Automorphic forms for unitary groups and Galois representations
Eigenvarieties of unitary groups

Three lectures at the Clay Mathematical Institute Summer School, Honolulu, Hawaii, 2009

Prerequisites: The prerequisites for these lectures are:

(i) Notions on algebraic groups.
(ii) If possible, classical modular forms, and their adelic interpretation.
(iii) For the eigenvarieties part, some notion of rigid analytic geometry.

Notations: During most of the lecture, F will be a number field. The ring of adèles wil be denoted by $\mathbb{A}_F$ or simply $\mathbb{A}$. It is a product $\mathbb{A}_F = \mathbb{A}_{F,f} \times \mathbb{A}_{F,\infty}$ of finite adèles $\mathbb{A}_{F,f} = \mathbb{A}_f$ and infinite adèles $\mathbb{A}_{F,\infty} = \mathbb{A}_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. When g is an adèle (or an idèle, or an adèle-valued point of a group scheme over F) we denote by g_f and g_∞ its finite and infinite componen. We often identify g_f with the adèle (or idèle, etc.) which ha has the same finite components and 0 (or 1) at all infinite component, and similarly for g_∞ so that $g = g_f + g_\infty$)or $g = g_f g_\infty$).

We shall denote by $\bar{\mathbb{Q}}$ the set of complex numbers that are algebraic over $\mathbb{Q}$. So $\bar{\mathbb{Q}}$ is supposed to be embedded in $\mathbb{C}$.

Contents

This lecture is about automorphic forms and representations for unitary groups, and their attached Galois representations. The very existence of those attached representations is a recent progress that constitutes one of the most important achievement in the still largely open Langlands' program. It has been the result of a huge collective work of many mathematicians over more than thirty years (realizing an initial sketch, incredibly accurate in retrospect, of Langlands in the seventies). It is absolutely out of question to give or even tho sketch the proof of this existence in this paper.

The aim of this lecture is only to provide a short and intuitive introduction to automorphic forms and representations for unitary groups, and to state the existence and main properties of their attached Galois representations. Even simply giving a proper and workable definition of automorphic representations for a general unitary group is a task that cannot be properly done in less than 100 pages. There are many technical difficulties (especially at the archimedean places), which, through important, are not essential to the intuitive understanding of the notions. To avoid most of those difficulties, I will only consider a special case, namely unitary groups that are definite, that is compact at all archimedean places. While there are many arithmetical applications for which this special case may be sufficient, considering the general case is necessary in many other (including some that might be explained by Chris using Eisenstein series, which only exists for non-definite unitary groups) and moreover, the construction of Galois representations requires to consider non-definite places.

A natural question is: why unitary groups? The short answer is simply: because they are (with a few sporadic other cases, most notably Sp_4) the only algebraic group for which we know, at this point, how to attach Galois representation to automorphic representations. We should be able to do the same to many other automorphic representations (e.g. all algebraic automorphic representations for GL_n) but it seems that some very difficult new ideas are missing in order to do that.

1. Unitary groups

1.1. **Generalities.** Let k be a field of characteristic 0 (to simplify), and E be an étale algebra of degree 2 of k (that is to say, either E is $k \times k$ or it is a extension of degree 2 of E). The algebra E has only one non-trivial k-automorphism that we note c. Let V be a free k-module of rank n, and $q : V \times V \to E$ be a non-degenrate

c-Hermitian form. (That is to say, $q(x,y)$ is E-linear in x and c-semilinear in y, we have $q(x,y) = c(q(y,x))$ for all $x, y \in V$, and $q(x,y) = 0\ \forall y \in V$ implies $x = 0$.)

To these data (k, E, V, q) we can attach an algebraic group over k.

Definition 1.1. The *unitary group G attached to* (k, E, V, q) is the algebraic group whose functor of points is

$$G(R) = \{g \in \mathrm{GL}_{E\otimes_k R}(V \otimes_k R), \quad q(gx, gy) = q(x,y) \forall x, y \in V \otimes_k R\}.$$

for all k-algebra R. In the case where E is a field (rather than $k \times k$) we shall say that G is a true unitary groups.

Exercise 1.1. This definition implicitly assumes that the functor $R \mapsto G(R)$ is representable by a scheme over k. Prove it.

Intuitively, this is not very complicated. The k-points of the unitary group are the set of E-linear automorphism of V that preserves the hermitian form q. Similarly for R-points.

Example 1.1. Take $k = \mathbb{R}$, $E = \mathbb{C}$, $V = \mathbb{C}^n$ and

$$q((x_1, \dots, x_n), (y_1, \dots, y_n) = x_1 c(y_1) + \dots x_p c(y_p) - x_{p+1} c(y_{p+1}) - \dots - x_n c(y_n).$$

Then the unitary group attached to these date is denoted $U(p, n-p)$. The set of real points $U(p, n-p)(\mathbb{R})$ is the classical unitary group of signature $(p, n-p)$ in the sense of undergraduate mathematics. It is compact if and only if $p = 0$ or $p = n$. Note that $U(p, n-p)$ is isomorphic to $U(n-p, p)$. It is not hard that any true unitary groups over $\mathbb{R}$ is isomorphic to one of those $U(p, n-p)$.

