Multiple Zeta Functions, Modular Forms and Adeles

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Abstract

In this paper we give an adelic interpretation of the multiple zeta functions, of multiple completed zeta functions and of multiple L-functions associated to modular forms.

Contents

2 Introduction

In this paper we give adelic interpretation of various multiple zeta functions, multiple L-functions of defined as iteration over a path in the complex plane.

Let us recall what some of these function are. Riemann zeta function is

$$\zeta(s) = \sum_{n>0} \frac{1}{n^s}.$$

It was examined first by Euler. Multiple zeta function of depth d is

$$\zeta(s_1, \ldots, s_d) = \sum_{0<n_1<\ldots<n_d} \frac{1}{n_1^{s_1} \ldots n_d^{s_d}}.$$

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This function was also examined first by Euler.

Multiple zeta values are the values of a multiple zeta functions at the positive integers. Kontsevich gave an interpretation of the multiple zeta values as iterated integrals.

Theorem 2.1 *Let $k_1,\ldots,k_d$ be d positive integers with $k_d>1$. Then*

$$\zeta(k_1,\ldots,k_d)=\int_0^{k_1+\ldots k_d}(\ldots(\int_0^{x_3}(\int_0^{x_2}\frac{dx_1}{1-x_1})\underbrace{\frac{dx_2}{x_2})\ldots\frac{dx_{k_1}}{x_{k_1}}}_{k_1-1})\frac{dx_{k_1+1}}{1-x_{k_1+1}})\ldots\frac{dx_{k_1+\ldots+k_d}}{x_{k_1+\ldots+k_d}}.$$

The iterated integral on the right hand side can be written as

$$\int\ldots\int_{0<x_1<\ldots<x_{k_1+\ldots+k_d}}\frac{dx_1}{1-x_1}\wedge\underbrace{\frac{dx_2}{x_2}\wedge\ldots\wedge\frac{dx_{k_1}}{x_{k_1}}}_{k_1-1}\wedge\frac{dx_{k_1+1}}{1-x_{k_1+1}}\ldots\wedge\frac{dx_{k_1+\ldots+k_d}}{x_{k_1+\ldots+k_d}}.$$

In his paper we express multiple zeta function as iterated integrals, where the integration is over the adeles.

Manin has defined multiple L-functions that come from iteration of modular forms. Given holomorphic cusp forms $f_1(z),\ldots,f_d(z)$ for $SL_2(\mathbb{Z})$, he defines

$$L(s_1,\ldots,s_d)=\int\ldots\int_{0<t_1<\ldots<t_d}f_1(it_1)t_1^{s_1}\frac{dt_1}{t_1}\wedge\ldots\wedge f_d(it_d)t_d^{s_d}\frac{dt_d}{t_d}.$$

We express these multiple L-functions as iterated integrals over the adeles.

In the last section we show that the iterated integrals over the adeles satisfy a shuffle relation.

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3 Iterated integrals over local fields

According to Tate's thesis [T] the Riemann zeta function can be written as a product of p-adic integrals. We want to examine how multiple zeta functions are related to iterated p-adic integrals.

Let x_p be a p-adic number. Let $|x_p|_p$ be the normalized p-adic norm so that $|p|_p=p^{-1}$. Let d_px_p be the additive p-adic Haar measure so that

$$\int_{\mathbb{Z}_p}d_px_p=1.$$

Let $d_p^\times x_p$ be the Haar measure on $\mathbb{Q}_p^\times$ normalized so that

$$\int_{\mathbb{Z}_p^\times}d_p^\times x_p=1.$$

The relation between the two measures is the following

$$d_p^\times x_p = \frac{p-1}{p}\frac{dx_p}{|x_p|_p}.$$

Indeed, $\frac{dx_p}{|x_p|_p}$ is a multiplicative Haar measure. Also,

$$\mathbb{Z}_p - \{0\} = \bigcup_{k=0}^{\infty} p^k \mathbb{Z}_p^\times,$$

and

$$\int_{p^k\mathbb{Z}_p^\times} dx_p = p^{-k}\int_{\mathbb{Z}_p^\times} dx_p.$$

We have,

$$1 = \int_{\mathbb{Z}_p} dx = \sum_{k=0}^{\infty}\int_{p^k\mathbb{Z}_p^\times} dx_p = \sum_{k=0}^{\infty} p^{-k}\int_{\mathbb{Z}_p^\times} dx_p = \frac{p}{p-1}\int_{\mathbb{Z}_p^\times} dx_p.$$

