1.– Let $K$ be a number field, $\mathbb{A}_K$ its adèle ring, and $w$ a place of $K$. The completion $K_w$ can be embeded in $\mathbb{A}_K$ by sending $x \in K_w$ to the adèle $(x_v)$ with $x_v = 1$ if $v \neq w$, and $x_v = x$. Show that $K_w$ thus embeded in $\mathbb{A}_K$ is a closed subspace.

What about the map $K_w \to \mathbb{A}_K/K$? Show that it is injective, and realize an homeomorphism of $K_w$ onto its image.

2.– Same question as in 1. but with $K^*$, $A^*_K$ and $K^*_w$.

3.– Let $X$ be a scheme of finite type over spec $\mathbb{Q}$. To simplify, we may assume that $X$ is affine, so

$$X = \text{spec } \mathbb{Q}[T_1, \ldots, T_n]/(P_1, \ldots, P_r)$$

though in a non canonical way.

a.– Since $\mathbb{Q}$ is a subfield of $\mathbb{A}_Q$ (by the diagonal embedding), it makes sense to talk of $X(\mathbb{A}_Q)$. A writing of $X$ as in (1) defines an embedding of $X(\mathbb{A}_Q)$ into $\mathbb{A}_Q^n$. Show that the subspace topology on $X(\mathbb{A}_Q)$ does not depend on the writing (1). Show that for that topology, $X(\mathbb{A}_Q)$ is locally compact.

b.– A model of $X$ over spec $\mathbb{Z}$ is a scheme $Y$ of finite type over spec $\mathbb{Z}$ together with an isomorphism of schemes over $\mathbb{Q}$ $Y \otimes \text{spec } \mathbb{Q} \simeq X$. Show that $X$ always have a model $Y$. Show that any two models $Y_1$ and $Y_2$ of $X$ become isomorphic on a non empty open subscheme of spec $\mathbb{Z}$. Deduce that $Y_1(Z_p) = Y_2(Z_p)$ for almost all primes $p$.

c.– Consider the restricted product $\prod'_v X(\mathbb{Q}_v)$ where the compact subspace are $Y(Z_p)$ for a chosen model $Y$ (where $v$ is the $p$-adic finite place). Show that this restricted product does not depend on the model $Y$ chosen. Show that it is in natural bijection with $X(\mathbb{A}_Q)$.

d.– Are the topology on $X(\mathbb{A}_Q)$ and on the restricted product of c. the same?

1. CHARACTERS

Those exercises are a preparation of the study of Hecke characters that will be detailed in the next exercises set.

Let $G$ be locally compact group. A character of $G$ is a continuous morphism from $G$ to $\mathbb{C}^*$. If the image of the character is in the unit circle of $\mathbb{C}$, the character is called unitary.

1.– Let $G^{\text{ab}}$ be the quotient of $G$ by the closure $C$ of the derived subgroup (i.e the group generated by commutators $aba^{-1}b^{-1}$) of $G$. Show that $C$ is normal, so $G^{\text{ab}}$ is a group. Show that any character of $G$ factors through $G^{\text{ab}}$.
2. Show that a character of a torsion group (in particular, a finite group) is always unitary. Same question for a compact group.

3. Show that all characters of $\mathbb{R}$ is of the form $t \mapsto e^{\alpha t}$ for some complex number $\alpha$. Such a character is unitary if and only if $\alpha$ is purely imaginary.

4. Show that all characters of $\mathbb{R}^*$ are of the form $x \mapsto \epsilon(x)^n |x|^\alpha$ where $n = 0$ or 1, $\epsilon(x)$ is the sign $\pm 1$ of $x$ and $\alpha$ is a complex number. When is such a character unitary?

5. Determine all characters of $\text{Gl}_n(\mathbb{R})$. (use 1. and 4. and the fact the derived subgroup of $\text{Gl}_n(\mathbb{R})$ is $\text{Sl}_n(\mathbb{R})$.)

6. Determine all characters of $\mathbb{C}^*$