivially on $L[a, b]$. So the cohomology of $GL_2(\mathbb{Z})$ vanishes. If $a = b = 2k$ then $L[a, b]$ is the trivial

representation of $GL_2(\mathbb{Z})$. So

$$H^i(GL_2(\mathbb{Z}), L[2k, 2k]) = H_{Eis}^i(GL_2(\mathbb{Z}), L[2k, 2k]) = \begin{cases} \mathbb{Q} & i = 0, \\ 0 & i = 1, \end{cases}$$

and

$$H_{cusp}^i(GL_2(\mathbb{Z}), L[2k, 2k]) = 0.$$

If $a = b = 2k + 1$ then

$$H^i(GL_2(\mathbb{Z}), L[2k + 1, 2k + 1]) = 0.$$

So the Eisenstein and the cusp cohomology also vanish.

The interesting cases are when both a and b are even or when both a and b are odd. For those cases we do not give a complete proof, but rather an interpretation of the cohomologies. It follows from considering modular forms for $GL_2(\mathbb{Z})$ of weight $2(a - b)$ or equivalently, holomorphic modular forms for $SL_2(\mathbb{Z})$. The Eisenstein cohomology is generated by the Eisenstein series and the dimension of the cusp cohomology $H_{cusp}^1(GL_2(\mathbb{Z}), L[a, b])$ is equal to the dimension of the cusp forms of weight $2(a - b)$. In any of these cases we have

$$H^0(GL_2(\mathbb{Z}), L[a, b]) = 0.$$

Also, if a and b are both odd, we have that the map

$$H^1(GL_2(\mathbb{Z}), L[a, b]) \rightarrow H^1(B, L[a, b]) = \mathbb{Q}$$

is surjective. Then

$$H_{Eis}^1(GL_2(\mathbb{Z}), L[a, b]) = \mathbb{Q},$$

and

$$\dim H_{cusp}^1(GL_2(\mathbb{Z}), L[a, b]) = -1 + \dim H^1(GL_2(\mathbb{Z}), L[a, b]).$$

If the weights a and b are both even, then the Eisenstein cohomology coincides with the whole group cohomology.

Here is one interpretation of the cohomology of $GL_2(\mathbb{Z})$ in cases when both a and b are either even or odd. We are not going to use the following interpretation, only the above formulas, but it is nice to keep it in mind.

Let a and b be both odd. Then

$$\begin{aligned} H^1(SL_2(\mathbb{Z}), L[a, b]) = \\ H_{cusp}^1(GL_2(\mathbb{Z}), L_{[a+1, b+1]}) \oplus H_{cusp}^1(GL_2(\mathbb{Z}), L_{[a, b]}) \oplus H_{Eis}^1(GL_2(\mathbb{Z}), L_{[a, b]}). \end{aligned}$$

The first direct summand corresponds to holomorphic cuspidal forms of weight $a - b - 2$. The second summand correspond to anti-holomorphic cusp forms of weight $a - b - 2$. And the last summand corresponds to the Eisenstein series of weight $a - b - 2$ (when bigger than 2).

Keeping in mind the above decompositions one can compute the dimensions of the cohomology groups (or dimensions of cusp forms) using theorem 2.1. Note that in theorem 2.1 the homological Euler characteristic is equal to minus the dimension of the first cohomology group, since the higher cohomology groups vanish as well as the zeroth.

4 Cohomology of $GL_3(\mathbb{Z})$

In this section we compute cohomology groups of $GL_3(\mathbb{Z})$ with coefficients in certain representations which are needed for our main problem. They arise as representations of the Levi quotients of two of the maximal parabolic subgroups of GL_4 , namely, P_{13} and P_{24} . They lead to computation of cohomology groups of $GL_3(\mathbb{Z})$ with coefficients in any of the representations $L[0, 0, 0] = \mathbb{Q}$, $L[w - 3, 1, 0]$, $L[w - 2, 2, 2]$ and $L[w - 2, 1, 1]$.

Theorem 4.1 *The cohomology of $GL_3(\mathbb{Z})$ with coefficients in the above representations are given by*

$$(a) H^i(GL_3(\mathbb{Z}), \mathbb{Q}) = \begin{cases} (0|0|0) & i = 0, \\ 0 & i \neq 0. \end{cases}$$

$$(b) H^i(GL_3(\mathbb{Z}), L[n - 3, 1, 0]) = \begin{cases} (\overline{n-3, -1|2}) & i = 2 \\ (-2|\overline{n-2, 2}) & i = 3 \\ 0 & i \neq 2, 3 \end{cases}$$

$$(c) H^i(GL_3(\mathbb{Z}), L[n - 2, 2, 2]) = \begin{cases} (0|\overline{n-1, 3}) & i = 3 \\ 0 & i \neq 3 \end{cases}$$

$$(d) H^i(GL_3(\mathbb{Z}), L[n - 2, 1, 1]) = \begin{cases} (0|\overline{n-1, 1}) & i = 2 \\ 0 & i \neq 2. \end{cases}$$

Before proving the above theorem, we examine the cohomology of $GL_3(\mathbb{Z})$ with coefficients in $L_{[a,b,c]}$

The algebraic group GL_3 has three parabolic subgroups: B , P_{12} and P_{23} . In order to find their cohomology groups, we need the explicit action of the Weyl group; more precisely we need the various $\omega(\lambda + \rho) - \rho$ that enter in Kostant's theorem. Note that half of the sum of the positive roots is $\rho = [1, 0, -1]$.

$\omega \in W$	length of ω	$\omega(\lambda)$	$\omega(\lambda + \rho) - \rho$
123	0	$[a, b, c]$	$[a, b, c]$
132	1	$[a, c, b]$	$[a, c - 1, b + 1]$
213	1	$[b, a, c]$	$[b - 1, a + 1, c]$
231	2	$[b, c, a]$	$[b - 1, c - 1, a + 2]$
312	2	$[c, a, b]$	$[c - 2, a + 1, b + 1]$
321	3	$[c, b, a]$	$[c - 2, b, a + 2]$

Using the Kostant's theorem we find the cohomology groups of the nilpotent radicals of the parabolic groups.

$$H^q(H, L[a, b, c]) = \begin{cases} L[a, b, c] & q = 0 \\ L[a, c - 1, b + 1] \oplus L[b - 1, a + 1, c] & q = 1 \\ L[b - 1, c - 1, a + 2] \oplus L[c - 2, a + 1, b + 1] & q = 2 \\ L[c - 2, b, a + 2] & q = 3 \end{cases}$$

$$H^q(N_{12}, L[a, b, c]) = \begin{cases} L[a, b, c] & q = 0 \\ L[a, c - 1, b + 1] & q = 1 \\ L[b - 1, c - 1, a + 2] & q = 2 \end{cases}$$

$$H^q(N_{23}, L[a, b, c]) = \begin{cases} L[a, b, c] & q = 0 \\ L[b - 1, a + 1, c] & q = 1 \\ L[c - 2, a + 1, b + 1] & q = 2 \end{cases}$$

In order to pass to cohomologies of the parabolic groups, we use the Hochschild-Serre spectral sequence relating the nil radical and the Levi quotient of a parabolic subgroup to the parabolic subgroup itself; namely the short exact sequence $N \rightarrow P \rightarrow S$. We recall the notation $H^n(L[a_1, \dots, a_k]) = H^n(GL_k \mathbb{Z}, L[a_1, \dots, a_k])$ and $(a|b|c) = H^0(L[a]) \otimes H^0(L[b]) \otimes H^0(L[c])$.