Then

$$\int_{\mathbb{Z}_p^\times} dx_p = \frac{p-1}{p}.$$

Therefore,

$$\int_{\mathbb{Z}_p^\times} \frac{dx_p}{|x_p|_p} = \int_{\mathbb{Z}_p^\times} dx_p = \frac{p-1}{p} = \frac{p-1}{p}\int_{\mathbb{Z}_p^\times} d_p^\times x_p.$$

Let $f_p(x_p)$ be a function defined on $\mathbb{Q}_p^\times$ by

$$f_p(x_p) = \begin{cases} 1 & x \in \mathbb{Z}_p - \{0\}, \\ 0 & \text{otherwise} \end{cases}$$

The local factor of the Riemann zeta function is given by

$$\frac{1}{1-p^{-s}} = \int_{\mathbb{Q}_p^\times} f_p(x_p)|x_p|_p^s d_p^\times x_p.$$

Indeed, for fixed value of k the integrant $f_p(x_p)|x_p|_p^s$ is constant on the set $p^k\mathbb{Z}_p^\times$. For $k < 0$ we have $f_p(x_p) = 0$. For $k \geq 0$ we have

$$\int_{p^k\mathbb{Z}_p^\times} f_p(x_p)|x_p|_p^s d_p^\times x_p = \int_{p^k\mathbb{Z}_p^\times} |x_p|_p^s d_p^\times x_p = p^{-ks}.$$

Also, $\mathbb{Z}_p - \{0\} = \bigcup_{k=0}^{\infty} p^k\mathbb{Z}_p^\times$.

$$\int_{\mathbb{Q}_p^\times} f_p(x_p)|x_p|_p^s d_p^\times x_p = \int_{\mathbb{Z}_p-\{0\}} |x_p|_p^s d_p^\times x_p = \sum_{k=0}^{\infty}\int_{p^k\mathbb{Z}_p^\times} |x_p|_p^s d_p^\times x_p = \sum_{k=0}^{\infty} p^{-ks} = \frac{1}{1-p^{-s}}$$

Now we are going to iterate the measures defining the local factor. We can iterate in the following way. Consider

$$I_p(s_1, s_2) = \int_{|x_{p,1}|_p > |x_{p,2}|_p} f_p(x_{p,1})|x_{p,1}|^{s_1} d^\times x_{p,1} f_p(x_{p,2})|x_{p,2}|^{s_2} d^\times x_{p,2}$$

Lemma 3.1

$$I_p(s_1,s_2) = \sum_{0\leq k_1<k_2} \frac{1}{p^{k_1 s_1} p^{k_2 s_2}}.$$

Proof. In the definition of $I_p(s_1,s_2)$ we integrate over a union of $(p^{k_1}\mathbb{Z}_p^\times, p^{k_2}\mathbb{Z}_p^\times)$ for $k_1 < k_2$. Also, if $k_1 < 0$ then the integrant is zero because $f_p(x_{p,1}) = 0$ from the definition of f_p. For $k_1 \geq 0$ we have

$$\int_{(p^{k_1}\mathbb{Z}_p^\times, p^{k_2}\mathbb{Z}_p^\times)} f_p(x_{p,1})|x_{p,1}|^{s_1} d^\times x_{p,1} f_p(x_{p,2})|x_{p,2}|^{s_2} d^\times x_{p,2} =$$

$$= \int_{(p^{k_1}\mathbb{Z}_p^\times, p^{k_2}\mathbb{Z}_p^\times)} |x_{p,1}|^{s_1} d^\times x_{p,1} |x_{p,2}|^{s_2} d^\times x_{p,2} =$$

$$= \frac{1}{p^{k_1 s_1} p^{k_2 s_2}}$$

Then we have to take the sum of the above quantities over $0 \leq k_1 < k_2$. And we obtain the formula in the lemma. Denote by x_∞ an element of $\mathbb{R}$. Let $|x_\infty|_\infty$ be the norm which by definition is the absolute value of the real number. (We save the notation $|x|$ for a norm of an idele.) Let dx_∞ be the Haar measure on the additive group of the real numbers. Consider the multiplicative Haar measure on $\mathbb{R}^\times$, namely,

$$\frac{dx_\infty}{|x_\infty|_\infty}$$

Let

$$f_\infty(x_\infty) = e^{-\pi x_\infty^2}.$$

We are going to integrate $f_\infty(x_\infty)$ with respect to the multiplicative measure. The Mellin transform of $f_\infty(x_\infty)$ gives the local factor at infinity of the completed Riemann zeta function

$$\zeta_\infty(s) = \int_{\mathbb{R}-\{0\}} f_\infty(x)|x|^s \frac{dx}{|x|} = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}).$$

