PROBLEMS SET 1
SOLUTIONS

1.– Let e and f be two idempotents in a commutative ring R. We have

$$\begin{aligned}(ef)^2 &= efef \\ &= e^2f^2 \text{ since } R \text{ is commutative} \\ &= ef \text{ since } e \text{ and } f \text{ are idempotents.}\end{aligned}$$

Hence ef is an idempotent.

2.– Let R be a domain, and e an idempotent of R : $e^2 = e$. So $e^2 - e = 0$, that is, using distributivity, $e(1-e) = 0$. Since R is a domain, either $e = 0$, or $1 - e = 0$, taht is $e = 1$. So 0 and 1 are the only idempotents in a domain.

3.– A very simple computation shows that $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ are idempotents but that AB is not ad idempotent.

4.– If R is a ring with a unity 1, and e is an idempotent, then $(1-e)^2 = 1 - 2e + e^2$ (note that in a ring, the general formula for expanding $(a+b)^2$ is $(a + b^2) = a^2 + ab + ba + b^2$ –check it–. Only when $ab = ba$ do we retrive the familiar high-school formula $(a+b)^2 = a^2 + 2ab + b^2$. In the acse at hand, $1(-e) = -e1 = -e$, so the usual formula hands). So $(1-e)^2 = 1-2e+e = 1-e$ and $1 - e$ is an idempotent.

5.– If $x \in R_e$, then $x = exe$. So x can be written eye : take $y = x$. Conversely, assume $x = eye$ for some $y \in R$. Then $exe = eeyee = e^2ye^2 = eye = x$, hence $x \in R_e$. This completes the first question.

To show that R_e is a subring of R, we have to check that it is closed by addition and multiplication. Indeed, if $x, y \in R_e$, $e(x+y)e = exe+eye = x+y$ so $x+y \in R_e$. And $xy = exeeye = e(xey)e$ so $xy \in R_e$ by the charcaterization of elements of R_e proved above. Hence $R)e$ is a subring of R.

Finally, note that $e = eee$ is in R_e. Let $x \in R_e$, we have seen that $x = eye$ for some $y \in R$. We have $ex = e(eye) = e^2ye = eye = x$, and similarly $xe = e$. Thus e is a unity of R. (Note that this holds even when R has no unity).

6.– A function from $\mathbb{R}$ to $\mathbb{R}$ is an idempotent if $f^2 = f$. This means $f^2(x) = f(x)$ for all $x \in \mathbb{R}$. By definition of the products of function, $f^2(x)$ is $f(x)^2$, so f is an idempotent if and only if $f(x)$ is an idempootent of $\mathbb{R}$ for all x, that is $f(x) \in \{0, 1\}$ for all x. So the idempotents are the function whose range is a subset of $\{0, 1\}$. There are imfinitely many of them.

Now if we look for idempotents in the ring of continuous functions, they are as above continuous function with range is a subset of $\{0, 1\}$. But the range of a continuous functions is an interval (intermediate value theorem) and the only subsets of $\{0,1\}$ taht are intervals are $\{0\}$ and $\{1\}$. Hence f has to be constant, and the only idempotents are the constant functions 0 and 1.

7.– For $x \in R$, $xe = xe^2 = exe$ using the fact thet e is an idempotent and the commutativity of R. Hence $xe \in R_e$. Similarly $x(1-e) \in R_{(1-e)}$ (recall that $1-e$ is an idempotent as well). Thus $x \mapsto (xe, (1-x)e)$ defines a map $f : R \to R_e \times R_{(1-e)}$.

If x, y are in R, then $(x+y)e$ is $xe + ye$ and $xye = xye^2 = xeye$. This shows that the map $R \mapsto R_e$ defined by $x \mapsto xe$ is a ring gomomorphism. The same holds of course for the map $x \mapsto x(1-e)$ form R to $R_{(1-e)}$ for the same reasons. Thus the product map f is also a ring homomorphism.

Let us prove that f is injective. Let $x, y \in R$ and assume that $f(x) = f(y)$. We thus have $xe = ye$ and $x(1-e) = y(1-e)$. Just adding those eqaluities gives $x = y$.

Let us now prove that f is surjective. Let $(a, b) \in R_e \in R_{(1-e)}$. We thus have $eae = a$ so $ae = a$, and thus $a(1-e) = 0$. and similarly $b(1-e) = b$ so $be = 0$. So consider the element $a + b \in R$. We get $(a+b)e = ae + be = a + 0 = a$, and $(a+b)(1-e) = a(1-e) + b(1-e) = 0 + b = b$. Hence $f(a+b) = (a, b)$. This shows that f is surjective.

Thus we see that f is an isomorphism. Note that this proves that any commutative ring with a non trivial idempotent (i.e. not 0 or 1) is (isomorphic to) a product of two (non-zeros) rings.