# TRAINING FINAL

Remember : thi final exam is cumulutative, though there will be more questions on the last third of the peogram. This is a set of training exercises on that last third (Groups action, rings and fields).

1.– Let $R$ be a ring with unity, $R^*$ the groups of unit.

a.– Show that the application $R^* \times R \to R$, $(g, x) \mapsto gx$ is an action of the group $R^*$ on the set $R$.

b.– Let $x$ in $R$, and let $G_x$ be its stabilizer for the action of $R^*$. Show that if $x$ is not a divisor of 0, then $G_x = \{1\}$.

c.– Take $R = \mathbb{Z}_{10}$. Write down the orbits of the action described above.

2.– Let $R$ be a ring of characteristic 2.

a.– Show that for all $x \in R$, we have $x = -x$.

b.– Show that if $R$ is commutative, then for all $x, y \in R$, $(x+y)^2 = x^2+y^2$.

c.– Conversely, show that if for all $x, y \in R$, $(x + y)^2 = x^2 + y^2$, then $R$ is commutative.

3.– Let $R$ be the ring $\mathbb{Z} \times Z$.

a.– Show that $R$ has a unity.

b.– What are the divisors of 0 in $R$.

c.– What are the units of $R$. Is the group $R^*$ cyclic ?

4.– Solve the congurence

$$7x \equiv 11 \pmod{20}$$

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5.– Solve the equation $\overline{12}x = \overline{23}$ in $\mathbb{Z}_{50}$.

6.– Solve the equation $\overline{4}x = \overline{6}$ in $\mathbb{Z}_{10}$.

7.– Compute $\phi(18)$. Compute $5^{1000} \pmod{18}$,

8.– Let $p$ be a prime number different from 2 and 5. Show that $p$ divides an integer with only 9 in its decimal writing (that is $9999\ldots999$.) You may use Fermat's little theorem.

9.– Let $C$ be the ring of all continuous function from $\mathbb{R}$ to $\mathbb{R}$. Is $C$ a domain?

10.– Let $R$ be the set of all rational numbers that can be written $\frac{a}{2^n}$ where $a$ is an integer and $n$ a non negtive integer.

a.– Show that $R$ is a subring of $\mathbb{Q}$. Show that it is a domain. Is it a field?

b.– What are the units of $R$? To which group is $R^*$ isomorphic?

c.– What is the field of fraction of $R$?

11.– Find all roots of the polynomial $X^2 + 2x + 1$ in $\mathbb{Z}_4$.

12.– Prove or disprove

a.– If $P$ and $Q$ are two non-zero polynomials such that $P + Q$ is non zero, then the degree of $P + Q$ is the maximum of the degree $P$ and the degree of $Q$.

b.– If $P$ has degree 5 and $Q$ has degree 3, then $P + Q$ has degree 5.

c.– In $\mathbb{R}[X]$, there exists a polynomial $P(X)$ such that $X^4 + 1 = (X^2 - 2)P(X)$.