EXERCISE SET 1

(1) Let $K$ be a number field, $\mathcal{O}_K$ its ring of algebraic integers. Show that $x \in \mathcal{O}_K^*$ if and only if $N_{K/\mathbb{Q}}(x) = \pm 1$.

(2) Determine $\mathcal{O}_K^*$ when $K$ is a quadratic field $\mathbb{Q}(\sqrt{d})$ with $d < 0$. (Be careful, there are three different answers according to the value of $d$)

(3) Let $R$ be a noetherian domain. Assume that all maximal ideals of $R$ are principal. Show that $R$ is a PID. (This result was the one needed to complete the proof that a Dedekind domain which is an UFD is a PID, if one wants to avoid using the theorem of decomposition of primes in a Dedekind domain to be proved next week)

(4) Let $K$ be a field and $L$ a $K$-algebra of finite dimension over $K$. We have proved in class that if $L$ is a product of fields that are separable over $K$, the $K$-bilinear map $\text{Tr}_{L/K}(xy)$ is non-degenerate. The converse also holds. Prove it (this is not obvious), or at least prove the easy result that if $\text{Tr}$ is non-degenerate, then $L$ has no nilpotent elements (except 0) – we will need this fact later.

(5) Show that if $x, y$ is in $\mathbb{Z}[i]$, with $y \neq 0$, there exist $q$ and $r$ in $\mathbb{Z}[i]$ such that

$$x = qy + r$$

and $N(r) < N(y)$. Deduce that $\mathbb{Z}[i]$ is a PID, hence a UFD.

(6) Exhibit a non principal ideal in $\mathbb{Z}[\sqrt{-5}]$.

(7) Let $d$ be a square free number, with $d < -1$ and $d \equiv -1 \pmod{4}$. Show that $\mathcal{O}_K$ for $K = \mathbb{Q}(\sqrt{d})$ is not a UFD.