

## TRAINING EXERCISES FOR MIDTERM 2 SOLUTIONS

1.– Let  $f : G \rightarrow G'$  be a morphism of groups, where  $G$  is a finite group. Show that  $|f(G)|$  divides  $|G|$ .

**Solution:** By the factorization theorem, we can write  $f = \bar{f} \circ \pi$ , where  $\pi$  is the natural surjective map from  $G$  to  $G/\ker f$  and  $\bar{f}$  is an injective map from  $G/\ker f$  to  $G'$ . The range of  $f$ , that is  $f(G)$ , is equal to the range of  $\bar{f}$  as it is easily seen. Therefore,  $\bar{f}$  defines an isomorphism from  $G/\ker f$  to  $f(G)$ , and  $|f(G)| = |G/\ker f|$ . But we have seen in class that  $|G/\ker f| = |G|/|\ker f|$  divides  $|G|$ .

2.– Let  $f : G \rightarrow G'$  be a morphism of group. Let  $H$  be a normal subgroup of  $G'$ . Show that  $f^{-1}(H)$  is a normal subgroup of  $G$ .

**Solution:** We have to show that for all  $x \in f^{-1}(H)$ , and all  $a \in G$ ,  $axa^{-1}$  is in  $f^{-1}(H)$ .

If  $x \in f^{-1}(H)$ , then  $f(x) \in H$ . We thus have  $f(axa^{-1}) = f(a)f(x)f(a)^{-1} \in H$  since  $H$  is normal in  $G'$ . Therefore,  $axa^{-1} \in f^{-1}(H)$ , QED.

3.– Let  $G = \mathbb{R}/\mathbb{Z}$ . Show that  $G$  has elements of any order  $n \geq 1$ .

**Solution:** Let us consider the element  $x = 1/n + \mathbb{Z}$  in the factor group  $\mathbb{R}/\mathbb{Z}$ . For  $k = 1, \dots, n-1$ ,  $kx = k/n + \mathbb{Z} \neq \mathbb{Z}$ , but  $nx = 1 + \mathbb{Z} = \mathbb{Z}$  which is the neutral element in  $\mathbb{R}/\mathbb{Z}$ . Therefore,  $x$  has order  $n$ .

4.– Let  $T$  be the set of upper triangular  $2 \times 2$ -matrices  $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$  with  $a, b, c \in \mathbb{R}$ ,  $a \neq 0$ ,  $c \neq 0$ .

a.– Show that  $T$  is a group for the multiplication of matrices.

**Solution:** The determinant of the matrix  $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$  is  $ac$ , which by assumption is not 0, so the matrices in  $T$  are invertible. We know that all invertible matrices are a group for multiplication (an example seen in class and in the book) so we only have to show that  $T$  is a subgroup of this group, which can be shown by checking that  $T$  is closed by multiplication, taking inverse (both an easy computation) and contain the identity matrix (obvious).

b.– What is the center  $Z$  of  $T$ ?

**Answer (without proof):** It is the subgroup of scalar matrices  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ .

c.– What is the commutator group  $C$  of  $T$ ?

**Answer (without proof):** it is the subgroup of matrices  $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

d.– Can you describe the group  $T/C$ ?

**Solution:** Let us define an application  $f : T \rightarrow \mathbb{R}^* \times \mathbb{R}^*$  by  $f\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = (a, c)$ . Then we see easily that  $f$  is a morphism of groups, and is surjective. Moreover, its kernel is  $C$ . Therefore, by the isomorphism theorem  $T/C$  is isomorphic to  $\mathbb{R}^* \times \mathbb{R}^*$ .

5.– What is  $(\mathbb{Z} \times \mathbb{Z})/\langle(2, 2)\rangle$ ?

**Answer:**  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .

6.– Let  $G$  be a group,  $H$  a normal subgroup of  $G$  of index  $n$ . Show that for all  $x \in G$ , we have  $x^n \in H$ .

**Solution:** Let  $x \in G$ , and let  $xH$  be the coset that contains  $x$ . This coset  $xH$  is an element in the factor group  $G/H$ . By Lagrange theorem, since  $G/H$  has order  $n$ , we have  $(xH)^n = H$ , the neutral element in  $G/H$ . Therefore  $x^n H = H$ , which implies  $x^n \in H$ .