$$H^i(B, L[a, b, c]) = \begin{cases} (a|b|c) & i = 0 \\ (a|c - 1|b + 1) \oplus (b - 1|a + 1|c) & i = 1 \\ (b - 1|c - 1|a + 2) \oplus (c - 2|a + 1|b + 1) & i = 2 \\ (c - 2|b|a + 2) & i = 3 \end{cases}$$

$$E_2^{p,q}(P_{12}, L[a, b, c]) = \begin{cases} H^p(L[a, b]) \otimes H^0(L[c]) & q = 0 \\ H^p(L[a, c - 1]) \otimes H^0(L[b + 1]) & q = 1 \\ H^p(L[b - 1, c - 1]) \otimes H^0(L[a + 2]) & q = 2 \end{cases}$$

$$E_2^{p,q}(P_{23}, L[a, b, c]) = \begin{cases} H^0(L[a]) \otimes H^p(L[b, c]) & q = 0 \\ H^0(L[b - 1]) \otimes H^p(L[a + 1, c]) & q = 1 \\ H^0(L[c - 2]) \otimes H^p(L[a + 1, b + 1]) & q = 2 \end{cases}$$

It is true that the above two spectral sequences stabilize at the E_2 -level. However, in any particular case the formulas will be much simpler, and one can use them to compute the boundary cohomology.

Let B, P_{12}, P_{23} be the parabolic subgroups of $GL_3 \mathbb{Z}$.

$H^i(GL_3(\mathbb{Z}), \mathbb{Q})$

For part (a) we have

$$H^i(B, \mathbb{Q}) = \begin{cases} (0|0|0) & i = 0 \\ (-2|0|2) & i = 3 \\ 0 & n \neq 0, 3 \end{cases}$$

$$H^0(P_{12}, \mathbb{Q}) = H^0(GL_2(\mathbb{Z}), \mathbb{Q}) \otimes H^0(GL_1(\mathbb{Z}), \mathbb{Q})$$

$$H^0(P_{23}, \mathbb{Q}) = H^0(GL_1(\mathbb{Z}), \mathbb{Q}) \otimes H^0(GL_2(\mathbb{Z}), \mathbb{Q})$$

From Mayer-Vietoris we obtain that the boundary cohomology of $GL_3(\mathbb{Z})$ is

$$H_{\partial}^i(GL_3(\mathbb{Z}), \mathbb{Q}) = \begin{cases} (0|0|0) & i = 0 \\ (-2|0|2) & i = 4 \\ 0 & i \neq 0, 4 \end{cases}$$

The homological Euler characteristic of $GL_3(\mathbb{Z})$ with trivial coefficients is 1 (see theorem 2.2 part (c) and theorem 2.1). That is,

$$\chi_h(GL_3(\mathbb{Z}), \mathbb{Q}) = 1.$$

Then the fourth cohomology of the boundary component disappears in the Eisenstein cohomology. Therefore,

$$H_{Eis}^i(GL_3(\mathbb{Z}), \mathbb{Q}) = \begin{cases} (0|0|0) & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Also, the cusp cohomology of $GL_3(\mathbb{Z})$ with trivial coefficients is zero. Therefore the Eisenstein cohomology coincides with the whole group cohomology.

We proceed to part(b).

$$H^i(B, L[n-3, 1, 0])$$

Using the computations in the beginning of this section, we obtain

$$H^i(B, L[n-3, 1, 0]) = \begin{cases} 0 & i = 0 \\ (0|n-2|0) & i = 1 \\ (-2|n-2|2) & i = 2 \\ 0 & i = 3 \end{cases}$$

$$H^i(P_{12}, L[n-3, 1, 0]) = \begin{cases} (n-3, 1|0) & i = 1 \\ (n-3, -1|2) & i = 2 \\ 0 & i \neq 1, 2 \end{cases}$$

$$H^i(P_{23}, L[w-3, 1, 0]) = \begin{cases} (0|n-2, 0) & i = 2 \\ (-2|n-2, 2) & i = 3 \\ 0 & i \neq 2, 3 \end{cases}$$

Using Mayer-Vietoris, for the cohomology of the boundary of the Borel-Serre compactification, we obtain

$$H_{\partial}^i(GL_3(\mathbb{Z}), L[n-3, 1, 0]) = \begin{cases} (\overline{n-3, 1|0}) & i = 1 \\ (\overline{n-3, -1|2}) \oplus (0|\overline{n-2, 0}) & i = 2 \\ (-2|\overline{n-2, 2}) & i = 3 \\ 0 & i \neq 1, 2, 3 \end{cases}$$

The representation $L[n-3, 1, 0]$ is not self dual. So the cohomology of $GL_3(\mathbb{Z})$ with coefficients in $L[n-3, 1, 0]$ coincides with the Eisenstein cohomology, which is a subspace of the cohomology of the boundary. The first cohomology of $GL_3(\mathbb{Z})$ with coefficients in any representation vanishes. For the homological Euler characteristic of $GL_3(\mathbb{Z})$ with coefficients in $L[n-3, 1, 0]$ (theorem 2.2 part (a)) we have

$$\chi_h(GL_3(\mathbb{Z}), L[n-3, 1, 0]) = \chi_h(GL_2(\mathbb{Z}), S^{n-4}V_2) - \chi_h(GL_2(\mathbb{Z}), S^{n-2}V_2).$$

We obtain that the dimension of the second cohomology is half of the dimension of the second cohomology of the boundary of the Borel-Serre compactification. That is,

$$\dim H_{Eis}^2(GL_3(\mathbb{Z}), L[n-3, 1, 0]) = \frac{1}{2} \dim H_{\partial}^2(GL_3(\mathbb{Z}), L[n-3, 1, 0]).$$

Also,

$$\dim H_{Eis}^3(GL_3(\mathbb{Z}), L[n-3, 1, 0]) = \dim H_{\partial}^3(GL_3(\mathbb{Z}), L[n-3, 1, 0]).$$

The second cohomology of the boundary is a direct sum of two spaces with the same dimensions. In order to find out which of the subspaces or which linear combination of the spaces enters in the Eisenstein cohomology, we have to consider the central characters of the two parabolic subgroups [Ha]. For the parabolic subgroup P_{12} we take the central torus

$$\begin{bmatrix} t & & \\ & t & \\ & & t^{-2} \end{bmatrix}.$$

The highest weight induces a character on it, namely $[n-3, -1, 2]$, whose evaluation on the above element is

$$n-3-1-2 \times 2 = n-8.$$

For the parabolic subgroup P_{23} we take the central torus

$$\begin{bmatrix} t^2 & & \\ & t^{-1} & \\ & & t^{-1} \end{bmatrix}.$$

The highest weight induces a character on it, namely $[0, n-2, 0]$, whose evaluation on the above element is

$$0 - (n-2) = -n+2.$$

Their sum is -6. The space which enters in the Eisenstein cohomology has higher weight. Thus we need to solve

$$n-8 > -n+2.$$

Thus for $n > 5$ we have

$$H^i(GL_3(\mathbb{Z}), L[n-3, 1, 0]) = \begin{cases} (\overline{n-3, -1|2}) & i = 2 \\ (-2|\overline{n-2, 2}) & i = 3 \\ 0 & i \neq 2, 3 \end{cases}$$