Similarly to the p-adic case we can iterate the measure that leads to a local factor of the Riemann zeta function. We obtain

$$\int_{|x_{\infty,1}|_\infty > |x_{\infty,2}|_\infty > 0} e^{-\pi(x_{\infty,1}^2 + x_{\infty,2}^2)} |x_{\infty,1}|_\infty^{s_1} |x_{\infty,2}|_\infty^{s_2} \frac{dx_{\infty,1}}{|x_{\infty,1}|_\infty} \frac{dx_{\infty,2}}{|x_{\infty,2}|_\infty}.$$

4 Adelic interpretation of the multiple zeta functions

Let $x \in \hat{\mathbb{Z}}$. Denote by $|x|_f$ the product of all the p-adic norms. Namely,

$$|x|_f = \prod_{p:finite} |x|_p.$$

Let also

$$f_f(x) = \prod_{p:finite} f_p(x_p).$$

Denote by

$$d_f^\times x_f = \prod_p d_p^\times x_p,$$

the multiplicative measure on the finite ideles given as product of all local multiplicative measures considered above. Let us iterate the function $f_f(x_f)|x_f|^s$. We have

$$I(s_1,s_2)=\int_{|x_1|_f>|x_2|_f} f_f(x_1)|x_1|_f^{s_1}d_f^\times x_1 f_f(x_2)|x_2|_f^{s_2}d_f^\times x_2.$$

Recall the definition of double zeta function,

$$\zeta(s_1,s_2)=\sum_{0<n_1<n_2}\frac{1}{n_1^{s_1}n_2^{s_2}}.$$

Theorem 4.1

$$I(s_1,s_2)=\zeta(s_1,s_2)$$

Proof. Restrict the integral to the domain where the support of f_f is not zero. For any such idele we have $|x|_f=1/n$ for some $n\in\mathbb{N}$. Therefore

$$\begin{aligned}
&I(s_1,s_2)=\\
&=\textstyle\int_{|x_1|_f>|x_2|_f} f_f(x_1)|x_1|_f^{s_1}d_f^\times x_1 f_f(x_2)|x_2|_f^{s_2}d_f^\times x_2=\\
&=\textstyle\sum_{0<n_1<n_2}\int_{n_1\hat{\mathbb{Z}}^\times\times n_2\hat{\mathbb{Z}}^\times}|x_1|_f^{s_1}d_f^\times x_1|x_2|_f^{s_2}d_f^\times x_2=\\
&=\textstyle\sum_{0<n_1<n_2}\frac{1}{n_1^{s_1}n_2^{s_2}}=\\
&=\zeta(s_1,s_2)
\end{aligned}$$

is a double zeta function.

If we iterate d times, we obtain a multiple zeta function of depth d. Namely, if we set

$$I(s_1,\dots,s_d)=\int_{|x_1|_f>\dots>|x_d|_f} f_f(x_1)|x_1|_f^{s_1}d_f^\times x_1\dots f_f(x_d)|x_d|_f^{s_d}d_f^\times x_d$$

and

$$\zeta(s_1,\dots,s_d)=\sum_{0<n_1<\dots<n_d}\frac{1}{n_1^{s_1}\dots n_d^{s_d}},$$

then we have

$$I(s_1,\dots,s_d)=\zeta(s_1,\dots,s_d).$$

5 Adelic interpretation of the iterated completed zeta functions

Let x be an element of the adeles over $\mathbb{Q}$. We are going to write x_∞ for the infinite coordinate of the adele x, and x_p for the p-adic coordinate. Consider the function

$$f(x)=f_\infty(x_\infty)\prod_p f_p(x_p)$$

Let $d^\times x$ be a multiplicative measure on the ideles given by the product of local multiplicative measures considered above for all of the local fields. Let $\mathbb{A}$ be the adels over the rational numbers $\mathbb{Q}$. We are going to integrate over $\mathbb{A}^\times$. For an idele $x \in \mathbb{A}^\times$ let

