EXERCISES SET 10

1.– Let $L/K$ be a Galois extension of number fields. Let $v$ be a place of $K$ corresponding to a prime $p$, and $w$ a place of $L$ corresponding to a prime $\mathfrak{q}$ above $p$. Show that any automorphism of $L$ over $K$ that fixes $\mathfrak{q}$ extends continuously to an automorphism of $L_w$ over $K_v$. Show that $L_w/K_v$ is Galois and that the decomposition group at $p$ in $\text{Gal}(L/K)$ is isomorphic to $\text{Gal}(L_w/k_b)$.

2.– Concept check: Is $\mathbb{Q}_p$ complete? locally compact? Is $\mathbb{C}_p$ complete? locally compact?

3.– What is the image of the absolute value on $\mathbb{Q}_p$? on $\mathbb{C}_p$?

4.– Let $L = \mathbb{Q}_p[X]/(X^n - a)$, where $a \in \mathbb{Q}_p$ is such that $v_p(a)$ is coprime to $n$. Show that $L$ is a field, and a totally ramified extension of $\mathbb{Q}_p$.

5.– Let $K$ be a local field, $A$ its ring of elements of absolute value less than 1, $\mathfrak{m}$ the maximal ideal of $A$. Let $P$ be an irreducible monic polynomial in $A[X]$. Show that $\overline{P}(X) \in A/\mathfrak{m}[X]$ is a power of an irreducible polynomial.

6.– Let $K$ be either a number field or a finite extension of $\mathbb{Q}_l$ for some prime $l$. Let $p$ be a fixed prime, and for every integer $n$ let $\zeta_n$ be a $p^n$-th primitive root of unity in $\overline{K}$ (a fixed algebraic closure of $K$). We choose the $\zeta_n$’s so that $\zeta_p^n = \zeta_{n-1}$. Let $K_n = K(\zeta_n)$.

a.– Show that $K_n$ is Galois over $K$, and construct an isomorphism from $\text{Gal}(K_n/K)$ on a subgroup of $(\mathbb{Z}/p^n\mathbb{Z})^*$ that sends $\sigma$ to $a$ if $\sigma(\zeta_n) = \zeta_n^a$.

b.– Let $G_K = \text{Gal}(\overline{K}/K)$. We note $\chi_n : G_K \to (\mathbb{Z}/p^n\mathbb{Z})^*$ the group homomorphism obtained by composing the Galois-theoretic map $G_K \to \text{Gal}(K_n/K)$ with the map $\text{Gal}(K_n/K) \to (\mathbb{Z}/p^n\mathbb{Z})^*$ of question a. Show that there exists a unique map $\chi : G_k \to \mathbb{Z}_p^*$ such that $\chi(\sigma) \equiv \chi_n(\sigma) \pmod{p^n}$. The map $\chi$ is called the cyclotomic character of $G_K$.

c.– Show that $\chi$ is continuous (for the $p$-adic topology on $\mathbb{Z}_p^*$ and the Krull topology on $G_K$ - look up in a book on Galois theory what it is if needed).

d.– Show that the image of $\chi$ has finite index in $\mathbb{Z}_p^*$. What is this index when $K = \mathbb{Q}$? When $K = \mathbb{Q}_l$?