The value of n is always even and greater or equal to 4. The other option for n is $n = 4$. Then

$$H^i(GL_3(\mathbb{Z}), L[1, 1, 0]) = \begin{cases} (\overline{0|4-2, 0}) & n = 2 \\ (-2|\overline{4-2, 2}) & n = 3 \\ 0 & n \neq 2, 3 \end{cases}$$

That is,

$$H^i(GL_3(\mathbb{Z}), L[1, 1, 0]) = 0.$$

$H^*(GL_3, L[n-2, 2, 2])$ **when n is even.**

Using the computation in the beginning of section 3 we obtain:

$$H^i(B, L[n-2, 2, 2]) = \begin{cases} (n-2|2|2) & i = 0 \\ 0 & i = 1 \\ 0 & i = 2 \\ (0|2|n) & i = 3 \end{cases}$$

$$H^i(P_{12}, L[n-2, 2, 2]) = \begin{cases} (n-2, 2|2) & i = 1 \\ 0 & i \neq 1 \end{cases}$$

$$H^i(P_{23}, L[n-2, 2, 2]) = \begin{cases} (n-2|2|2) & i = 0 \\ (0|n-1, 3) & i = 3 \\ 0 & i \neq 0, 3 \end{cases}$$

Using Mayer-Vietoris we obtain

$$H_{\partial}^i(GL_3(\mathbb{Z}), L[n-2, 2, 2]) = \begin{cases} (\overline{n-2, 2|2}) & i = 1 \\ (0|\overline{n-1, 3}) & i = 3 \\ 0 & i \neq 1, 3 \end{cases}$$

The first cohomology of $GL_3(\mathbb{Z})$ vanishes, Therefore,

$$H^i(GL_3, L[n-2, 2, 2]) = \begin{cases} (0|\overline{n-1, 3}) & i = 3 \\ 0 & i \neq 3 \end{cases}$$

$H^*(GL_3, L[n-2, 1, 1])$ **when n is even.**

Using the computation in the beginning of section 3 we obtain:

$$H^i(B, L[n-2, 1, 1]) = \begin{cases} 0 & i = 0 \\ (n-2|0|2) & i = 1 \\ (0|0|n) & i = 2 \\ 0 & i = 3 \end{cases}$$

$$H^i(P_{12}, L[n-2, 1, 1]) = \begin{cases} (n-2, 0|2) \oplus (0|0|n) & i = 2 \\ 0 & i \neq 2 \end{cases}$$

$$H^i(P_{23}, L[n-2, 1, 1]) = \begin{cases} (0|n-1, 1) & i = 2 \\ 0 & i \neq 2 \end{cases}$$

Using Mayer-Vietoris we obtain

$$H_{\partial}^i(GL_3(\mathbb{Z}), L[n-2, 1, 1]) = \begin{cases} (0|0|n) \oplus (n-2, 0|2) \oplus (0|\overline{n-1, 1}) & i = 2 \\ 0 & i \neq 2, \end{cases}$$

From the homological Euler characteristic of $GL_3(\mathbb{Z})$ with coefficients in $L[n-2, 1, 1]$ we obtain that

$$\dim H_{Eis}^2(GL_3(\mathbb{Z}), L[n-2, 1, 1]) = \frac{1}{2}(-1 + \dim H_{\partial}^2(GL_3(\mathbb{Z}), L[n-3, 1, 0])).$$

For the parabolic subgroup P_{12} we take the central torus

$$\begin{bmatrix} t & & \\ & t & \\ & & t^{-2} \end{bmatrix}.$$

The highest weight induces a character on it, namely $[n - 2, 0, 2]$, whose evaluation on the above element is

$$n - 2 + 0 - 2 \times 2 = n - 6.$$

For the parabolic subgroup P_{23} we take the central torus

$$\begin{bmatrix} t^2 & & \\ & t^{-1} & \\ & & t^{-1} \end{bmatrix}.$$

The highest weight induces a character on it, namely $[0, n - 1, 1]$, whose evaluation on the above element is

$$0 - (n - 1) - 1 = -n.$$

Their sum is -6 . The space which enters in the Eisenstein cohomology has higher weight. Thus we need to solve

$$n - 6 > -n.$$

Thus for $n > 3$, which is always the case, we have

$$H^i(GL_3(\mathbb{Z}), L[n - 2, 1, 1]) = \begin{cases} (n - 2, 0|2) \oplus (0|0|n) & i = 2 \\ 0 & i \neq 2 \end{cases}$$

5 Cohomologies of the parabolic subgroups of GL_4 .

This section consists of computation of cohomology of the parabolic subgroups of $GL_4(\mathbb{Z})$ with coefficients in the representation $S^{n-4}V_4 \otimes det$. We use Kostant's theorem in order to compute these cohomology groups. In the process we reduce the question to computation of the cohomology groups of the Levi quotients which have factors $GL_1(\mathbb{Z})$, $GL_2(\mathbb{Z})$ or/and $GL_3(\mathbb{Z})$. For the last three groups we use the computation from the sections on cohomology of $GL_2(\mathbb{Z})$ and of $GL_3(\mathbb{Z})$.

Recall the notation of the parabolic subgroups: We choose the Borel subgroup B to be the group of upper triangular matrices. Let N be its unipotent radical of B . Let P_{ij} be the smallest parabolic subgroup containing B and containing a non zero a_{ji} -entry. Similarly, $P_{12,34}$ is the smallest (parabolic) subgroup containing B and containing non zero a_{21} - and a_{43} -entries. The unipotent radicals of P_{ij} will be denoted by N_{ij} ; and the Levi quotient by $S_{ij} = P_{ij}/N_{ij}$.

Proposition 5.1 (cohomologies of the parabolic subgroups) *Let $V = S^{n-4}V_4 \otimes det$. Then*

$$H^i(B, V) = \begin{cases} (0|n - 2|0|2) & i = 2 \\ (0|0|0|n) \oplus (-2|w - 2|2|2) & i = 3 \\ (-2|0|2|n) & i = 6 \\ 0 & i \neq 2, 3, 6. \end{cases}$$

$$\begin{aligned}
H^i(P_{12}, V) &= \begin{cases} (n-3, 1|0|2) & i=2, \\ (0|0|0|w) \oplus (n-3, -1|2|2) & i=3, \\ 0 & i \neq 2, 3. \end{cases} \\
H^i(P_{23}, V) &= \begin{cases} (0|n-2, 0|2) \oplus (0|0, 0|n) & i=3 \\ (-2|n-2, 2|2) & i=4, \\ 0 & i \neq 3, 4. \end{cases} \\
H^n(P_{34}, V) &= \begin{cases} (0|0|n-1, 1) \oplus (-2|n-2|2|2) & i=3, \\ (-2|0|n-1, 3) & i=6 \\ 0 & i \neq 3, 6. \end{cases} \\
H^i(P_{13}, V) &= \begin{cases} (0|0|0|n) \oplus (\overline{n-3, -1|2|2}) & i=3, \\ (-2|\overline{n-2, 2|2}) & i=4, \\ 0 & i \neq 3, 4. \end{cases} \\
H^i(P_{12,34}, V) &= \begin{cases} (n-3, -1|2|2) \oplus (0|0|n-1, 1) & i=3 \\ 0 & i \neq 3. \end{cases} \\
H^i(P_{24}, V) &= \begin{cases} (0|0|0|n) \oplus (0|n-2, 0|2) & i=3, \\ (-2|0|\overline{n-1, 3}) & i=6, \\ 0 & i \neq 3, 6.. \end{cases}
\end{aligned}$$