$$|x| = |x_\infty|_\infty \prod_p |x_p|_p$$

be the product of all the local valuations. Let

$$w_{1,\mathbb{A}}(x) = f(x)|x|^{s_1} d^\times x$$

and let

$$w_{2,\mathbb{A}}(x) = f(x)|x|^{s_2} d^\times x$$

be measures on $\mathbb{A}^\times$ that give the Mellin transform of $f(x)$. From Tate's thesis [T] we know that

$$\int w_{1,\mathbb{A}}(x) = \pi^{-\frac{s_1}{2}} \Gamma(\frac{s_1}{2}) \zeta(s_1)$$

is the completed Riemann zeta function. Consider the following integral

$$I(s_1, s_2) = \int_{|x_1|>|x_2|} f(x_1)|x_1|^{s_1} d^\times x_1 f(x_2)|x_2|^{s_2} d^\times x_2.$$

Theorem 5.1

$$I(s_1, s_2) = \int_{r_1>r_2>0} (\theta(ir_1) - 1) r_1^{\frac{s_1}{2}} \frac{dr_1}{r_1} (\theta(ir_2) - 1) r_2^{\frac{s_2}{2}} \frac{dr_2}{r_2},$$

where

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 \tau}.$$

Proof. We want to modify the function $f(x)$ in the definition of $I(s_1, s_2)$ so that the two new functions are defined over $\mathbb{A}^\times/\mathbb{Q}^\times$ and over $\mathbb{R}^\times_{>0}$, respectively. In this modifications we want that suitable Mellin transforms of the new functions lead to the same function in terms of s_1 and s_2 for the iterated integral. We shall notice that the new function defined over $\mathbb{R}^\times_{>0}$ is essentially the classical theta function.

Denote by $\bar{x}$ the projection of an idele x to an element of $\mathbb{A}^\times/\mathbb{Q}^\times$. Let

$$\bar{f}(\bar{x}) = \sum_{q\in\mathbb{Q}^\times} f(qx).$$

Note that $|qx| = |x|$ for $q \in \mathbb{Q}$ and $x \in \mathbb{A}$. Denote by $|\bar{x}| := |x|$ the norm of an element in $\mathbb{A}^\times/\mathbb{Q}^\times$. Recall that $\mathbb{Q}^\times$ is a discrete subgroup of $\mathbb{A}^\times$. For that reason we can take the same measure $d^\times x$ to the set $\mathbb{A}^\times/\mathbb{Q}^\times$. We have

$$\int_{|x_1|>|x_2|} f(x_1)|x_1|^{s_1} d^\times x_1 f(x_2)|x_2|^{s_2} d^\times x_2 = \int_{|\bar{x}_1|>|\bar{x}_2|} g(\bar{x}_1)|\bar{x}_1|^{s_1} d^\times x_1 g(\bar{x}_2)|\bar{x}_2|^{s_2} d^\times x_2.$$

Now we define the corresponding function $f_0(t)$ for $t \in \mathbb{R}$ and $t > 0$. The function f is constant on each set $(qt, q\hat{\mathbb{Z}}^\times)$, where $t \in \mathbb{R}^\times_{>0}$ is fixed, $q \in \mathbb{Q}^\times$ varies and $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. We can set $F(t)$ to be the value of $\bar{f}$ on any element of the above set $(qt, q\hat{\mathbb{Z}}^\times)$. Then

$$\int_{|x_1|>|x_2|} f(x_1)|x_1|^{s_1} d^\times x_1 f(x_2)|x_2|^{s_2} d^\times x_2 = \int_{t_1>t_2>0} f_0(t_1) t_1^{s_1} \frac{dt_1}{t_1} f_0(t_2) t_2^{s_2} \frac{dt_2}{t_2}.$$

Let us examine more carefully the relation between $f(x)$ and $F(t)$. Let x be an idele. Then for some $q \in \mathbb{Q}^\times$ and $r \in \mathbb{R}^\times$ we have $x \in (qr, q\hat{\mathbb{Z}}^\times)$. If q is not an integer then $f(x) = 0$. Also, $f(-x) = f(x)$. For these reasons we can sum over all positive integers. Let x_f be the finite adele of x. That is, x_f consists of all coordinates of x except the coordinate corresponding to the infinite place. Let, also

$$f_f(x_f) = \prod_p f_p(x_p),$$

where the product is over all primes (finite places) p. Denote by $d^\times x_f$ the product of multiplicative Haar measure of $\mathbb{Q}_p^\times$ over all primes. Denote, also, by $\hat{\mathbb{Z}}$ the product of the p-adic integers over all primes. Then

$$F(|x|) = 2\sum_{n\in\mathbb{N}} f_\infty(nr) \times \int_{n\hat{\mathbb{Z}}^\times} f_f(x_f) d^\times x_f.$$

