TRAINING EXERCISES FOR MIDTERM II

1.– Show that in a field F, the only ideals are (0) and F.

2.– Let F be a field, and let $R = F \times F$. Is R a field ? A domain? Describe all the ideals of R.

3.– Is $X^3 + 3X^2 + 27X - 3$ irreducible over $\mathbb{Q}$? over $\mathbb{R}$?

4.– Is $\mathbb{Q}[X]/(X^2 - 1)$ a field ? Why ?

5.– Is $\mathbb{Q}[X]/(X^2 + 1)$ a field ? Why ?

6.– One of the two rings in questions 4 and 5 above is isomorphic to $\mathbb{Q}(i)$. Which one and why ?

7.– Let F be the field $\mathbb{Q}(\sqrt{2})$.

a.– Show that any element in F can be written uniquely $a + b\sqrt{2}$. with a, b in $\mathbb{Q}$.

b.– If $x = a + b\sqrt{2}$, with $a, b \in \mathbb{Q}$, set $\tau(x) = a - b\sqrt{2}$. Show that τ is a homomorphism of F to F and that it is bijective.

c.– Show that for $x \in F$, $x\tau(x)$ and $x + \tau(x)$ are in $\mathbb{Q}$.

d.– Let A be the subset of F of elements $a + b\sqrt{2}$ with $a, b \in \mathbb{Z}$. Show that A is a commutative ring with unit. Show that its fraction field is F.

e.– Let $x \in A$. Show that $x\tau(x)$ and $x + \tau(x)$ are in $\mathbb{Z}$.

Solution: If $x = a + b\sqrt{2}$, then $x + \tau(x) = 2a$ and $x\tau(x) = a^2 - 2b^2$ are in $\mathbb{Z}$.

f.– Conversely, let $x \in F$ such that $x\tau(x)$ and $x + \tau(x)$ be in $\mathbb{Z}$. Show that $x \in A$.

g.– The ideal $2\mathbb{Z}$ of $\mathbb{Z}$ is prime. Is the ideal $2A$ of A prime?

Solution: We have $\sqrt{2}\sqrt{2} = 2 \in 2A$, so if $2A$ was prime, $\sqrt{2}$ would be in $2A$, and it is not.

h.– Show that $1 + \sqrt{2}$ is a unit of A.

Solution: We compute $(1 + \sqrt{2})(\sqrt{2} - 1) = 1$, so $1 + \sqrt{2}$ is a unit.

i.– (Difficult). Compute

$$\lim_{n\to\infty} (2 + \sqrt{2})^n - E((2 + \sqrt{2})^n),$$

where $E(x)$ is the integral part of x, that is the greatest integer smaller or equal to x.

Solution: Set $x = 1+\sqrt{2}$. Then $x \in A$, and so is $x^n = (2+\sqrt{2})^n$. Now $x^n + \tau(x^n)$ is an integer a_n in $\mathbb{Z}$. by question e. What is $\tau(x^n)$? It is $\tau(x)^n = (2-\sqrt{2})^n$. Note that $0 < 2-\sqrt{2}^n < 1$, so $0 < (2-\sqrt{2})^n < 1$ and since $x^n = a_n - (2-\sqrt{2})^n$, it follows that $E(x^n) = a_n - 1$. Thus $x^n - E(x^n) = 1-(2-\sqrt{2})^n$ and the limit of this sequence is clearly 1. That is the anmswer.

8.– Determine if the following element are algebraic or transcendetal over the given fields (and if it is, try to compute its degree) :

π^2 over $\mathbb{Q}$? No. Consider $\mathbb{Q} \subset \mathbb{Q}(\pi^2) \subset \mathbb{Q}(\pi)$. If π^2 was algebraic over $\mathbb{Q}$, $\mathbb{Q}(\pi^2)$ would be finite over $\mathbb{Q}$ (important result seen in class, cf corollary 31.7) and since $\mathbb{Q}(\pi)$ is of degree (at most 2) over $\mathbb{Q}(\pi^2)$, then by the theorem of multiplicativity of degree (theorem 31.4 in the book), $\mathbb{Q}(\pi)$ would be finite of $\mathbb{Q}$, so π would be algebraic over $\mathbb{Q}$ (theorem 31.3) which we know is not.

i over $\mathbb{Q}$? Yes, since X^2+1 is a polynomial in $\mathbb{Q}[X]$ that kills i. And since this polynomial is irreducible, it is the irreducible polynomial of i over $\mathbb{Q}$, and the degree is 2.