The main tool in the proof will be Kostant's theorem and Hochschild-Serre spectral sequence. In terms of weights representation $S^{n-4} \otimes \det$ is $L[n-3, 1, 1, 1]$. We shall denote the representation $L[n-3, 1, 1, 1]$ simply by V . We identify the Weyl group of GL_4 with the permutation group of four elements. We also need the length of the permutation which we denote by l .

w	l	$w(\lambda + \rho) - \rho$
1234	0	$[w - 3, 1, 1, 1]$
1243	1	$[w - 3, 1, 0, 2]$
1324	1	$[w - 3, 0, 2, 1]$
1342	2	$[w - 3, 0, 0, 3]$
1423	2	$[w - 3, -1, 2, 2]$
1432	3	$[w - 3, -1, 1, 3]$
2134	1	$[0, w - 2, 1, 1]$
2143	2	$[0, w - 2, 0, 2]$
2314	2	$[0, 0, w - 1, 1]$
2341	3	$[0, 0, 0, w]$
2413	3	$[0, -1, w - 1, 2]$
2431	4	$[0, -1, 1, w]$
3124	2	$[-1, w - 2, 2, 1]$
3142	3	$[-1, w - 2, 0, 3]$
3214	3	$[-1, 1, w - 1, 1]$
3241	4	$[-1, 1, 0, w]$
3412	4	$[-1, -1, w - 1, 3]$
3421	5	$[-1, -1, 2, w]$
4123	3	$[-2, w - 2, 2, 2]$
4132	4	$[-2, w - 2, 1, 3]$
4213	4	$[-2, 1, w - 1, 2]$
4231	5	$[-2, 1, 1, w]$
4312	5	$[-2, 0, w - 1, 3]$
4321	6	$[-2, 0, 2, w]$

Now we can consider particular parabolic subgroup P . In order to apply Kostan's theorem we need to find good representatives $W_P \backslash W$; more precisely representatives of minimal length. This can be done by choosing the elements of the permutation group that preserve the ordered subsets corresponding to P_{ij} . For example, when we consider P_{23} the minimal representatives of $W_{P_{23}} \backslash W$ are the permutations w such that $w(2) < w(3)$. When we consider $P_{12,34}$ we need the permutations w such that $w(1) < w(2)$ and $w(3) < w(4)$. And for the group P_{13} the needed permutations are the ones such that $w(1) < w(2) < w(3)$. Thus, using Kostant's theorem we obtain:

It is easier to describe the cohomology

$$H^n(N, V),$$

than to write it down. One can think of it in the following way. Consider the last column of the above table. If it is with weight $[a, b, c, d]$ and with length l then $H^l(N, V)$ contains the representation $L[a, b, c, d]$. Also, all components of the cohomology are obtained in this way.

$$H^i(N_{12}, V) = \begin{cases} L_{[n-3,1,1,1]}, & i = 0, \\ L_{[n-3,1,0,+2]} \oplus L_{[n-3,0,+2,1]}, & i = 1, \\ L_{[n-3,0,0,+3]} \oplus L_{[n-3,-1,+2,+2]} \oplus L_{[0,0,n-1,1]}, & i = 2, \\ L_{[n-3,-1,1,+3]} \oplus L_{[0,-1,n-1,2]} \oplus L_{[0,0,0,n]}, & i = 3, \\ L_{[-1,-1,n-1,+3]} \oplus L_{[0,-1,1,n]}, & i = 4, \\ L_{[-1,-1,2,n]}, & i = 5. \end{cases}$$

$$H^i(N_{23}, V) = \begin{cases} L_{[n-3,1,1,1]}, & i = 0, \\ L_{[n-3,1,0,+2]} \oplus L_{[0,n-2,1,1]}, & i = 1, \\ L_{[n-3,0,0,3]} \oplus L_{[0,n-2,0,+2]} \oplus L_{[-1,n-2,2,1]}, & i = 2, \\ L_{[-2,n-2,2,2]} \oplus L_{[-1,n-2,0,3]} \oplus L_{[0,0,0,n]}, & i = 3, \\ L_{[-2,n-2,1,3]} \oplus L_{[-1,1,0,n]}, & i = 4, \\ L_{[-2,1,1,n]}, & i = 5. \end{cases}$$

$$H^i(N_{34}, V) = \begin{cases} L_{[n-3,1,1,1]}, & i = 0, \\ L_{[n-3,0,+2,1]} \oplus L_{[0,n-2,1,1]}, & i = 1, \\ L_{[n-3,-1,2,2]} \oplus L_{[-1,n-2,2,1]} \oplus L_{[0,0,n-1,1]}, & i = 2, \\ L_{[-2,n-2,2,2]} \oplus L_{[0,-1,n-1,2]} \oplus L_{[-1,1,n-1,1]}, & i = 3, \\ L_{[-2,1,n-1,2]} \oplus L_{[-1,-1,n-1,+3]}, & i = 4, \\ L_{[-2,0,n-1,3]}, & i = 5. \end{cases}$$

$$H^i(N_{13}, V) = \begin{cases} L_{[n-3,1,1,1]}, & i = 0, \\ L_{[n-3,1,0,+2]}, & i = 1, \\ L_{[n-3,0,0,3]}, & i = 2, \\ L_{[0,0,0,n]}, & i = 3. \end{cases}$$

$$H^i(N_{12,34}, V) = \begin{cases} L_{[n-3,1,1,1]} & i = 0 \\ L_{[n-3,0,+2,1]} & i = 1 \\ L_{[n-3,-1,2,2]} \oplus L_{[0,0,n-1,1]} & i = 2 \\ L_{[0,-1,n-1,2]} & i = 3 \\ L_{[-1,-1,n-1,+3]} & i = 4 \end{cases}$$

$$H^i(N_{24}, V) = \begin{cases} L_{[n-3,1,1,1]}, & i = 0, \\ L_{[0,n-2,1,1]}, & i = 1, \\ L_{[-1,n-2,2,1]}, & i = 2, \\ L_{[-2,n-2,2,2]}, & i = 3. \end{cases}$$

Now we apply the Hochschild-Serre spectral sequence to the exact sequences $0 \rightarrow N_{ij} \rightarrow P_{ij} \rightarrow S_{ij} \rightarrow 0$. Thus, the spectral sequence is of the form

$$H^p(S, H^q(N, V)) \Rightarrow H^{p+q}(P, V).$$

In the computation we are going to use the Kunnetth formula $H^*(G_1 \times G_2, V_1 \otimes V_2) = H^*(G_1, V_1) \otimes H^*(G_2, V_2)$. A substantial simplification comes from the facts that $H^p(GL_m(\mathbb{Z}), \det) = 0$, for $m = 1, 2, 3$. It can be proven by the Hochschild-Serre

spectral sequence relating GL_n to SL_n and G_m . One more observation about the computation of the cohomology of the parabolic subgroups. The cohomology groups of the nilpotent radical $H^*(N, V)$ are representations of the Levi quotient. For example, $H^0(N_{12}, S^{w^{-4}}V_4 \otimes \det) = L_{[w^{-3}, 1, 1, 1]} = L_{[w^{-3}, 1]} \otimes L_{[1]} \otimes L_{[1]}$, since the Levi quotient is $S_{12} = GL_2(\mathbb{Z}) \times GL_1(\mathbb{Z}) \times GL_1(\mathbb{Z})$.

