COMPUTATION OF THE CRITICAL $p$-ADIC $L$-FUNCTIONS OF CM MODULAR FORMS

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Abstract. In this paper, we compute the critical $p$-adic $L$-function of a CM modular form.

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Conventions: We fix a prime number $p$. The cyclotomic character is denoted by $\omega_p : G_{\mathbb{Q}} \to \mathbb{Q}_p^\times$. The Frobeniuses are arithmetic, so $\omega_p(\text{Frob}_l) = l$ if $l$ is any prime different from $p$. The Hodge-Tate weight of $\omega_p$ is $-1$.

We fix two embeddings $\mathbb{Q} \to \mathbb{C}$, and $\mathbb{Q} \to \bar{\mathbb{Q}}_p$, and we denote by $v_p$ the $p$-adic valuation on $\bar{\mathbb{Q}}_p$, normalized as $v_p(p) = 1$.

1. Introduction

Let $f$ be a modular eigenform of weight $k + 2$, level $\Gamma_1(N)$ (where $N$ is prime to $p$) and nebentypus $\varepsilon$. Choose a root $\beta$ of the equation $X^2 - a_p X + \varepsilon(p)p^{k+1} = 0$, and call $f_\beta$ the $p$-refinement of $f$ of $U_p$-eigenvalues $\beta$ (see below for precise definitions). It is possible to define in a natural way a $p$-adic $L$-function $L(f_\beta, \sigma)$ of $f$: the case
where \( v_p(\beta) < k + 1 \) has been known since the seventies (works of Swinnerton-Dyer and Mazur, Manin, Visik, Amice-Velu), the more general case where \( f_\beta \) is not in the image of the operator \( \theta^{k+1} \) (where \( \theta \) acts on the \( q \)-development of an overconvergent \( p \)-adic modular form by \( q \frac{d}{dq} \)) was treated more recently by Stevens and Pollack ([PS2]), and the general case was solved by the author in [B2].

A typical example where \( f_\beta \) is in the image of the \( \theta^{k+1} \)-operator is the case of a form that has CM by a quadratic imaginary field \( K \) in which \( p \) is split, and the chosen root \( \alpha \) is the one of valuation \( k + 1 \). The first aim of this paper is to compute the \( p \)-adic \( L \)-function of \( f_\beta \). This is done in Theorem 2 below. The method requires a careful study of the geometry of the eigencurve at both the point \( x \) to \( f_\beta \), and the point \( y \) corresponding to the \( p \)-adic modular form \( g \) such that \( f_\beta = \theta^{k+1}g \). The result of this study, some of which are due to G. Chenevier, are given in Proposition 1.

The rest of this introduction describes in more details those results.

1.1. **Reminder about CM forms.** Let \( N \) be a natural number, not divisible by \( p \), and \( k \) a non-negative integer. Let \( f = \sum_{n=1}^{\infty} a_n q^n \) be a cuspidal newform of level \( \Gamma_1(N) \), nebentypus \( \epsilon \) and weight \( k + 2 \). We denote by \( \rho_f : G_\mathbb{Q} \rightarrow GL_2(\overline{\mathbb{Q}}_p) \) the \( p \)-adic Galois representation attached to \( f \). It is irreducible, of determinant \( \omega_p^{k+1} \epsilon \), crystalline at \( p \) of Hodge-Tate weights 0 and \(-1 - k\), and unramified outside \( Np \).

If \( K \) is an imaginary quadratic field, then we say that \( f \) has complex multiplication by \( K \) if \( a_l \) is 0 for almost all prime numbers \( l \) that are inert in \( K \).

**We assume that \( f \) has complex multiplication by \( K \).**

For \( l \) prime to \( Np \), one has \( \text{tr} \rho_f(\text{Frob}_l) = a_l \), hence \( \text{tr} \rho_f(\text{Frob}_l) = 0 \) for almost all \( l \) inert in \( K \). It follows easily that \( \rho_f \simeq \text{Ind}^G_K \chi \), where \( \chi \) is a continuous character \( G_K \rightarrow \overline{\mathbb{Q}}_p^* \). We thus have

\[(1) \quad (\rho_f)_{G_K} \simeq \chi \oplus \chi^c,\]

where \( \chi^c(g) := \chi(cg \chi^{-1}) \), where \( c \) is any element of \( G_K, G_\mathbb{Q} \). Note that the character \( \chi \) is well determined by \( f \) up to the change \( \chi \leftrightarrow \chi^c \). We shall say that \( \chi \) (and \( \chi^c \)) is "the" character attached\(^1\) to \( f \).

Let \( \alpha \) and \( \beta \) be the roots of the polynomial \( X^2 - a_p X + p^{k+1} \epsilon(p) \). If \( p \) is inert in \( K \), then \( a_p = 0 \) and \( v_p(\alpha) = v_p(\beta) = (k + 1)/2 \). But we are interested in the critical case, so we assume that \( p \) is split in \( K \).

Then one of the root, say \( \beta \), satisfies \( v_p(\beta) = k + 1 \). The refined modular form \( f_\beta(z) := f(z) - \alpha f(pz) \) is an eigenform for the group \( \Gamma = \Gamma_0(p) \cap \Gamma_1(N) \) and is critical in the sense of [B2, Def. 2.13], and there is a unique overconvergent

\(^1\)There are of course many ways to see the character attached to the CM modular form \( f \). One can see it a character on the idele class group of \( K \), or a character on \( f \), or as a Hecke character in the usual sense, ... There are of course trivial but cumbersome translations between the different points of view. I prefer working in the Galois representations world, and see \( \chi \) as a character of \( G_K \).
eigenform \( g \in M^\dagger_{k}(\Gamma) \) such that \( f = \theta^{k+1}(g) \) (cf. [Co],[B2, Prop. 2.14]). The form \( g \), which is ordinary, is called the companion of \( f \).