Proposition 1.1. *If E is $k \times k$, then the unitary group G is isomorphic to* GL_n. *The is isomorphism is well-defined up to inner automorphism if we chose one of the two k-morphism $E \to k$.*

Proof — Let $p : E \to k$ be a k-morphism. The choice of p makes the isomorphism $E = k \times k$ (as k-algebra canonical, by saying tjat p is the first projection on $k \times k$. Let $V_0 = V \otimes_{E,p} k$. We construct an morphism of functors $G(R) \to \mathrm{GL}_R(V_0 \otimes_k R)$ by sending $g \in G(R)$ (an E-automorphism on $V \otimes_E R$) to its action on $V \otimes_E R \otimes_E k = V_0 \otimes_k R$. We leave to the reader to check that is an isomorphism. □

Now, let k' be any extension of k. We can consider $E' = E \otimes_k k'$ which is still an etale algebra over k' (but not that if E was a field, this may fail for E', hence the interest of the generality adopted), $V' = V \otimes_E E'$ which is still free of rank n over E, and q' the natural extension of q to V'. To the data (k', E', V', q') one can attach an unitary group G' over k'. It is clear from the definition that G' is $G \times_k k'$.

1.2. **Unitary groups over totally real number fields.** Let F be a totally real finite extension of $\mathbb{Q}$, and E be a quadratic extension of E that is imaginary.. Let c be the non-trivial automorphism of E over F. let V be an E-vector space of dimension n and q be a non-degenerate c-hermitian form on V. To (F, E, V, q) one can attach a unitary group G over F.

To get a better understanding of G we analyze the groups $G_v = G \times_F F_v$ for all places v of F.

If v is **archimedean**, then by assumption v is real, that is $F_v = \mathbb{R}$, and $E_v = \mathbb{C}$, while c induces the complex conjugation on $\mathbb{C}$. The group G_v is therefore $\mathbb{R}$-isomorphic to a group $\mathrm{U}(p_v, n-p_v)$ for some $0 \le p_v \le n$ as in example 1.1.

Definition 1.2. We shall say that G is *definite* if G_v is compact (that is $p_v = 0$ or $p_v = n$) for all real places v of F.

If v is **finite and split** in E, then $E \otimes_F F_v$ is not a field, but is isomorphic to $F_v \times F_v$. Therefore, the group G_v which is the unitary group attached to $(F_v, E \otimes_F F_v, \dots, \dots)$ is by Prop 1.1 isomorphic to GL_n over F_v. The choice of one of the two F_v-morphisms$E \otimes_F F_v$ to F_V, that is of one of the two places of E above v makes this isomorphism canonical, up to conjugation.

If v is *finite and either inert or ramified* in E, then E_v is a quadratic extension of F_v, so G_v is a true unitary groups of v.

The point to remember (for beginners) is that even a true unitary group over F becomes GL_n at basically one place of F over two: the places that are split in E. This will be very helpful later.

1.3. **Adelic points of G.** Let G be a unitary group as in the preceding §. Since the ring of adèles $\mathbb{A} = \mathbb{A}_F$ is an F-algebra (via the diagonal embeding), it make sense to talk of the group $G(\mathbb{A}_F)$. SInce G is linear, that is a subgroup of GL_m for some m, $G(\mathbb{A}_f) \subset M_m(\mathbb{A}_F) = \mathbb{A}_F^{m^2}$ which allots to give a natural topology on $G(\mathbb{A}_F)$, namely the coarsest that is finer that the topology of $\mathbb{A}_F^{m^2}$ and for which $x \mapsto x^{-1}$ is continuous. This topology is easily been seen to be independent of the choices. We shall need a good understanding of this group and of its natural topology.

Let $\mathcal{G}$ be a model of G over $\operatorname{Spec} \mathcal{O}_F[1/f]$ where $f \in \mathcal{O}_F - \{0\}$, that is a group scheme over this scheme whose generic fiber is G. Then for all v prime to f, the modeles define a subgroup $\mathcal{G}(\mathcal{O}_v)$ of $G(F_v)$ which is compact when v is finite. We can therefore form the restricted product $\prod'_{v \text{ places of } F} G(F_v)$ with respect to the subgroups $\mathcal{G}(\mathcal{O}_v)$.

Lemma 1.1. *This restricted product with its restricted product locally compact topology is independant of the model $\mathcal{G}$ chosen and is naturally isomorphic, as a topological group, with $G(\mathbb{A}_F)$.*

The same lemma of course holds for the finite adèles $G(\mathbb{A}_{F,f}) = \prod'_{v \text{ finite}} G(F_v)$ We have $G(\mathbb{A}_F) = G(\mathbb{A}_{F,f}) \times G(\mathbb{A}_{F.\infty})$.

The group $G(F)$ can be embedded diagonally in $G(\mathbb{A}_F)$. Its image is discrete. If G is definite, then $G(F)$ is discrete in $G(\mathbb{A}_{F,f})$ and the quotient is compact.

1.4. **Local theory.** In order to understand automorphic representations, we need to remind (without any proof) a very little part of the very important and still active theory of representations of p-adic Lie group.

1.4.1. *Brief review on smooth representations.* In this § only, G will be a group where there is basis of neighborhood of 1 made of compact open subgroups U.

Exercise 1.2. Prove that such a group is locally compact and totally disconnected. What about the converse?

Let k be a field of characteristic 0. By a *smooth representation* V of G over k we mean a k-vector space (not finite dimensional in general) with a continuous action of G such that for every vector $v \in V$, there exists a compact open subgroup U in G such that v is invariant by U.