We are going to use some abbreviation in the computation that follows. More precisely, by $H^p(L[a, b])$ we mean $H^p(GL_2\mathbb{Z}, L_{[a, b]})$, similarly, by $H^p(L[a])$ we mean $H^p(GL_1\mathbb{Z}, L_{[a]})$ and by $H^p(L[a, b, c])$ we mean $H^p(GL_3\mathbb{Z}, L_{[a, b, c]})$. Also, we set $V = S^{w^{-4}}V \otimes \det = L[n - 3, 1, 1, 1]$.

5.1 Cohomology of B

The Levi quotient of a Borel subgroup is a Cartan subgroup. Thus the representations obtained from Kostan's theorem decompose into tensor product of one dimensional representations.

$$E_2^{p, q} = H^p(S, H^q(N, V)) = \begin{cases} H^p(S, L_{[0, n-2, 0, 2]}) & q = 2 \\ H^p(S, L_{[0, 0, 0, n]}) \oplus H^p(S, L_{[-2, n-2, 2, 2]}) & q = 3 \\ H^p(S, L_{[-2, 0, 2, n]}) & q = 6 \\ 0 & q \neq 2, 3, 6. \end{cases}$$

All other representations of S do not contribute to the cohomology of the Borel subgroup because at least one of the entries of the weight is an odd number. The ones that are left contain only even coefficients. Thus, they are trivial representations of $GL_1(\mathbb{Z})$. Then the E_2 -terms of the spectral sequence can be simplified to:

$$E_2^{p, q} = \begin{cases} (0|n-2|0|2) & p = 0, q = 2 \\ (0|0|0|n) \oplus (-2|n-2|2|2) & p = 0, q = 3 \\ (-2|0|2|n) & p = 0, q = 6 \\ 0 & otherwise \end{cases}$$

The only non-zero entries of the above spectral sequence occur only when $p = 0$. Therefore the sequence degenerates at the E_2 -level, and cohomology of the Borel subgroup is

$$H^i(B, V) = \begin{cases} (0|n-2|0|2) & i = 2 \\ (0|0|0|n) \oplus (-2|n-2|2|2) & i = 3 \\ (-2|0|2|n) & i = 6 \\ 0 & i \neq 2, 3, 6 \end{cases}$$

5.2 Cohomology of P_{12}

We proceed similarly with the other parabolic subgroups. Recall, the Levi quotient of P_{12} is $S_{12} = GL_2(\mathbb{Z}) \times GL_1(\mathbb{Z}) \times GL_1(\mathbb{Z})$. Thus, the spectral sequence becomes:

$$E_2^{p, q} = H^p(S_{12}, H^q(N_{12}, V))$$

$$E_2^{p,q} = \begin{cases} H^p(S_{12}, L_{[n-3,1,0,+2]}) = H^p(S_{12}, L_{[n-3,1]} \otimes L_{[0]} \otimes L_{[2]}) & q = 1, \\ H^p(S_{12}, L_{[n-3,-1,+2,+2]}) = H^p(S_{12}, L_{[n-3,-1]} \otimes L_{[2]} \otimes L_{[2]}) & q = 2, \\ H^p(S_{12}, L_{[0,0,0,n]}) = H^p(S_{12}, L_{[0,0]} \otimes L_{[0]} \otimes L_{[n]}) & q = 3, \\ 0 & q \neq 1, 2, 3. \end{cases}$$

The representations of the $GL_1(\mathbb{Z})$ quotients that give a contribution to the cohomology groups are the trivial representations. Thus,

$$E_2^{p,q} = \begin{cases} H^p(L[w-3, 1]) \otimes H^0(L[0]) \otimes H^0(L[2]) & q = 1, \\ H^p(L[w-3, -1]) \otimes H^0(L[2])^{\otimes 2} & q = 2, \\ H^p(L[0, 0]) \otimes H^0(L[0]) \otimes H^0(L[w]) & q = 3, \\ 0 & q \neq 1, 2, 3. \end{cases}$$

We are going to use the fact that $H^p(GL_2(\mathbb{Z}), L) = 0$ for $p > 1$, for any representation L of $GL_2(\mathbb{Q})$. In particular the differential $d_2 : E_2^{p,q} \rightarrow E_2^{p+2,q-1}$ is zero, since $E_2^{p,q}$ is non-zero only when $p = 0$ or $p = 1$. Therefore, the spectral sequence degenerates at the E_2 -level, and the cohomology of P_{12} is the following.

$$H^i(P_{12}, V) = \begin{cases} (n-3, 1|0|2) & i = 2, \\ (0|0|0|n) \oplus (n-3, -1|2|2) & i = 3, \\ 0 & i \neq 2, 3. \end{cases}$$

5.3 Cohomology of P_{23}

Recall that the Levi quotient S_{23} of P_{23} is $GL_1(\mathbb{Z}) \times GL_2(\mathbb{Z}) \times GL_1(\mathbb{Z})$. Then

$$E_2^{p,q} = H^p(S_{23}, H^q(N_{23}, V)) = \begin{cases} H^p(S_{23}, L_{[0,n-2,0,2]}) & q = 2, \\ H^p(S_{23}, L_{[-2,n-2,2,2]} \oplus L_{[0,0,0,n]}) & q = 3, \\ 0 & q \neq 2, 3. \end{cases}$$

Using similar arguments, we obtain

$$H^i(P_{23}, V) = \begin{cases} (0|n-2, 0|2) \oplus (0|0|0|n) & i = 3 \\ (-2|n-2, 2|2) & i = 4, \\ 0 & i \neq 3, 4. \end{cases}$$

5.4 Cohomology of P_{34}

Recall that the Levi quotient S_{34} of P_{34} is $GL_1(\mathbb{Z}) \times GL_1(\mathbb{Z}) \times GL_2(\mathbb{Z})$. Then

$$E_2^{p,q} = H^p(S_{34}, H^q(N_{34}, V)) = \begin{cases} H^p(S_{34}, L_{[0,0,n-1,1]}) & q = 2, \\ H^p(S_{34}, L_{[-2,n-2,2,2]}) & q = 3, \\ H^p(S_{34}, L_{[-2,0,n-1,3]}) & q = 5. \quad 0 \quad q \neq 2, 3, 5 \end{cases}$$

Similarly, we obtain

$$E_2^{p,q} = \begin{cases} H^0(L[0]) \otimes H^0(L[0]) \otimes H^p(L[n-1, 1]) & q = 2, \\ H^0(L[-2]) \otimes H^0(L[n-2]) \otimes H^p(L[2, 2]) & q = 3, \\ H^0(L[-2]) \otimes H^0(L[0]) \otimes H^p(L[n-1, 3]) & q = 5, \\ 0 & q \neq 2, 3, 5 \end{cases}$$

