GALOIS REPRESENTATIONS
PROBLEMS SET 6

1. one-parameter subgroups

1.– A 1-paramater subgroup of $\mathrm{Gl}_n(\mathbb{R})$ is a continuous morphism of group f from $(\mathbb{R},+)$ to $\mathrm{Gl}_n(\mathbb{R})$. Show that every such morphism has the form $t \to f(t) = e^{tA}$ where $A \in M_n(\mathbb{R})$. (Hint: use the logarithm of matrices –defined only on a neighborhood of Id– to show that on some neighborhood of 0 in $\mathbb{R}$, $f(t) = e^{tA}$ for some matrix A, and conclude.) More generally, if G is a real Lie group and $\mathfrak{g}$ is its Lie algebra, show that a 1-parameter subgroup has the form $t \mapsto \exp(tA)$ for $A \in \mathfrak{g}$.

2.– A 1-parameter subgroup of $\mathrm{Gl}_n(\mathbb{Q}_p)$ is a morphsim $f : \mathbb{Z}_p \to \mathrm{Gl}_n(\mathbb{Q}_p)$. Show that on some open subgroup of $\mathbb{Z}_p$, f has the form $t \to \exp(tA)$, for some unique matrix $A \in \mathrm{Gl}_n(A)$.

2. Non-abelian cohomology

1.– Let G be a group which acts on another group A, and H be a normal subgroup of G.

a.– Let a_h be a cocycle form H to A. If $s \in G$, define $s(a)_h = s(a_{s^{-1}hs})$. Show that $s(a)$ is again a cocycle from H to A, and that if (a_h) and (b_h) are cohomologous, so are $(s(a)_h)$ and $(s(b)_h)$. You have thus defined an action of G/H on $H^1(H,A)$.

b.– Show that there is a natural exact sequence of pointed set $H^1(G/H, A^H) \to H^1(G,A) \to H^1(H,A)^{G/H}$. The first morphism is called *inflation* and the second is called *restriction* (ane the exact sequence is known as the *inflation-restriction exact sequence*.

c.– Show that the inflation map is injective.

d.– Assume that G and A are topological groups, that the action of G on A is continuous and that H is closed in G. Show that the results above holds if we replace H^1 by H^1_{cont}.

2.– The precise statement of Hilbert 90 for infinite extensions is the following: Let K'/K be a Galois extension (finite or not), and let $\mathrm{Gal}(K'/K)$ be given its natural Krupp topology, while K' is given the discrete topology. The action of $\mathrm{Gal}(K'/K)$ over K' is then continuous (why?). Then $H^1_{\mathrm{cont}}(\mathrm{Gal}(K'/K), \mathrm{Gl}_n(K'))$ is trivial.

a.– Show this statement, by proving that any element in $H^1_{\mathrm{cont}}(\mathrm{Gal}(K'/K), \mathrm{Gl}_n(K'))$ belongs to the image by inflation of $H^1(\mathrm{Gal}(K_0/K), \mathrm{Gl}_n(K_0))$ for some $K_0 \subset K'$ finite extension of K – thus reducing the infinite case to the finite case.

b.– Check that this is exactly the statement we used at the end of the proof of Serre's proposition in class Friday.

3.– Is $H^1_{\mathrm{cont}}(G_{\mathbb{Q}_p}, \mathrm{Gl}_n(\bar{\mathbb{Q}}_p))$ trivial when we give $G_{\mathbb{Q}_p}$ its Krull topology and $\bar{\mathbb{Q}}_p$ its p-adic topology? (Hint: If it is, what does this imply on $\bar{\mathbb{Q}}_p$-admissibility of representations?) Same question for $H^1(G_{\mathbb{Q}_p}, \mathrm{Gl}_n(\bar{\mathbb{Q}}_p))$, all topologies forgotten?