A smooth representation is *admissible* if for every open subgroup U of G, V^U is finite dimensional over k.

Let us fix a (left invariant, but since in practice our G will be *unimodular*, this doesn't really matter) Haar measure dg on G, normalized such that the measure of some open compact subgroup U_0 is 1. Then is is easily seen that the measure of any other compact open subgroup will be a rational, hence an element in k.

We denote by $\mathcal{H}(G,k)$ the spaces of function from G to k that are locally constant and have compact support. This space as a natural product : the convolution of functions $(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g)\ dx$. Indeed, since f_1 and f_2 are in $\mathcal{H}(G,k)$, the integral is actually a finite sum, an what is more a finite sum of terms that are products of a value of f_1, a value of f_2, and the measure of a compact open subgroup. So the integral really defines a k-valued function, and it is easy to check that $f_1 * f_2 \in \mathcal{H}(G,k)$. This product makes $\mathcal{H}(G,k)$ an algebra, which is not commutative if G is not, and which is general has no unity (the unity would be a Dirac at 1, which is not a function on G if G is not discrete). If V is a smooth representation of G it has a natural structure of $\mathcal{H}(G,k)$-module by $f.v = \int_G f(g) g.v\ dg$. The algebra $\mathcal{H}(G,k)$ is called the *algebra of the group* G over k.

Let U be a compact open subgroup of G. Let $\mathcal{H}(G,U,k)$ be the set of functions form G to k that have compact support and that are both left and right invariant by U. This is easily seen to be a subalgebra of $\mathcal{H}(G,k)$, with unity (the unity is the characteristic function of U times a normalization factor depending on the Haar measure). The algebra $\mathcal{H}(G,U,k)$ is called the *Hecke algebra of* G *w.r.t* U over k. We have $\mathcal{H}(G,k) = \cup_U \mathcal{H}(G,U,k)$ and in particular we see that if G is not commutative, the $\mathcal{H}(G,U,k)$ are not for U small enough. The Hecke algebra are important because of the following trivial property:

Lemma 1.2. *Let V be a smooth representation of G (over k). Then the spaces of invariant V^U has a natural structure of $\mathcal{H}(G,U,k)$-modules.*

Indeed, if $v \in V^U \subset$, and $f \in \mathcal{H}(G,U,k) \subset \mathcal{H}(G,k)$, $f.v$ is easily seen to be in V^U.

If there is one thing to remember about Hecke algebras it is this lemma, not the definition. In Jeopardy style : define the Hecke Algebra of G relatively to U... "What acts on V^U when V is a (smooth) representation of G? "

We will apply this theory to groups $G(F_v)$ when G is a unitary group as above and v is a finite place of F.

Exercise 1.3. Let k' be an extension of k.

a.— Show that if V is a smooth representation of G over k, then $V \otimes_k k'$ is a smooth representation of G over k'.

b.– Show that the formation of V^U commutes to the extesions k'/k.

c.– Show that $\mathcal{H}(G,U,k) \otimes_k k' = \mathcal{H}(G,U,k')$.

Exercise 1.4. Let $\mathcal{C}(G,k)$ and $\mathcal{C}(G/U,k)$ be the space of smooth functions from G and G/U to k.

a.– Show that they both are smooth representation of G (for the left translations)

b.– Show $\mathcal{C}(G,k)^U = \mathcal{C}(G/U,k)$. Therefore $\mathcal{H}(G,U,k)$ acts on $\mathcal{C}(G,U,k)$. Show that this action commutes to the G-action.

c.– Show that $\mathcal{H}(G,U,k)$ is actually naturally isomorphic to $\mathrm{End}_G(\mathcal{C}(G/U,k))$.

Exercise 1.5. a.– Show that $V \mapsto V^U$ is an exact functor form the category from smooth representation of G over k to $\mathcal{H}(G,U,k)$-module. This functor takes admissible representations to $\mathcal{H}(G,U,k)$ that are finite dimensional.

b.– Let $W = V^U$, $W' \subset W$ a sub-$\mathcal{H}(G,U,k)$-module, and $V' \subset V$ the subrepresentation of V generated by W'. Show that $V'^U = W$.

c.– Deduce that if V is irreducible as a representation of G, than V^U is irreducible as a $\mathcal{H}(G,U,k)$-module

Note that in general the functor defined in a.– above is not fully faithful (even on admissible rep.) and that the converse of c.– is false.

1.4.2. *Maximal compact subgroups.* If v is a finite place of F that splits in E, then $G(F_v) \simeq \mathrm{GL}_n(F_v)$. It is an easy fact that all maximal compact subgroups of $\mathrm{GL}_n(F_v)$ are conjugate (and conjugate to $\mathrm{GL}_n(\mathcal{O}_v)$.

If v is a finite place that does not spit in E, then $G(F_v)$ is a true unitary groups. It is not true that all maximal compact subgroups of $G(F_v)$ are conjugate, though there are only finitely many conjugacy class. If v is inert, then there is one class whose elements have maximal volume (for a fixed Haar measure), and we call compact of this class *maximal hyperspecial* compact subgroup of $G(F_v)$. We shall neglect places of F that ramify in E since there only in finite number.

Back to the case v split, we shall call any maximal compact subgroup hyperspecial.