$\sqrt{7}$ over $\mathbb{Q}$? Yes. Same as above with X^2-7.

π over $\mathbb{Q}(\pi^3)$? Yes, the polynomial $X^3-\pi^3$ is in $\mathbb{Q}(\pi^3)$, and kills π. We also see that the degree of π over $\mathbb{Q}(\pi^3)$ is at most 3. To see that the degree is exabtly 3, we would have to prove that $X^3-\pi^3$ is irreducible over $\mathbb{Q}(\pi^3)$ – which is true, but harde r than anything I will ask you to prove.

$\sqrt{2}-\sqrt{3}$ over $\mathbb{Q}(\sqrt{2})$? Yes, the polynomial $(X-\sqrt{2})^2-3$ is $\mathbb{Q}(\sqrt{2})$ and kills $\sqrt{2}-\sqrt{3}$. The degree is thus at most 2. It can't be 1, because it would mean that $\sqrt{2}-\sqrt{3} \in \mathbb{Q}(\sqrt{2})$, hence $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$ which is not true (could you tell why?). So the degree is 2.

$\sqrt{2}-\sqrt{3}$ over $\mathbb{Q}(\sqrt{6})$? Same answer as above, with polynomial $X^2+2\sqrt{6}-5$, and the degree is 2.

$\sqrt{2}-\sqrt{3}$ over $\mathbb{Q}(\sqrt{5})$? Again, the answer is yes, but we cannot find a polynomial of degree 2. Howemer, we have seen in class how to find a polynomial with coefficient in $\mathbb{Q}$ that kills $\sqrt{2}-\sqrt{3}$, namely $(X^2-5)^2-24$. So $\sqrt{2}-\sqrt{3}$ is algebraic over $\mathbb{Q}$ of degree at most 4, and it is also algebraic over $\mathbb{Q}(\sqrt{5})$ of degree at most 4. (this method could also have been used in the two above questions, but it gives less precise information on the degree). The degree is 4 actually but this is difficult.

9.– Let p be a prime number, let α be a solution in $\mathbb{R}$ of X^p-2, and let $F = \mathbb{Q}(\alpha)$. a.– Show that $[F:\mathbb{Q}] = p$.

Solution: X^p-2 is irreducible by Eisentein criterion for the prime 2, so it is the irreducible polynomial of α, and α has degree p over $\mathbb{Q}$, so $[F:\mathbb{Q}] = p$.

b.– Show that the only subfield of F are $\mathbb{Q}$ and F.

Solution: Let F' be a subfield of F. Certainly F contains $\mathbb{Q}$ (it contains 0 and 1, so every natural integer n by summing n times 1, and every integer as well by taking the opposite of a positive integer, and finally every rational by taking the quotients of two integers). So we have $\mathbb{Q} \subset F' \subset F$, so $p = [F : F'][F' : \mathbb{Q}]$. Since p is prime, one of the factors in the above product has to be 1. If $[F : F'] = 1$, then $F' = F$, and if $[F : \mathbb{Q}] = 1$, $F = \mathbb{Q}$.

c.– Let β be in F. Assume that $\beta \notin \mathbb{Q}$. Show that there exists a polynomial $P(X) \in \mathbb{Q}[X]$ such that $\alpha = P(\beta)$.

Solution: Consider the field $\mathbb{Q}(\beta)$. It is a subfield of F, and it is not $\mathbb{Q}$ since $\beta \notin \mathbb{Q}$. Hence by the question above, it is F. This means that any element of F, especially α, is in $\mathbb{Q}(\beta)$. On the other hand, since β is algebraic over $\mathbb{Q}$ (it is algebraic because it is in F which is finite, hence algebraic over $\mathbb{Q}$), every element of $\mathbb{Q}(\beta)$ is of the form $P(\beta)$ for $P \in \mathbb{Q}[X]$ so $\alpha = P(\beta)$ for some $P \in \mathbb{Q}[X]$.

10.– Starting with a segment of length 1, can you construct a segment of length $\sqrt{3}$ with a straightedge and a compass ?

Solution: Construct a square of side 1, and call $[AB]$ its diagonal. On the line perpendicular to (AB) through A (that can be constructed with a straightedge and a compass – do you remember how?), take C at a distance 1 form A. Then by Pythagoras theorem, the distance from B to C is $\sqrt{3}$.

Same question for the element α of the preceding exercise.

Solution: α is algebraic of degree p. A prime number p is power of 2 only if $p = 2$. So the answer is no if p is odd, but yes if $p = 2$. In that case, $\alpha = \sqrt{2}$, and is easy to construct explictly.