For $n \in \mathbb{N}$ the integral becomes

$$\int_{n\hat{\mathbb{Z}}^\times} f_f(x_f) d^\times x_f = 1.$$

For the infinite place we have

$$f_\infty(nr) = e^{-n^2r^2\pi}.$$

Therefore,

$$F(t) = 2\sum_{n=1}^{\infty} e^{-n^2t^2\pi} = -1 + \sum_{n\in\mathbb{Z}} e^{-n^2t^2\pi} = -1 + \theta(it^2),$$

where $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{n^2\tau\pi i}$. Iteration of $f(x)|x|^s$ can be written as iteration of $F(t)t^s$ which can be expressed in terms of iteration of theta function minus 1.

$$\int_{|x_1|>|x_2|} f(x_1)|x_1|^{s_1} d^\times x_1 f(x_2)|x_2|^{s_2} d^\times x_2 =$$

$$= \int_{t_1>t_2>0} F(t_1)t_1^{s_1}\tfrac{dt_1}{t_1} F(t_2)t_2^{s_2}\tfrac{dt_2}{t_2} =$$

$$= \int_{r_1>r_2>0} (\theta(ir_1)-1)r_1^{\frac{s_1}{2}}\tfrac{dr_1}{r_1}(\theta(ir_2)-1)r_2^{\frac{s_2}{2}}\tfrac{dr_2}{r_2}$$

The changes of variables are $t_j = |x_j|$ and $r_j = t_j^2$ for $j = 1, 2$. Note that the last integral is an iteration of theta function minus 1 over the geodesic in the upper half plane connecting 0 and $i\infty$.

6 Adelic interpretation of multiple L-functions associated to modular forms of even weight

Similarly to the previous sections, we are going to iterate only two functions. If we iterate more times all the statements will be analogous to the two-function case.

Let $F(\tau)$ and $G(\tau)$ be two cusp forms of even weight with respect to the group $SL_2(\mathbb{Z})$. Let

$$F(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}$$

and

$$G(\tau) = \sum_{n=1}^{\infty} b_n e^{2\pi i n\tau}$$

be the Fourier expansion of cusp forms of even weight. Let $L(F,s)$ and $L(G,s)$ be the corresponding L-functions defined by

$$L(F,s) = \sum_{n=0}^{\infty} a_n n^{-s}$$

and

$$L(G,s) = \sum_{n=0}^{\infty} b_n n^{-s}.$$

And let

$$\Lambda(F,s) = (2\pi)^{-s}\Gamma(s)L(F,s) = \int_0^{\infty} F(it)t^s \frac{dt}{t}$$

and

$$\Lambda(G,s) = (2\pi)^{-s}\Gamma(s)L(G,s) = \int_0^{\infty} G(it)t^s \frac{dt}{t}$$

be the completed L-functions. Define the iterated L-function

$$L(F,G,s_1,s_2) = \sum_{0<n_1<n_2} \frac{a_{n_1} b_{n_2}}{n_1^{s_1} n_2^{s_2}},$$

and the completed iterated L-functions

$$\Lambda(F,G,s_1,s_2) = \int_{t_1>t_2>0} F(it_1)t_1^{s_1} G(it_2)t_2^{s_2} \frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

The iterated completed L-function was considered by Manin in his paper [M]. The iteration takes place over the geodesic connecting 0 and $i\infty$ in the hyperbolic geometry of the upper half plane.

We are going to give an idelic interpretation of $L(F,G,s_1,s_2)$ and of $\Lambda(F,G,s_1,s_2)$.