Proposition 1.2 (Tits)**.** *Let $\mathcal{G}$ be a model of G over Spec $\mathcal{O}_F[1/f]$. Then $\mathcal{G}(\mathcal{O}_v)$ is a maximal compact hyperspecial subgroup of $G(F_v)$ for almost all finite places v.*

Corollary 1.1. *Any compact open subgroup of $G(\mathbb{A}_f)$ contains a compact open subgroup of the form $\prod_v U_v$ with U_v a compact open subgroup of $G(F_v)$ for all v, with U_v maximal hyperspecial for almost all v.*

1.4.3. *Spherical Hecke algebras and unramified representations.* Let v be a place of F (not ramified in E), and K_v be a maximal compact hyperspecial subgroup of $G(F_v)$. Let k be any field of characteristic 0.

Proposition 1.3. *The algebra $\mathcal{H}(G(F_v), K_v, k)$ is commutative*

This fact is not a tautology. As we have seen, $\mathcal{H}(G(F_v), U)$ will certainly be non commutative for small enough compact open subgroups. The fact that it is commutative for K_v a well chosen maximal compact subgroup is what relates ultimately the theory of automorphic forms to commutative algebra, and is at the basis of all modern development ($R = T$. eigenvarieties, etc.) Instance of this phenomenon were first noticed by Poincare in his paper on the shape of Saturn's rings. The proposition above is due I guess, to Bruhat, and dates back to the early fifties.

Actually we can even determine the structure of the Hecke algebra $\mathcal{H}(G(F_v), K_v, k)$. We shall only need it in the case v split, so $\mathcal{H}(G(F_v), K_v, k) = \mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_v), k)$.

Proposition 1.4 (Satake)**.** *There exists an isomorphism of k-algebras*

$$\mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_v), k) \simeq k[T_1, \dots, T_{n-1}, T_n, T_n^{-1}]$$

that sends T_i to the characteristic function of

$$\mathrm{GL}_n(\mathcal{O}_v)\, diag(\pi, \dots, \pi, 1, \dots, 1) \mathrm{GL}_n(\mathcal{O}_v),$$

where the number of π's is i.

Lemma and Definition 1.1. Let V be a smooth irreducible representation of $G(F_v)$ over k, and K_v a maximal hyperspecial subgroup of G. The statements $V^{K_v} \neq 0$ and $\dim_1 V^{K_v} = 1$ are equivalent. We call V *unramified* if they hold.

Actually this lemma follows from the proposition 1.3 above as follows: we can easily (see exercise 1.3) reduce to the case k algebraically closed and then since if V is irreducible, V^K is as a $\mathcal{H}(G, K_v, k)$-module (see exercise 1.5), hence has dimension one since the latter is commutative.

Here is an important observation: If V is unramified, V^{K_v} has dimension 1 and $\mathcal{H}(G(F_v), K_v, k)$ acts on V^{K_v}. Therefore, V determines a character

$$\psi_{V,v} : \mathcal{H}(G(F_v), K_v, k) \to k. \tag{1}$$

It is a theorem (that we shall not use) that V is uniquely determined by ψ_V.

The information containing in $\psi_{V,v}$ (that is the list of values $\psi_V(T_1), \dots, \psi_V(T_n)$ in k, the latter being non-zero), can be summarized in a polynomial, the Satake polynomial.

Definition 1.3. The Satake polynomial of V is the polynomial

$$P_{V,v}(X) = X^n - \psi_{V,v}(T_1)X^{n-1} + \psi_{V,v}(T_2)X^{n-2} - \cdots + (-1)^n \psi_{V,v}(T_n) \in k[X].$$

Important caveat: This is not the correct normalization (for what follows). Actually coefficients should be multiplied by suitable integral power (or perhaps half-integral power) of the cardinality of the residue field. The correct form is to be found in Harris-Taylor's book. Without access to this book here , this would be an excessive effort (so close to the beach) to retrieve the correct coefficients. I'll put them after the conference.

Exercise 1.6. Prove Proposition 1.4 for $n = 2$, as follows. Replace first $\mathrm{GL}_2(K_v)$ by $G = \mathrm{PGL}_2(K_v)$ and K by the image of $\mathrm{GL}_2(\mathcal{O}_v)$ in G. Proceed as follows:

a.– Construct an isomorphism of G-representations $\mathcal{C}(G/K, k) = \mathcal{C}(X, k)$ where X is the tree of $\mathrm{PGL}_2(K_v)$ defined in the lectures note son Ribet's lemma, and $\mathcal{C}(X, k)$ the set of functions from X to k with finite support. Deduce (use exercise 1.4) an ismorphism of algebras $\mathcal{H}(G, K, k) \simeq \mathrm{End}_{k[G]}(\mathcal{C}(X, k))$.

b.– Let T_1 be the characteristic function of $K \mathrm{diag} \pi, 1 K$. Show that this element of $\mathcal{H}(G, K, k)$, seen as an operator on $\mathcal{C}(X, k)$ by the baoce isomorphsim, sends a function f on X to the function $f'(x) = \sum_{y \text{ neighbor of } x} f(y)$.

c.– Deduce that $\mathcal{H}(G, K, k) = \mathbb{C}[T_1]$. Conclude.

2. Automorphic representations for G

We now fix a data (F, E, V, q) as in the preceding §and we assume that the group G is definite.

2.1. Automorphic forms.

Definition 2.1. A function $f : G(\mathbb{A}_F) = G(\mathbb{A}_{F,f}) \times G(\mathbb{A}_{F,\infty}) \to \mathbb{C}$ is said *smooth*, if it is continuous and if $f(g_f, g_\infty)$ is $\mathbb{C}^\infty$ as a function of g_∞ (for g_f fixed) and is locally constant with compact support as a function of g_f (for g_∞ fixed).

