EXERCISES SET 2

In all this set, $A$ is a Dedekind Domain with fraction field $K$.

1.– Recall that we defined a discrete valuation domain as a local Dedekind domain, and we shown that the unique maximal ideal $\mathfrak{p}$ was principal. A generator of this ideal is called a uniformizer.

Assuming that $A$ is a discrete valuation domain show that for $x \in K^*$, we have

a.– $x \in A$ if and only if $\text{ord}_p(x) \geq 0$

b.– $x \in A^*$ if and only if $\text{ord}_p(x) = 0$

c.– $x \in \mathfrak{m}$ if and only if $\text{ord}_p(x) \geq 1$

d.– $x$ is an uniformiser if and only if $\text{ord}_p(x) = 1$.

2.– Show that in a Dedekind domain, every fractional ideal has a generating family of at most 2 elements. (hint : use the theorem of weak approximation.)

3.– Show that if $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ are maximal ideals in $A$, and $a_1, \ldots, a_k$ are in $\mathbb{Z}$, there is an $x \in K^*$ with $\text{ord}_{\mathfrak{p}_i}(x) = a_i$ for $i = 1, \ldots, k$, and with nonnegative valuation at any other prime. (in class I have proved it for nonnegative $a_i$’s. Prove that the result for nonnegative $a_i$’s implies the result for arbitrary $a_i$. You have to apply twice the result for nonnegative $a_i$’s, and carefully).

Two characterizations of Dedekind domains:

4.– Let $A$ be a Noetherian domain, such that for every prime ideal $\mathfrak{p}$ of $A$, the localization $A_{(\mathfrak{p})}$ is a PID. Show that $A$ is a Dedekind domain.

5.– Let $A$ be a Noetherian domain, such that the set of non-zero fractional ideal is a group for the multiplication of ideals. Show that $A$ is a Dedekind domain. (Difficult. See Lang, Algebra, Third edition, exercises of chapter II for hints).