And finally, the cohomology of P_{34} is

$$H^i(P_{34}, V) = \begin{cases} (0|0|n-1, 1) \oplus (-2|n-2|2|2) & i = 3, \\ (-2|0|n-1, 3) & i = 6 \\ 0 & i \neq 3, 6. \end{cases}$$

5.5 Cohomology of P_{13}

Recall that the Levi quotient S_{13} of P_{13} is $GL_3\mathbb{Z} \times GL_1\mathbb{Z}$.

$$H^p(S_{13}, H^i(N_{13}, V)) = \begin{cases} 0 & q = 0, \\ H^p(S_{13}, L_{[n-3, 1, 0, 2]}) & q = 1, \\ 0 & q = 2, \\ H^p(S_{13}, L_{[0, 0, 0, n]}) & q = 3. \end{cases}$$

We can simplify it to

$$H^p(S_{13}, H^q(N_{13}, V)) = \begin{cases} 0 & q = 0, \\ H^p(L[n-3, 1, 0]) \otimes H^0(L[2]) & q = 1, \\ 0 & q = 2, \\ H^p(L[0, 0, 0]) \otimes H^0(L[n]) & q = 3. \end{cases}$$

From the section "Cohomology of $GL_3(\mathbb{Z})$ " we know that for $n > 5$ we have

$$H^p(GL_3(\mathbb{Z}), L[n-3, 1, 0]) = \begin{cases} (\overline{n-3, -1}|2) & p = 2 \\ (-2|\overline{n-2, 2}) & p = 3 \\ 0 & p \neq 2, 3 \end{cases}$$

And for $n = 4$

$$H^p(GL_3(\mathbb{Z}), L[n-3, 1, 0]) = 0.$$

Also

$$H^p(GL_3(\mathbb{Z}), \mathbb{Q}) = \begin{cases} (0|0|0) & p = 0 \\ 0 & p \neq 0. \end{cases}$$

Therefore

$$H^p(S_{13}, H^q(N_{13}, V)) = \begin{cases} (0|0|0|n) & p = 0 \text{ and } q = 3, \\ (\overline{n-3, -1}|2|2) & p = 2 \text{ and } q = 1, \\ (-2|\overline{n-2, 2}|2) & p = 3 \text{ and } q = 1, \\ 0 & \text{for all other } p \text{ and } q. \end{cases}$$

The above spectral sequence degenerates at E_2 level. Therefore

$$H^i(P_{13}, V) = \begin{cases} (0|0|0|n) \oplus (\overline{n-3, -1}|2|2) & i = 3, \\ (-2|\overline{n-2, 2}|2) & i = 4, \\ 0 & i \neq 3, 4. \end{cases}$$

5.6 Cohomology of $P_{12,34}$

For the last parabolic subgroup we can obtain a better answer in terms of cohomology of $GL_2(\mathbb{Z})$. However, the d_2 differential might be non-trivial. Recall that the Levi quotient $S_{12,34}$ of $P_{12,34}$ is $GL_2(\mathbb{Z}) \times GL_2(\mathbb{Z})$. Then the spectral sequence is

$$E_2^{p,q} = H^p(S_{12,34}, H^q(N_{12,34}, V))$$

$$E_2^{p,q} = \begin{cases} H^p(S_{12,34}, L_{[n-3, -1, 2, 2]}) \oplus H^p(S_{12,34}, L_{[0, 0, n-1, 1]}) & q = 2 \\ 0 & q \neq 2 \end{cases}$$

Therefore

$$E_2^{1,2} = [H^1(L[n-3, -1]) \otimes H^0(L[2, 2])] \oplus [H^0(L[0, 0]) \otimes H^1(L[n-1, 1])].$$

And

$$E_2^{p,q} = 0 \text{ for } p \neq 1 \text{ or } q \neq 2$$

Finally,

$$H^i(P_{12,34}, V) = \begin{cases} (n-3, -1|2|2) \oplus (0|0|n-1, 1) & i = 3 \\ 0 & i \neq 3 \end{cases}$$

5.7 Cohomology of P_{24}

Recall that the Levi quotient S_{24} of P_{24} is $GL_1(\mathbb{Z}) \times GL_3(\mathbb{Z})$. We have the spectral sequence

$$H^p(S_{24}, H^q(N_{24}, V)) = \begin{cases} 0 & q = 0, \\ H^p(S_{24}, L_{[0, n-2, 1, 1]}) & q = 1, \\ 0 & q = 2, \\ H^p(S_{24}, L_{[-2, n-2, 2, 2]}) & q = 3. \end{cases}$$

We can simplify it to

$$H^p(S_{24}, H^q(N_{24}, V)) = \begin{cases} 0 & q = 0, \\ H^0(L[0]) \otimes H^p(L[n-2, 1, 1]) & q = 1, \\ 0 & q = 2, \\ H^0(L[-2]) \otimes H^p(L[n-2, 2, 2]) & q = 3. \end{cases}$$

From the section "Cohomology of $GL_3(\mathbb{Z})$ " we know that

$$H^i(GL_3(\mathbb{Z}), L[n-2, 1, 1]) = \begin{cases} (n-2, 0|2) \oplus (0|0|n) & i = 2 \\ 0 & i \neq 2 \end{cases}$$

And also

$$H^i(GL_3, L[n-2, 2, 2]) = \begin{cases} (0|\overline{n-1}, 3) & i = 3 \\ 0 & i \neq 3 \end{cases}$$

Therefore,

$$H^p(S_{24}, H^q(N_{24}, V)) = \begin{cases} (0|0|0|n) \oplus (0|n-2, 0|2) & p = 2, q = 1, \\ (-2|0|\overline{n-1}, 3) & p = 3, q = 3, \\ 0 & \text{for all other } p \text{ and } q. \end{cases}$$

The spectral sequence degenerates. Therefore,

$$H^i(P_{24}, V) = \begin{cases} (0|0|0|n) \oplus (0|n-2, 0|2) & i = 3, \\ (-2|0|\overline{n-1}, 3) & i = 6, \\ 0 & i \neq 3, 6. \end{cases}$$

6 Boundary cohomology of $GL_4(\mathbb{Z})$

In this section we compute the cohomology of the boundary of the Borel-Serre compactification associated to $GL_4(\mathbb{Z})$ with coefficients in

$$V = S^{n-4} \otimes \det = L[n-3, 1, 1, 1].$$

The Eisenstein cohomology, which in our case is the whole group cohomology, injects into the cohomology of the boundary.

We recall briefly several statements about Borel-Serre compactification associated to $GL_m(\mathbb{Z})$. Let

$$X = GL_m(\mathbb{R})/SO_m(\mathbb{R}) \times \mathbb{R}_{>0}^\times.$$

And let

$$Y = GL_m(\mathbb{Z}) \backslash X.$$

Then the Borel-Serre compactification of Y , denoted by \overline{Y} , is a compact space, containing Y , and of the same homotopy type. The space \overline{Y} is obtained by attaching cell σ_P to X , corresponding to each parabolic subgroup P . Denote by Y_P the projection of σ_P to X . Let \overline{Y}_P be the closure of Y_P . Then $\overline{Y}_Q \subset \overline{Y}_P$ when $Q \subset P$. The boundary of \overline{Y} is obtained by gluing together the spaces \overline{Y}_P . In the following computation we shall denote by Y_{ij} the space $Y_{P_{ij}}$. For these spaces we have

$$H_{top}^i(\overline{Y}_{ij}, i^* F_V) = H_{group}^i(P_{ij}, V),$$

for a suitable sheaf F_V on \overline{Y} , where i is the inclusion of \overline{Y}_{ij} into \overline{Y} . For simplification we will not write the restriction functor i^* .