Definition 2.2. A function $f : G(\mathbb{A}_F) \to \mathbb{C}$ is called *automorphic* (or an *automorphic form*) if it is smooth, left-invariant by $G(F)$, and if it generates a finite dimensional spaces under $G(\mathbb{A}_{F,f})$. The space of all automorphic forms is called $A(G)$.

The space $A(G)$ has a natural hermitian product

$$(f, f') = \int_{G(F)\backslash G(\mathbb{A}_F} f(g)\bar{f}'(g)\ dg,$$

, which makes it a pre-Hermitian space (not a Hermitian space, since it is not complere). It has a natural action of $G(\mathbb{A}_F)$ by right translation, which preserves the hermitian product, so $A(G)$ is a pre-unitary representation.

2.2. **Automorphic representations.** An irreducible representation π of $G(\mathbb{A})$ is said *admissible* if, writing $\pi = \pi_f \otimes \pi_\infty$ where π_f is an irreducible representation of $G(\mathbb{A}_f)$ and π_∞ is an irreducible representation of $G(\mathbb{A}_\infty)$, then π_f is admissible.

Theorem 2.1. *The representation $A(G)$ is the direct sum of irreducible admissible representations of $G(\mathbb{A})$:*

$$A(G) = \bigoplus_{\pi} m(\pi)\pi, \tag{2}$$

where π describes all the (isomorphism classes of) irreducible admissible representations of $G(\mathbb{A})$, and $m(\pi)$ is the (always finite) multiplicity of π in the above space.

It will be convenient to denote by Irr the set (of isomorphism classes) of irreducible complex continuous (hence finite dimensional) representations of $G(\mathbb{A}_{F,\infty})$. For $W \in \text{Irr}$, we define $A(G, W)$ to be the $G(\mathbb{A}_{F,f})$-representation by right translation on the space of smooth vector valued functions $f : G(\mathbb{A}_{F,f}) \longrightarrow W^*$ such that $f(\gamma g) = \gamma_\infty f(g)$ for all $g \in G(\mathbb{A}_{F,f})$ and $\gamma \in G(\mathbb{F})$.

Proof — (Sketch) As $G(\mathbb{A}_{F,\infty})$ is compact the action of this group on $A(G)$ is completely reducible, hence as $G(\mathbb{A}_F) = G(\mathbb{A}_{F,\infty}) \times G(\mathbb{A}_{F,f})$-representation we have:

$$A(G) = \bigoplus_{W \in Irr} W \otimes (A(G) \otimes W^*)^{G(\mathbb{A}_{F,\infty})}.$$

But we check at once that the restriction map $f \mapsto f_{|1 \times G(\mathbb{A}_{F,f})}$ induces a $G(\mathbb{A}_{F,f})$-equivariant isomorphism

$$(A(G) \otimes W^*)^{G(\mathbb{A}_{F,\infty})} \simeq A(G, W).$$

As a consequence, the compactness of $G(F)\backslash G(\mathbb{A}_f)$ shows, by classical arguments that $A(G)$ is admissible, which together with the pre-unitariness of $A(G,W)$ proves the lemma. $\square$

Definition 2.3. An irreducible representation π of $G(\mathbb{A})$ is said to be *automorphic* if $m(\pi) \neq 0$.

Automorphic representations (for definite unitary groups) are always algebraic, in the following sense.

Proposition 2.1. *If π is automorphic, the representation π_f has a model over $\bar{\mathbb{Q}}$.*

Proof — Let $W \in \mathrm{Irr}$ and let us restrict it to $G(F) \hookrightarrow G(\mathbb{A}_{F,\infty})$. As is well known W comes from an algebraic representation of G, hence the inclusion $\bar{\mathbb{Q}} \subset \mathbb{C}$ equips W with a $\bar{\mathbb{Q}}$-structure $W(\bar{\mathbb{Q}})$ which is $G(\bar{\mathbb{Q}})$-stable. As a consequence, the obviously defined space $A(G, W(\bar{\mathbb{Q}}))$ provides a $G(\mathbb{A}_f)$-stable $\bar{\mathbb{Q}}$-structure on $A(G,W)$, and the results follows. $\square$

Definition 2.4. We say that a compact open subgroup U of $G(\mathbb{A}_f)$ is a *level* for an automorphic form π if $\pi^U \neq 0$. Equivalently $\pi_f^U \neq 0$. Or we say simply that π has level U. The weight of π is simply the finite dimensional representation π_∞.

Of course, if $U' \subset U$ and then if π has level U it has also level U'.

It is not hard to see that there exists only a finite number of automorphic representations with a fixed level and weight. (We shall not use it, but it fixes the ideas).

2.3. **Decomposition of automorphic representations.** Recall that if $(V_i)_{i\in I}$ is a family of vector spaces, with $W_i \subset V_i$ a given dimension 1 subspace defined for almost all i (that is forall i except for a finite set J_0 of I), then the restricted tensor product $\bigotimes'_{i\in I} V_i$ is defined as the inductive limit of $\bigotimes'_{i\in J} V_i \otimes \bigotimes_{i \in I-J} W_i$ over the filtering set ordered by inclusion, of finite subsets J (containing J_0) of I.