Denote by $\hat{\mathbb{Z}}$ the product of all rings of p-adic integers. That is,

$$\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p.$$

Then the invertible elements of $\hat{\mathbb{Z}}$ are the product of all groups of p-adic units. That is,

$$\hat{\mathbb{Z}}^\times = \prod_p Z_p^\times.$$

Denote by $n\hat{\mathbb{Z}}^\times$ the subset of $\hat{\mathbb{Z}}$ which consists of elements of the form n times an elements of $\hat{\mathbb{Z}}^\times$. For a finite idele x_f let

$$f_f(x_f) = \begin{cases} a_n & \text{if } x_f \in n\hat{\mathbb{Z}}^\times, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly for the modular form G, we define

$$g_f(x_f) = \begin{cases} b_n & \text{if } x_f \in n\hat{\mathbb{Z}}^\times, \\ \\ 0 & \text{otherwise.} \end{cases}$$

For the infinite place, we define

$$f_\infty(x_\infty) = g_\infty(x_\infty) = e^{-2\pi|x_\infty|_\infty}.$$

For an idele $x = (x_f, x_\infty)$, where x_f is the finite part of the idele and x_∞ the infinite place of the idele, we define

$$f(x) = f_f(x_f)f_\infty(x_\infty)$$

and

$$g(x) = g_f(x_f)g_\infty(x_\infty).$$

Similarly to the previous sections, we define $|x|$ to be the product over all places of the norms of each coordinate of the idele x. Namely,

$$|x| = \prod_{\text{all places } p} |x_p|_p.$$

Also, we define $d^\times x$ to be the multiplicative measure on the ideles, which is a product of the local multiplicative measures for each local field defined in the first two sections. We also define a norm of a finite idele. If x_f is a finite idele then

$$|x_f|_f = \prod_{\text{all primes } p} |x_p|_p.$$

We define two iteration of f and g - one using the finite ideles, and the other using the whole idele group. Let

$$I(s_1, s_2) = \int_{|x_{f,1}|_f > |x_{f,2}|_f} f_f(x_{f,1})|x_{f,1}|_f^{s_1} d_f^\times x_{f,1} g(x_{f,2})|x_{f,2}|_f^{s_2} d_f^\times x_{f,2},$$

where $x_{f,1}$ and $x_{f,2}$ are finite ideles.

Theorem 6.1

$$I(s_1, s_2) = L(F, G, s_1, s_2).$$

Proof. Restrict the integral to the domain where the support of f_f is not zero. For any such idele we have $|x|_f = 1/n$ for some $n \in \mathbb{N}$. Therefore

$$I(s_1, s_2) =$$

$$= \int_{|x_1|_f > |x_2|_f} f_f(x_1)|x_1|_f^{s_1} d_f^\times x_1 f_f(x_2)|x_2|_f^{s_2} d_f^\times x_2 =$$

$$= \sum_{0<n_1<n_2} \int_{n_1\hat{\mathbb{Z}}^\times \times n_2\hat{\mathbb{Z}}^\times} a_{n_1}|x_1|_f^{s_1} d_f^\times x_1 b_{n_2}|x_2|_f^{s_2} d_f^\times x_2 =$$

$$= \sum_{0<n_1<n_2} \frac{a_{n_1} b_{n_2}}{n_1^{s_1} n_2^{s_2}} =$$

$$= L(F, G, s_1, s_2)$$

is the desired iterated L-function.

Let also

$$J(s_1,s_2)=\int_{|x_1|>|x_2|} f(x_1)|x_1|^{s_1}d^\times x_1 g(x_2)|x_2|^{s_2}d^\times x_2,$$

where x_1 and x_2 are ideles.

Theorem 6.2

$$J(s_1,s_2)=\Lambda(F,G,s_1,s_2).$$

Proof. The proof is essentially the same as the one for iteration of Melin transforms of theta functions minus 1.

We want to modify the functions $f(x)$ and $g(x)$ in the definition of $I(s_1,s_2)$ so that the two pairs of new functions are defined over $\mathbb{A}^\times/\mathbb{Q}^\times$ and over $\mathbb{R}^\times_{>0}$, respectively. In this modifications we want that suitable Mellin transforms of the new functions lead to the same function in terms of s_1 and s_2 for the iterated integral. We shall notice that the new functions defined over $\mathbb{R}^\times_{>0}$ are the modular functions $F(\tau)$ and $G(\tau)$ restricted to the geodesic connecting 0 and $i\infty$.