The cohomology of the boundary can be computed the spectral sequence of the type 'Mayer-Vietoris'.

$$E_1^{*,q} : \begin{array}{ccccc} & & H^q(\overline{Y}_{13}, F_V) & \longrightarrow & H^q(\overline{Y}_{12}, F_V) & & \\ & & \searrow & \nearrow & \searrow & & \\ & H^q(\overline{Y}_{12,34}, F_V) & & & H^q(\overline{Y}_{23}, F_V) & \longrightarrow & H^q(\overline{Y}_B, F_V) \\ & & \nearrow & \searrow & \nearrow & & \\ & & H^q(\overline{Y}_{24}, F_V) & \longrightarrow & H^q(\overline{Y}_{34}, F_V) & & \end{array}$$

The direct sum of the first column will be $E_1^{0,q}$; the direct sum of the second column will be $E_1^{1,q}$; and $E_1^{2,q} = H^q(\overline{Y}_B, F_V)$. We have non-zero terms when $q = 2, 3, 4$ or 6 . Similarly, to the Mayer-Vietoris sequence, we want every square at the E_1 level to be anti-commutative. It can be achieved in the following way. First,

consider the maps induced by the inclusion of the boundary components. Then the squares will commute. Then change the sign of every other arrow mapping a subspace of $E_1^{0,q}$ to a subspace of $E_1^{1,q}$ as it is done in the definition of the spectral sequence. Then the squares will anti-commute.

Theorem 6.1 *The above spectral sequence stabilizes at E_2 level. It converges to the cohomology of the boundary of the Borel-Serre compactification associated to $GL_4(\mathbb{Z})$, which is*

$$H_{\partial}^i(GL_4(\mathbb{Z}), V) = \begin{cases} (0|0|0|n) \oplus (\overline{n-3, -1}|2|2) \oplus (\overline{n-3, 1}|0|2) & i = 3, \\ 0 & i \neq 3, \end{cases}$$

where

$$(a_1|a_2|\dots|a_k) = \otimes_{i=1}^k H^0(GL_1(\mathbb{Z}), L[a_i]),$$

and

$$(\overline{a_1, a_2}|a_3|a_4) = H_{cusp}^1(GL_2(\mathbb{Z}), L[a_1, a_2]) \otimes (a_3|a_4).$$

Proof. We consider all non-vanishing terms of the spectral sequence at E_1 level. The non-vanishing terms occur at $q = 2, 3, 4$ and 6 . For a fixed q we have arrows going in direction of the index p induced by the inclusion of the parabolic subgroups. We compute *kernel/image* for these arrows in order to find the E_2 level of the spectral sequence. As a consequence we find that the spectral sequence degenerates at E_2 level. Then we compute the cohomology to which it converges, which is the cohomology of the boundary.

6.1 Computation of $E_2^{*,2}$

For the $E_1^{p,2}$ -terms the only non-zero cohomologies come are $H^2(P_{12}, V)$ and $H^2(B, V)$. We have

$$(n-3, 1|0|2) \rightarrow (0|n-2|0|2).$$

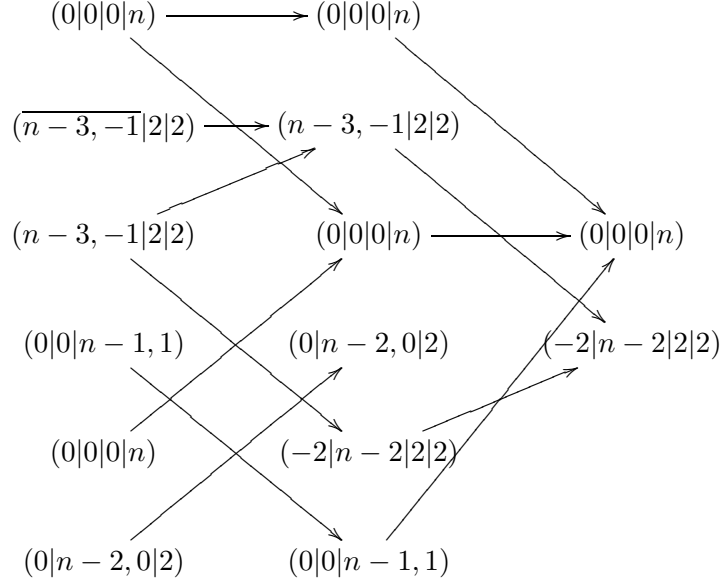
Therefore,

$$E_2^{p,2} = \begin{cases} (\overline{n-3, 1}|0|2) & p = 1 \\ 0 & p \neq 1 \end{cases}$$

6.2 Computation of $E_2^{*,3}$

First we consider the case $n > 5$. Now we describe the $E_1^{*,3}$ terms. Consider the columns of the diagram below. Break each column into pairs of vector spaces. Each pair comes one parabolic subgroup. For example $(0|0|0|n)$ and $(\overline{n-3, -1}|2|2)$ come from third cohomology of P_{13} . The two vector spaces below come from the third cohomology of $P_{12,34}$. The maps correspond to the inclusion of the parabolic

subgroups.



There are many cancellation which occur when passing to E_2 level. In order to follow the cancellation one considers the connected graph of the above diagram. There are 3 connected graphs: one containing the space $(0|0|0|n)$ coming from the 3rd cohomology of the Borel subgroup, and another containing $(-2|n-2|2|2)$ again from the 3rd cohomology of the Borel subgroup, and the 3rd containing $(0|n-2, 0|2)$ from the 3rd cohomology of P_{24} . Consider the graph containing $(0|0|0|n)$. The only term that is not cancelled at E_2 level is the vector space $(0|0|0|n)$ which comes from the parabolic group P_{24} . Now consider the second connected graph, containing $(-2|n-2|2|2)$. After cancellation the only vector space left is $(\overline{n-3, -1|2|2})$ coming from P_{13} . For the 3rd connected graph, there are two vertices corresponding to $(0|n-2, 0|2)$. So they cancel and do not contribute to the E_2 level. Thus, for $n > 4$ we have

$$E_2^{p,3} = \begin{cases} (0|0|0|n) \oplus (\overline{n-3, -1|2|2}) & p = 0, \\ 0 & p \neq 0. \end{cases}$$

Now we have to examine the case $n = 4$. The vector spaces are all the same as in the case $n > 4$ except the exchange of $(\overline{n-3, -1|2|2})$ with $(0|n-2, 0|2)$ in the 3rd cohomology of P_{13} . Note also that for $n = 4$, we have $(0|n-2, 0|2) = 0$. Then

the $E_1^{*,3}$ terms form the following anticommutative diagram:

$$\begin{array}{ccccc}
(0|0|0|4) & \longrightarrow & (0|0|0|4) & & \\
& \searrow & & \searrow & \\
0 & & (4-3, -1|2|2) & & \\
& \nearrow & & \nearrow & \\
(4-3, -1|2|2) & & (0|0|0|4) & \longrightarrow & (0|0|0|4) \\
& \searrow & & \searrow & \\
(0|0|4-1, 1) & & 0 & & (-2|4-2|2|2) \\
& \nearrow & & \nearrow & \\
(0|0|0|4) & & (-2|4-2|2|2) & & \\
0 & & (0|0|4-1, 1) & &
\end{array}$$