Theorem 2.2. *Every admissible irreducible representation π_f of $G(\mathbb{A}_{F,f})$ can be written in a unique way as a restricted tensor product $\pi_f = \otimes'_{v \text{ finite place}} \pi_v$ where π_v is a irreducible admissible representation of $G(F_v)$, and π_v is unramified for almost all v. More precisely, if $\pi_f^U \neq 0$ where $U = \prod_v U_v$, then $\pi_v^{U_v} \neq 0$.*

In particular, an automorphic representation $\pi = \pi_f \otimes \pi_\infty$ has components π_v at all places v of F: For finite v these are the components π_v of π_f in the above sense, and for infinite v, simply the component of π_∞ in the usual sense. If π_f has level $U = \prod_v U_v$, with U_v hyperspecial for all v except those in a finite set of places Σ, then π_v is unramified for $v \notin \Sigma$.

3. Galois representations

3.1. **set-up.** Let π be an automorphic representation of level U, and assume that U contains $\prod_v U_v$. Let $\Sigma(U)$ be the set of places v of F such that

(i) the place v splits in E
(ii) U_v is a compact maximal (necessariy hyperspecial) of $G(F_v)$

If $v \in \Sigma(U)$, then π_v is an unramified representation of $G(F_v)$, defined over $\bar{\mathbb{Q}}$. The choice of one of the two places w of v defines an isomorphism (up to conjugacy) $G(F_v) \simeq \mathrm{GL}_n(F_v)$, so allows us to see π_v as a well-determined up to isomorphism representation of $\mathrm{GL}_n(F_v)$. To such a representation one can attach its Satake polynomial, that we shall denote by $P_{\pi,w}(X)$.

3.2. **Existence.** The following theorem may now be considered as proven (even if some details have not yet appeared in print).

Let us fix an embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$.

Theorem 3.1. *Let p be a prime. There exists a unique semi-simple Galois representations $\rho_\pi : G_E \to \mathrm{GL}_n(\mathbb{Q}_p)$ such that at all places v of F in $\Sigma(U)$ such that v does not divide p, then for the two places w of E above v, ρ_π is unramified at w, and the characteristic polynomial of $\rho_\pi(\mathrm{Frob}_w)$ has coefficients in $\bar{\mathbb{Q}}$ and is equal to $P_{\pi,w}$.*

Note that the representation ρ_π is a representation of G_E, not of G_F Let us prove the uniqueness: The set of w above a place in $\Sigma(\pi)$ has density one in G_E. Therefore by Cebotarev, the set of such Frob_w in G_E is dense, and in particular the character of ρ_π is well determined by our condition. Therefore so is ρ_π since it is assumed semi-simple. (the proof of existence is about five thousand times longer).

3.3. **Properties.** The representation ρ_π enjoys many more properties. Let us give the most important ones.

(i) The non trivial automorphic $c \in \mathrm{Gal}(E/F)$ induces by conjugation an outer automorphism of G_E, still denoted c. We have $\rho_\pi^c = \rho_\pi^*(1-n)$. In particular ρ_π is polarized in the sense of my notes on Bloch-Kato.

This is easy by looking at the form of the characteristic polynomials $P_{\pi,w}$ and $P_{\pi,w'}$ where w and w' are the two places above v, where v is as in the theorem. Of course, I have not been explicit enough about the normalization so that you can check the details.

(ii) If v is any place of F where π is unramified, and w is a place of E above v, then ρ_π is unramified at v. (this is contained in the theorem for v split, but this also true for v inert.) In particular ρ_π is unramified almost everywhere.

(iii) For v split, and w above v, the restriction of ρ_π to G_{E_w} corresponds by Local Langlands to the representation π_v of $G(F_v)$ seen as a representation of $\mathrm{GL}_n(F_v)$ using the isomorphism determined by w. This determines what happens at all split v, for π unramified or not at v. (the analog statement for v non-split is not known in full generality so far).

(iv) The representations ρ_π is de Rham at all places dividing p and the Hodge-Tate weights are determined by π_∞. The representation ρ_π is even crystalline at those places w of E that are unramified above places v of F such that π_v is unramified. The crystalline Frobenius slope are determined by π_∞ and $P_{\pi,w}$.

(v) The representation ρ_π is geometric. (This follows from (ii) and (iv)). In most cases (technically, if π_∞ is regular – see below), it is know by construction that ρ_π actually comes from geometry. If ρ_π is irreducible, it is pure of motivic weight, $1-n$.

(vi) The L-function $L(\rho_\pi, s)$ satisfy all conjectures about L-functions sated in my BK notes (continuation, no zeros on the boundary of the domain of convergence, no poles exceot trivial case, functional equation).

Note that ρ_π is not irreducible in general. We can construct examples (*endoscopic* forms and *C.A.P* forms) of π for which ρ_π is reducible, and even some where its constituents have not all the same motivic weights (in those cases though, the set of motivic weights is an arithmetic progression of ratio 1). However, for a large class of representations π called *stable* (defined as those π whose base change to GL_n/E is cuspidal), it is expected that ρ_π is irreducible. It is known so far only if $n \leq 3$ (Blasius-Rogawsky for $n=3$), $n=4$ if $F=\mathbb{Q}$ and $\pi^c=\pi$ (Ramakrishna) or any n and E but π_v square integrable at some place v of F split in E (Taylor-Yoshida).