Denote by $\bar{x}$ the projection of an idele x to an element of $\mathbb{A}^\times/\mathbb{Q}^\times$. Let

$$\bar{f}(\bar{x})=\sum_{q\in\mathbb{Q}^\times} f(qx)$$

and

$$\bar{g}(\bar{x})=\sum_{q\in\mathbb{Q}^\times} g(qx)$$

Note that $|qx|=|x|$ for $q\in\mathbb{Q}$ and $x\in\mathbb{A}$. Denote by $|\bar{x}|:=|x|$ the norm of an element in $\mathbb{A}^\times/\mathbb{Q}^\times$. Recall that $\mathbb{Q}^\times$ is a discrete subgroup of $\mathbb{A}^\times$. For that reason we can take the same measure $d^\times x$ to the set $\mathbb{A}^\times/\mathbb{Q}^\times$. We have

$$\int_{|x_1|>|x_2|} f(x_1)|x_1|^{s_1}d^\times x_1 g(x_2)|x_2|^{s_2}d^\times x_2=\int_{|\bar{x}_1|>|\bar{x}_2|} \bar{f}(\bar{x}_1)|\bar{x}_1|^{s_1}d^\times x_1 \bar{g}(\bar{x}_2)|\bar{x}_2|^{s_2}d^\times x_2.$$

Now we define the corresponding functions $f_0(t)$ and $g_0(t)$ for $t\in\mathbb{R}$ and $t>0$. The function f is constant on each set $(qt,q\hat{\mathbb{Z}}^\times)$, where $t\in\mathbb{R}^\times_{>0}$, $q\in\mathbb{Q}^\times$ and $\hat{\mathbb{Z}}^\times=\prod_p\mathbb{Z}_p^\times$. We set $f_0(t)$ to be the value of $\bar{f}$ on the set $(qt,q\hat{\mathbb{Z}}^\times)$. Then

$$f_0(t)=\bar{f}(\bar{x})$$

for $t=|x|$. Similarly, we define $g_0(t)$ to be the value of $\bar{g}$ on the set $(qt,q\hat{\mathbb{Z}}^\times)$, $q\in\mathbb{Q}$. Then

$$\int_{|x_1|>|x_2|} f(x_1)|x_1|^{s_1}d^\times x_1 g(x_2)|x_2|^{s_2}d^\times x_2=\int_{t_1>t_2>0} f_0(t_1)t_1^{s_1}\frac{dt_1}{t_1}g_0(t_2)t_2^{s_2}\frac{dt_2}{t_2}.$$

Let us examine more carefully the relation between $f(x)$ and $f_0(t)$. Let x be an idele. Then for some $q\in\mathbb{Q}^\times$ and $r\in\mathbb{R}^\times$ we have $x\in(qr,q\hat{\mathbb{Z}}^\times)$. If q is not an integer then $f(x)=0$. Also, $f(-x)=f(x)$. For these reasons we can sum over all positive integers. Let x_f be the finite adele of x. That is all coordinates of x except the infinite one which is the real numbers.

Denote by $d^{\times}x_f$ the product of multiplicative Haar measure of $Q_p^{\times}$ over all primes. Denote, also, by $\hat{\mathbb{Z}}$ the product of the p-adic integers over all primes. Then

$$f_0(|x|) = 2\sum_{n\in\mathbb{N}} f_\infty(nr) \times \int_{n\hat{\mathbb{Z}}^{\times}} f_f(x_f)d^{\times}x_f.$$

For $n \in \mathbb{N}$ the integral becomes

$$\int_{n\hat{\mathbb{Z}}^{\times}} f_f(x_f)d^{\times}x_f = a_n.$$

For the infinite place we have

$$f_\infty(nr) = e^{-2\pi|nr|}.$$

Therefore,

$$f_0(t) = 2\sum_{n=1}^{\infty} a_n e^{-2nt\pi}.$$

Note that $f_0(t) = F(it)$, for $t \in \mathbb{R}$, where $F(\tau)$ is the cusp form of even weight. Iteration of $f(x)|x|^s$ and $g(x)|x|^s$ can be written as iteration of $f_0(t)t^s$ and $g_0(t)t^s$ which can be expressed in terms of iteration of the cusp forms $F(\tau)$ and $G(\tau)$.

$$\begin{aligned} &J(s_1,s_2) = \\ &= \textstyle\int_{|x_1|>|x_2|} f(x_1)|x_1|^{s_1}d^{\times}x_1 g(x_2)|x_2|^{s_2}d^{\times}x_2 = \\ &= \textstyle\int_{t_1>t_2>0} f_0(t_1)t_1^{s_1}\frac{dt_1}{t_1} g_0(t_2)t_2^{s_2}\frac{dt_2}{t_2} = \\ &= \textstyle\int_{r_1>r_2>0} F(ir_1)r_1^{s_1}\frac{dr_1}{r_1} G(ir_2)r_2^{s_2}\frac{dr_2}{r_2}. \end{aligned}$$

The changes of variables are $t_j = |x_j|$ and $r_j = t_j^2$ for $j = 1, 2$. Note that the last integral is an iteration of $F(\tau)$ and $G(\tau)$ over the geodesic in the upper half plane connecting 0 and $i\infty$.