There are 2 connected graphs in the above diagram. One containing the vector space $(0|0|0|4)$ coming from the Borel subgroup. The other containing the vector space $(-2|4-2|2|2)$ again coming from the Borel subgroup. Consider the graph containing $(0|0|0|4)$. The only terms that is not canceled at E_2 level is the vector space $(0|0|0|4)$ which comes from the parabolic group P_{24} . Now consider the second connected graph, containing $(-2|4-2|2|2)$. All of its terms of that graph cancel when passing to E_2 level. Thus, for $w = 4$ we have

$$E_2^{p,3} = \begin{cases} (0|0|0|4) & p = 0, \\ 0 & p \neq 0. \end{cases}$$

6.3 Computation of $E_2^{*,4}$

For $q = 4$ the only non-zero terms at E_1 level come from P_{13} and P_{23} . We have

$$H^4(P_{13}, V) \rightarrow H^4(P_{23}, V).$$

From the first theorem (theorem 5.1) in the section "Cohomology of the parabolic subgroups of GL_4 " we obtain

$$(-2|n-2, 2|2) \rightarrow (-2|n-2, 2|2).$$

Therefore,

$$E_2^{*,4} = 0.$$

6.4 Computation of $E_2^{*,6}$

When $q = 6$, for all even w , the non-zero terms give

$$E_1^{*,6} : H^6(P_{24}, V) \rightarrow H^6(P_{34}, V) \rightarrow H^6(B, V),$$

which are isomorphic to

$$(-2|0|\overline{w-1, 3}) \rightarrow (-2|0|w-1, 3) \rightarrow (-2|0|2|w)$$

from theorem 5.1. The above sequence is exact. Therefore,

$$E_2^{*,6} = 0.$$

The spectral sequence degenerates at E_2 level. Therefore, we can find what is the cohomology of the boundary of the Borel-Serre compactification associated to $GL_4(\mathbb{Z})$ with coefficients in the sheaf F_V associated to

$$V = S^{n-4}V_4 \otimes \det.$$

Let us recall the notation that we are going to use. By $H_{\partial}^i(GL_4(\mathbb{Z}), V)$ we mean the cohomology of the boundary of the Borel-Serre compactification associated to $GL_4(\mathbb{Z})$ with coefficients the sheaf F_V . For even n greater than 4 we have

$$H_{\partial}^i(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes \det) = \begin{cases} (0|0|0|n) \oplus (\overline{n-3, -1}|2|2) \oplus (\overline{n-3, 1}|0|2) & i = 3, \\ 0 & i \neq 3. \end{cases}$$

Note that the first two summands for the 3rd cohomology of the boundary come from 3rd cohomology of the maximal parabolic subgroups. And the last summand comes from the 2nd cohomology of a non-maximal parabolic subgroup. Since it comes from second cohomology of a parabolic subgroup, but it contributes in the 3rd cohomology of the boundary, it is called a ghost class.

7 Cohomology of $GL_4(\mathbb{Z})$

We are going to show that the ghost class do not enter in the Eisenstein cohomology of $GL_4(\mathbb{Z})$ which coincides with the whole cohomology of $GL_4(\mathbb{Z})$.

Since the cohomology of the boundary is concentrated in degree 3, it is enough to compute homological Euler characteristic of $GL_4(\mathbb{Z})$ with coefficients in $S^{n-4}V_4 \otimes \det$. Recall the homological Euler characteristic of an arithmetic group Γ with coefficients in a finite dimensional representation is

$$\chi_h(\Gamma, V) = \sum_i (-1)^i \dim H^i(\Gamma, V).$$

Note that $S^{n-4}V_4 \otimes \det = L[n-3, 1, 1, 1]$ and $S^{n-2}V_2 \otimes \det = L[n-1, 1] = L[n-3, -1]$. From [?] we know that

$$\chi_h(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes \det) = \chi_h(GL_2(\mathbb{Z}), S^{n-2}V_2 \otimes \det).$$

Therefore, for even n greater than 4, we have

$$H^i(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes \det) = \begin{cases} (0|0|0|n) \oplus (\overline{n-3, -1}|2|2) & i = 3, \\ 0 & i \neq 3. \end{cases}$$

In the case $n = 4$ we use the same argument.

$$H_{\partial}^i(GL_4(\mathbb{Z}), \det) = \begin{cases} (0|0|0|4) & i = 3, \\ 0 & i \neq 3. \end{cases}$$

Also, the homological Euler characteristic gives

$$\chi_h(GL_4(\mathbb{Z}), \det) = -1.$$

Therefore, for $n = 4$ the cohomology of the boundary coincides with the Eisenstein cohomology. And we have

$$H_{Eis}^i(GL_4(\mathbb{Z}), \det) = \begin{cases} (0|0|0|4) & i = 3, \\ 0 & i \neq 3. \end{cases}$$

On the other hand,

$$H_{cusp}^i(SL_4(\mathbb{Z}), \mathbb{Q}) = 0.$$

Therefore,

$$H_{cusp}^i(GL_4(\mathbb{Z}), \det) = 0.$$

And we conclude that

$$H^i(GL_4(\mathbb{Z}), \det) = \begin{cases} (0|0|0|4) & i = 3, \\ 0 & i \neq 3. \end{cases}$$

References

- [BoSe] Borel, A., Serre, J.-P.: Corners and arithmetic groups, *Comment. Math. Helv.* 48 (1973), 436-491.
- [Br] Brown, K.: *Cohomology of Groups*, Graduate Text in Mathematics, Springer-Verlag: New York, 1982.
- [G1] Goncharov, A.: The double logarithm and Manin's complex for modular curves, *Math. Res. Lett.* 4 (1997) no.5, 617-636.
- [G2] Goncharov, A.: Multiple polylogarithms, cyclotomy and modular complexes, *Math. Res. Lett.* 1998 no.4, 497-516 .
- [G3] Goncharov, A.: The dihedral Lie algebras and the Galois symmetries of $\pi_1(P^1 - \{0, \infty\} \cup \mu_n)$, *Duke Math. J.* vol. 110, No. 3 (2001), 397-487.
- [Ha] Harder, G.: *Der Rang-Eins Beitrag Eisensteinkohomologie*, preprint, 43 pages.
- [Ho1] Horozov, I.: Euler characteristics of arithmetic groups, *Math. Res. Lett.* 2005 no.12, p.275-291.
- [Ho2] Horozov, I.: Euler characteristics of arithmetic groups, preprint in arXiv, math.GR/0311117, 94p.
- [K] Kostant.: Lie algebra cohomology and generalized Borel-Weil theorem, *Ann. of Math.* (2) 74, 1961, 329-387.
- [Se] Serre, J.-P.: Cohomologie des groupes discretes, *Ann. of Math. Studies* 70 (1971), 77-169.