

Research Statement

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My main research interests include Number Theory and Algebraic Geometry, more specifically themes related to arithmetic groups. Recently, I became interested in application of higher categories to Algebraic Topology and the relation of the latter with arithmetic groups.

The products of my research up to now will be discussed in section "Completed results". They include results on non-abelian modular symbol for Hilbert modular forms associated to real quadratic fields, on adelic interpretation of multiple zeta functions and on cohomology of $GL_4(\mathbb{Z})$ with coefficients in certain representations. Also, I will describe some of the computational results about Euler characteristics of arithmetic groups.

Currently, I am working on two themes. One of them is on multiple Dedekind zeta functions and the other - on orbifold Euler characteristics of unitary groups. More details on these current projects will be presented in the section "Current research". Each of the themes from my completed results has a potential for new research. That will be discussed in the section "Future plans".

1 Completed results

My newest result in the paper [H5] has applications to iterated Hilbert modular forms. Also, I define multiple completed Dedekind zeta functions for certain fields.

These results were inspired by Manin's construction of [M2] on non-abelian modular symbol. He considered iterated integrals of modular forms over a path for a finite index subgroups of $SL_2(\mathbb{Z})$. I wanted to generalize this to SL_2 over the ring of integers in a real quadratic extension of the rational numbers. Iteration over a path is not sufficient for this task. I defined a new ingredient. It is iterated integrals over a two-dimensional manifold. I call this manifold a membrane. Under reasonable assumptions the iterated integrals over a membrane are homotopy invariant. We use also fundamental 2-groupoid in the sense of Ronald Brown and Higgins [BH]. This consists of the homotopy classes of maps of a rectangle to a manifold, where the homotopy fixes the boundary. Under some restrictions the structure of iterated integrals over a 2-dimensional membrane has the following structure - a strict cubical 2-category which fibers over the fundamental 2-groupoid with fiber a Hopf algebra. This construction can easily be generalized to higher dimensions.

I give an interpretation of an iterated integral over an admissible membrane as a period of mixed Hodge structure. All the varieties in this construction are algebraic, so the iterated integrals have interpretation of a period of a motive. I used that to construct non-commutative modular symbol for Hilbert modular forms for real quadratic fields. This is the generating series of all iterated integrals of modular forms for groups $SL_2(\mathcal{O}_K)$, where \mathcal{O}_K is the ring of integers in a real quadratic field. I call it non-abelian because it captures the structure of the fundamental 2-groupoid, not just homology.

Another application is to use certain theta functions whose Mellin transform gives Dedekind zeta function of a real quadratic field of class number one. I have iterated the Mellin transforms of these theta functions to obtain multiple completed Dedekind zeta functions. I have motivic interpretations of the multiple completed Dedekind zeta functions for real quadratic fields of class number 1, when the values of the arguments are positive even integers. I have constructed also multiple completed Dedekind zeta functions for imaginary quadratic fields and for CM fields of degree 4 over \mathbb{Q} . In these cases I have motivic interpretations when the arguments are any positive integers.

I also have partial results on non-commutative modular symbol for $SL_3(\mathbb{Z})$.

In the paper [H4] I give adelic interpretation of the multiple zeta functions as well as adelic interpretation of multiple L-functions associated to modular forms of $SL_2(\mathbb{Z})$. The technique I use is based on Tate's thesis. I give also adelic interpretation of multiple completed zeta functions, which I express as iterated Mellin transforms of the Jacobi theta function, also considered in the paper described above. I show that multiple zeta functions, multiple completed zeta functions and multiple L-functions associated to modular forms for $SL_2(\mathbb{Z})$ satisfy certain shuffle relation which can be expressed adelically as well.

My next result is on cohomology of $GL(4, \mathbb{Z})$ in [H3]. This paper is very technical. It uses Hochschild-Serre spectral sequences, Borel-Serre compactification [BoSe], Eisenstein cohomology, cusp cohomology, Kostant's theorem on the cohomology groups of a nilpotent Lie subalgebra of a reductive Lie algebra with coefficients in a representation. I have used some ideas of professor Harder [Ha1]. I computed the cohomology groups of $GL_4(\mathbb{Z})$ with coefficients in the symmetric powers of the standard representation twisted by the determinant. This is a problem that Goncharov needs for motivic multiple zeta values. At the last step I used results from my thesis to show that a cohomological class in the cohomology of the boundary is not present in the cohomology at the infinity.