7 Shuffle relations

In this section we describe algebraically how a product of iterated integrals over the adeles can be expressed as a sum of iterated integrals. This product formula involves permutation which resembles shuffling of a deck of cards. For that reason we call them shuffle relations. Similar shuffle relations occur among multiple polylogarithms and among multiple zeta values.

Definition 7.1 *5.1 By* **ordering** σ *we mean an enriched permutation it the following sense. Given d positive real numbers* $n_1, \cdots, n_d$*,* σ *acts by a permutation on the indices so that* $n_{\sigma(1)} \leq \cdots \leq n_{\sigma(d)}$*. The permutation is enriched in the following way. It keep track whether after the permutation two consecutive numbers are equal. To set the notation, given* 5 *numbers, if*

$$\sigma = (5(31)24)$$

then

$$n_5 < n_3 = n_1 < n_2 < n_4.$$

Also, $\sigma(1) = 5$, $\sigma(2) = 3$, $\sigma(3) = 1$, $\sigma(4) = 2$, $\sigma(5) = 2$. *And*

$$(5(31)24) = (5(13)24).$$

We set some more notation about ordering. Given a set of positive real numbers $N = \{n_1, \cdots, n_d\}$, let $\sigma[N] = \sigma[n_1, \cdots, n_d]$ be the ordering σ that gives $n_{\sigma(1)} \leq \cdots \leq n_{\sigma(d)}$. Let Σ_d denotes the set of all orderings of d positive integers. Let

$$\Sigma_{(21)(354)}$$

denotes all orderings σ such that

$$n_2 < n_1 \text{ and } n_3 < n_5 < n_4.$$

Let σ_1 be an ordering of d_1 positive real numbers $\{n_1, \ldots, n_{d_1}\}$. And let σ_2 be an ordering of d_2 positive real numbers $\{n_{d_1+1}, \ldots, n_{d_1+d_2}\}$. We denote by

$$\Sigma_{(\sigma_1)(\sigma_2)}$$

the set of all orderings such that the first d_1 real numbers are ordered by σ_1 and the next d_2 real numbers are ordered by σ_2.

Let $L_i(s_i)$ for $i = 1, \ldots, d$ be L-functions, possibly different, defined an integral over the adeles or over the finite adeles by

$$L_i(s_i) = \int f_i(x)|x|^{s_i} d^\times x,$$

where x is either an invertible adele or a finite invertible adele. Given an ordering σ_0, let

$$(\prod_{i=1}^{d} L_i(s_i))_{\sigma_0} = \int_{\substack{x_1, \ldots, x_d : \\ \sigma(|x_1|, \ldots, |x_d|) = \sigma_0}} f_1(x_1)|x_1|^{s_1} d^\times x_1 \ldots f_d(x_d)|x_d|^{s_d} d^\times x_d.$$

Theorem 7.2 *Let* σ_1 *and* σ_2 *be orderings of* d_1 *and* d_2 *elements, respectively. Let* $L_i(s_i)$ *for* $i = 1, \ldots, d_1 + d_2$ *be* L*-functions of zeta functions or completes zeta functions, which have adelic interpretation. Then*

$$(\prod_{i=1}^{d_1} L_i(s_i))_{\sigma_1} (\prod_{i=d_1+1}^{d_1+d_2} L_i(s_i))_{\sigma_2} = \sum_{\sigma \in \Sigma_{(\sigma_1)(\sigma_2)}} (\prod_{i=1}^{d_1+d_2} L_i(s_i))_\sigma$$

The proof comes directly from the definitions of the L-series and the orderings. We remark that if all the L-series have an Euler factor at any of the infinite places the the orderings are just permutation, in the sense that all inequalities are strict.

I would like to remark that the above shuffle relation applied to multiple zeta functions corresponds to the shuffle relation obtained by the infinite sum representation.

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