3.4. **Hodge-Tate weights.** We here explain how the weight π_∞ of π determine the HT weights of ρ_π. For simplicity, we do so only in the case where $F=\mathbb{Q}$ and p splits in F. In this case π_∞ is simply a representation of the compact Lie group $U(n)(\mathbb{R})$, necessarily finitely dimensional.

If $\underline{m} := (m_1, \dots, m_n) \in \mathbb{Z}^n$ satisfies $m_1 \geq m_2 \geq \cdots \geq m_n$, we denote by $W_{\underline{m}}$ the rational (over $\mathbb{Q}$), irreducible, algebraic representation of GL_m whose highest weight relative to the upper triangular Borel is the character[1]

$$\delta_{\underline{m}} : (z_1, \cdots, z_n) \mapsto \prod_{i=1}^{n} z_i^{m_i}.$$

For any field F of characteristic 0, we get also a natural irreducible algebraic representation $W_{\underline{m}}(F) := W \otimes_{\mathbb{Q}} F$ of $\mathrm{GL}_n(F)$, and it turns out that they all have this form, for a unique $\underline{m}$.

Let us fix an embedding $E \hookrightarrow \mathbb{C}$, which allows us to see $\mathrm{U}(n)(\mathbb{R})$ as a subgroup of $\mathrm{GL}_n(\mathbb{C})$ well defined up to conjugation (see Prop.1.1). So for $\underline{m}$ as above, we can view $W_{\underline{m}}(\mathbb{C})$ as a continuous representation of $\mathrm{U}(n)(\mathbb{R})$. As is well known, the set of all $W_{\underline{m}}(\mathbb{C})$ is a system of representants of all equivalence classes of irreducible

[1] This means that the action of the diagonal torus of GL_n on the unique $\mathbb{Q}$-line stable by the upper Borel is given by the character above.

continuous representations of $\mathrm{U}(m)(\mathbb{R})$. We will say that $W_{\underline{m}}$ has *regular weight* if $m_1 > m_2 > \cdots > m_n$.

So by the above $\pi_\infty = W_{\underline{m}}(\mathbb{C})$. The identification depends on an embedding of E to $\mathbb{C}$ hence an embedding of E to the field $\bar{\mathbb{Q}}$ of algebraic number in $\mathbb{C}$, hence via the chosen embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$, an embedding $E \hookrightarrow \bar{\mathbb{Q}}_p$, that is a place w of E above p.

Proposition 3.1. *The Hodge-Tate weights of* $(\rho_\pi)_{|G_{E_w}}$ *are* $k_1 = -m_1 + 1, k_2 = -m_2 + 2, \dots, k_n = -m_n + n$.

Note that the Hodge-Tate weights are always distinct (this is a consequence of our working with a definite unitary group, analog to the fact that "modular forms of weight 1 (that is of HT weights 0 and 0) are not quaternionic modular forms"). When π_∞ is regular, two Hodge-Tate weights are never consecutive numbers.

Exercise 3.1. a.– If w' is the other place of E above p, what are the Hodge-Tate weights of $(\rho_\pi)_{|G_{E_{w'}}}$?

b.– Is your answer conform to prediction 2.1 in the BK notes ?

4. The set of all automorphic forms of level U

We begin to move slowly toward the definition of eigenvariety.

From now, for simplicity we shall assume that $F = \mathbb{Q}$.

We shall fix an open compact subgroup U of $G(\mathbb{A}_f)$ as in 3.1, of which we keep all notations. We consider automorphic representations of level U, but any weight.

Let $\Sigma = \Sigma(U)$ be as above the set of places of F that are split in v and such that U_v is hyperspecial. Let $\mathcal{H}_\Sigma = \otimes_{v \notin \Sigma} \mathcal{H}(G(\mathbb{Q}_v), U_v, \bar{\mathbb{Q}})$. This is a commutative $\bar{\mathbb{Q}}$-algebra. Every automorphic representation π of level U defines a character

$$\psi_\pi : \mathcal{H}_\Sigma \to \bar{\mathbb{Q}} :$$

This characters sends a $T \in \mathcal{H}(G(\mathbb{Q}_v), U_v, \bar{\mathbb{Q}})$ to its eigenvalues on $\pi_v^{U_v}$ for all $v \in \Sigma$.

The character ψ_π contains a lot of the information of interest about π. In particular it determines the Galois representation ρ_π.

Let $\mathcal{Z}_U$ be the set of all characters of the form $\psi_\pi : \mathcal{H}_\Sigma \to \bar{\mathbb{Q}}$ of the form ψ_π for some automorphic π of level U. The set $\mathcal{Z}_U$ is enumerable, and there is of course a surjective map $\pi \to \psi_\pi$ from the set of automorphic forms of level U to $\mathcal{Z}_U$. This map is not injective in general (its fiber are called (approximately) the L-packets for $G(\mathbb{A})$.) As we have just noticed, the map $\pi \mapsto \rho_\pi$ factors through $\mathcal{Z}_U$.

We want to understand the set $\mathcal{Z}_U$.

Let us choose a topology on $\mathcal{Z}_U$ as follows. Let $|\ |$ be an absolute value of $\bar{\mathbb{Q}}$. We put a metric (that could take infinite value) on $\mathcal{Z}_U$ by saying that $d(\psi, \psi') = \sup_{v \in \Sigma,\ w|v,\ i=1,\dots,n} |\psi(T_{i,w}) - \psi'(T_{i,w})|$. This determines a topology of $\mathcal{Z}_U$.