In [H1] and [H2] I have developed a method for computing the homological Euler characteristics for important classes of arithmetic groups. The homological Euler characteristic of an arithmetic group Γ is defined by

$$\chi_h(\Gamma, V) = \sum_i (-1)^i \dim(H^i(\Gamma, V)),$$

where V is a finite dimensional representation of Γ over a field of characteristic zero. Such an invariant and other types of Euler characteristics have been considered by many people (see [S], [Ba], [Br]). The technique in the paper is based on K. Brown's formula [Br2] expressing the homological Euler characteristic of an arithmetic group as a sum of orbifold Euler characteristics of centralizers of each of the torsion elements. Using linear algebra over number rings, I express the homological Euler characteristic of an arithmetic group as a sum over very few torsion elements. The new formula gives a computationally effective method for computing the homological Euler characteristics of $GL_m(\mathcal{O}_K)$ and $SL_m(\mathcal{O}_K)$ where \mathcal{O}_K is the ring of integers in a number field K . The method also works for some congruence subgroups of the above groups.

Some of the arithmetic groups, for which my method works, are related to spaces of certain motivic multiple polylogarithms (see [G2], [G3], [G4]).

For SL_2 over the ring of integers in a totally real number field I obtain relations Dedekind zeta function at -1 and algebraic K-groups [Mi].

Using the computational method we can find, for example, that

$$\chi_h(SL_2(\mathbb{Z}[(1 + \sqrt{5})/2]), \mathbb{Q}) = 4,$$

and

$$\chi_h(GL_{10}(\mathbb{Z}), \mathbb{Q}) = 1.$$

2 Current research

Currently, I am working on two themes. One of them is to define Multiple Dedekind Zeta function for real quadratic fields of class number one. I would like to give a motivic interpretation of these functions when the arguments are positive integers. I am approaching this problem with the technique developed in my last paper [H5], using iteration over membranes and Hilbert modular forms.

The other theme is finding the homological Euler characteristics of arithmetically defined symplectic groups. It is in the spirit of my thesis [H1]. As a part of this theme one needs to find the orbifold Euler characteristic of arithmetically defined unitary groups. I have assigned this part of the problem to Alexander Charis, a graduate student at Brandeis University. The main tools are Riemann curvature, Tamagawa measures and p -adic integration.

3 Future plans

There are four themes on which I am planning to work on. One of the problems is finding the homological Euler characteristic of $Sp_n(\mathcal{O}_K)$ with coefficients in a finite dimensional representation of $Sp_n(K)$, where \mathcal{O}_K is the ring of integers in a number field K .

There are two main steps in solving this problem. One of them is finding the orbifold Euler characteristic of unitary groups defined over totally real number rings, which, as I mentioned above, I assigned to Alexander Charis, a graduate student at Brandeis University. The other step is linear algebra over number rings with respect to a symplectic pairing. It has to be done in the spirit of my thesis [H1].

I am also planning to continue working on iterations over a membrane, which are the main tool in [H5]. These ideas lead to many directions for further research. They are a new type of iterated integrals, which I think have many applications. I am going to apply these ideas to pair of simplexes in a projective space. This problem is related to mixed Tate motives and I think that my ideas will give a different view on this problem. Another problem is to define Hilbert modular symbol for all totally real number fields in terms of iteration over a higher dimensional membrane. This problem is solved in my last paper [H5] for real quadratic fields. What remain to be done is to show that certain geodesically defined manifolds are algebraic. There is also a topological side of the problem. The question is what part of the fundamental 2-groupoid is captured by considering iterated integrals over a membrane. Similar question in the one dimensional case were done by Chen - the iterated integrals over a path capture the pro-unipotent completion of the fundamental groupoid.

In the paper [H3] I considered the cohomology groups of $GL_4(\mathbb{Z})$. This is related to motivic multiple zeta values. An important problem, which I will try to compute, is the cohomology of $\Gamma_1(4, p)$. The latter one is the subgroup of $GL_4(\mathbb{Z})$ that fixes the vector $(1, 0, 0, 0)$ modulo a prime p . This problem is related to spaces of multiple polylogarithms at the p -th root of unity, as explained to me by Goncharov. The techniques for this problem should be similar to the ones for the cohomology groups of $GL_4(\mathbb{Z})$.

I the paper [H4] I considered adelic iteration for various functions. I expect that there should be an adelic interpretation of multiple L -functions associated to Hilbert modular forms and of multiple completed Dedekind zeta functions.

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