It is a fact that if $|\ |$, the set $\mathcal{Z}_U$ for that topology is discrete. There is not much to say about this: There is no continuous archimedean families of automorphic forms for a definite unitary group. (Of course, for a non-definite unitary groups, there are the families of Eisensteins series studied by Langlands).

Now let p be a prime and assume that $|\ |$ is a p-adic absolute value of $\bar{\mathbb{Q}}$. (Say for compatibility the one induced by the embedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ we have already chosen). Then it is a fundamental fact, the basis of the theory of eigenvarieties, that $\mathcal{Z}_U$ is **not discrete**. Actually, we shall see that no point in $\mathcal{Z}_U$ is isolated. (I believe this point was first observed by Serre, though Ramanujan and Swinnerton-Dyer are precursors)

In that respect, $\mathcal{Z}_U$ is closely analog to $\mathbb{Z}$: $\mathbb{Z}$ has an archimedean topology, for which it is discrete, and a p-adic topology (for each p), for which it is not discrete, and even without isolated point. Can we put the analogy further. By the completion process, the points of $\mathbb{Z}$ (the integers) can be p-adically interpolated to define points of $\mathbb{Z}_p$ (p-adic numbers). Can the points of $\mathcal{Z}_U$ (the automorphic forms) can be p-adically interpolated to define more general object (p-adic automorphic forms)? As we shall see, yes. Let us put the the analogy further again. The topological space $\mathbb{Z}$ is a Zariski-dense subset of the set of $\mathbb{Q}_p$-points $\mathbb{Z}_p = B^1(\mathbb{Q}_p)$ of the rigid analytic variety $B^1 = \operatorname{Sp} \mathbb{Q}_p < T >$. Can we find a natural rigid analytic variety $\mathcal{E}$ such that $\mathcal{Z}_U \subset \mathcal{E}(\mathbb{Q}_p)$ in the same way? Until we precise our requirement on $\mathcal{E}$, the question is a little bit too vague to have an interesting answer. So let us try to precise it.

We fix an embedding $E \subset \mathbb{C}$, and we assume that p is split in E, and that U_p is a maximal compact (so that $p \in \Sigma$). As we have seen, this determines a place w of E above p and for every π of level U, a set of integers $k_1(\pi) < \cdots < k_n(\pi)$ (as in Prop **??**). Let $\mathbb{Z}^n_<$ be the set of such n-uples of integers. The integers $k_1(\pi), \ldots, k_n(\pi)$ are the weights of $(\rho_\pi)_{|G_{E_w}}$. Therefore, they only depends on ρ_π, so they only depend on ψ_π. We thus have a map $\kappa : \mathcal{Z}_U \to \mathbb{Z}^n_<$, which attaches to ψ_π the uple $(k_1(\pi), \ldots, k_n(\pi)$.

Proposition 4.1. *The map κ has finite fibers. If we put on $\mathbb{Z}^n_<$ the topology such that $(k_1, \ldots, k_n)$ is close to $(k'_1, \ldots, k'_n)$ if and only if $k_i \equiv k'_i \pmod{(p-1)p^m}$ for big m, then κ is continuous.*

Proof — The fact that κ has finite fibers results from the fact that there exits only a finite number of automorphic forms with fixed level and weight. To prove that it is continuous, we note that $\psi_\pi \mapsto \rho_\pi$ is by definition continuous for the p-adic topology. The fact that the weights of ρ are continuous in ρ is a result of Wittenberger (with a recent different proof by Berger-Colmez). The proposition follows. $\square$

Now it easy, and standard, to interpolate the topological $\mathbb{Z}^n_<$ by defining the rigid analytic space $\mathcal{W}$ over $\mathbb{Q}_p$ whose set of R-points are $\mathcal{W}(R) = \mathrm{Hom}_{\mathrm{cont}}((\mathbb{Z}_p^*)^n, R^*)$ where R is any $\mathbb{Q}_p$-affinoid algebra. Indeed, we can see $\mathbb{Z}^n_<$ as a subset of $W(\mathbb{Q}_p)$ by sending $(k_1, \ldots, k_n)$ to the continuous morphim of groups $(z_1, \ldots, z_n) \mapsto z_1^{k_1} \ldots z_n^{k_n}$ from $(\mathbb{Z}_p^*)^n$ to $\mathbb{Q}_p^*$, and the induced topology on $\mathbb{Z}^n_<$ is precisely the $\pmod{(p-1)p^m}$ we have put on it. The space $\mathcal{W}$ (geometrically a disjoint union of $(p-1)^n$ copies of a unit ball) is called the *the weight space*.

Now in view of Prop 4.1, it is natural to ask the question : can we find a natural rigid analytic space $\mathcal{E}$ over $\mathbb{Q}_p$ with a locally finite map $\kappa : \mathcal{E} \to W$ such that we can see $\mathcal{Z}_U$ (with its p-adic topology) as a subset of $\mathcal{E}(\mathbb{Q}_p)$, Zariski-dense in U, such that the restricion of κ to $\mathcal{Z}_U$ has image in $\mathbb{Z}^n_<$ and is just the map κ defined above?

Unfortunately, the answer is no. I have missed a turn somewhere...

To be continued

References

E-mail address: jbellaic@brandeis.edu

Math Department, MS 050, Brandeis University, 415 South Street, Waltham, MA 02453