1.2. Results concerning the eigencurve. We call \( C \) the Coleman-Mazur-Buzzard eigencurve of tame level \( \Gamma_1(N) \). The modular form \( f_\beta \) and the \( p \)-adic ordinary modular form \( g \) correspond to points on the eigencurve \( C \) that we will call respectively \( x \) and \( y \).

**Proposition 1.** The eigencurve \( C \) is smooth at \( x \) and Gorenstein at \( y \).

Let \( e \) be the ramification degree of \( C \) over the weight space at the point \( x \) corresponding to \( f_\beta \). Then the degree of ramification of \( C \) over the weight space at the point \( y \) corresponding to \( g \) is \( e - 1 \). In particular \( e \geq 2 \).

Moreover, the following are equivalent:

(i) \( \kappa \) is étale at \( y \).
(ii) \( C \) is smooth at \( y \).
(iii) \( C \) is irreducible at \( y \).
(iv) \( e = 2 \).
(v) The unique non trivial extension (in the category of continuous \( p \)-adic \( G_K \)-representations) of \( \chi \) by \( \chi^e \) stays non split after restriction to the decomposition groups at both places of \( K \) above \( p \).

If Jannsen's conjecture (see [J]) is true then the conditions (i) to (v) are satisfied.

**Remark 1.1.** In [BC1], analogous results are proved for Eisenstein series. The smoothness of \( C \) at \( x \), the second paragraph and the equivalence between (i) to (iv) were given to the author by G. Chenevier, with partial proofs, shortly after the publication of their common article [BC1]. Most of the proofs we shall give here are different.

1.3. Notations concerning the weight space. As usual, we denote by \( W \) the weight space: \( W(L) = \text{Hom}(Z^*_p, L^*) \) for all Banach \( \mathbb{Q}_p \)-algebra \( L \), where the Hom means here \( L \)-valued character on \( Z^*_p \), that is to say \( \text{continuous group homomorphisms from } Z^*_p \text{ to } L^* \).

Let \( \mu_{p-1} \) be the group of \((p - 1)\)-th roots of unity in \( Z_p \). There is a canonical isomorphism \( Z^*_p = \mu_{p-1} \times (1 + qZ_p) \), where \( q = p \) if \( p \) is odd and \( q = 4 \) if \( p = 2 \). Choosing a topological generator \( \gamma \) of \( 1 + qZ_p \), we can write any character \( \sigma : Z^*_p \to L^* \) in \( W(L) \) as \( \psi \chi_u \), where \( \psi : \mu_{p-1} \to L^* \) and \( \chi_u \) is the character \( 1 + pZ_p \to L^* \) which sends \( \gamma \) to \( u \), where \( u \) is an element of \( L \) such that \( |u - 1| < 1 \).

**Definition 1.2.** We define an analytic function on \( W \) by the formula

\[
\log_p^{[k]}(\sigma) := \prod_{i=0}^{k-1} \frac{\log_p(\gamma^{-j}u)}{(\log_p \gamma)^k},
\]

if \( \sigma = \psi \chi_u \).
It is clear that the definition of $\log_p^{[k]}$ is independent of the choice of the generator $\gamma$.

We call $\mathcal{W}^+$ (resp. $\mathcal{W}^-$) the subspace of even (resp. odd) characters, defined as those which sends $-1$ to $+1$ (resp. to $-1$). For $k \in \mathbb{Z}$, we denote by $z^k$ the character $\mathbb{Z}_p^* \mapsto L^*$ which sends $z$ to $z^k$.

1.4. **Results concerning the $p$-adic $L$-function.** Let $\sigma \mapsto L_p(f_\beta, \sigma)$ (\$W(\mathbb{C}_p) \to \mathbb{C}_p\$) denotes the critical $p$-adic $L$-functions of $f_\beta$ defined in [B2, Theorem 1 and §1.4].

Let us denote by $L_{p,\text{Katz}}(\tau)$ Katz’ two-variable $p$-adic $L$-function of $K$. This is a function on the set of continuous characters $\tau : \text{Gal}(\bar{K}/K) \to \mathbb{C}_p^*$. (This function is implicitly defined in [Ka], but it appears explicitly in [Gr], where our $L_{p,\text{Katz}}(\tau)$ is denoted $L_p(0, \tau)$).

If $\sigma \in \mathcal{W}(\mathbb{C}_p)$, then $\sigma \circ \omega_p : \text{Gal}(\bar{K}/K) \to \mathbb{C}_p^*$ will be denoted $\sigma_K$.

**Theorem 2.** Let $f$ be a newform of level $\Gamma_1(N)$, weight $k + 2 \geq 2$, and complex multiplication by $K$. We assume that $p$ is split in $K$ and relatively prime to $N$. Let $\chi$ be the $p$-adic character of $G_K$ attached to $f$. Let $f_\beta$ be the $p$-refinement of $f$ that is of critical slope (hence critical). Let $g$ be the companion form of $f_\beta$. Then for every $\sigma \in \mathcal{W}(\mathbb{C}_p)$, we have

$$L_p(f_\beta, \sigma) = \log_p^{[k+1]}(\sigma)L_{p,\text{Katz}}(\chi \sigma^{-1})$$

**Remark 1.3.**

(i) We recall that the $p$-adic $L$-function $L(f_\beta, \sigma)$ is well-defined only up to non-zero multiplicative constants, one on $\mathcal{W}^+$ and one on $\mathcal{W}^-$. The equality in the theorem is to be understood up to that indeterminacy.

(ii) It is interesting to note that the result is the same whether $e = 2$ or not. The proof is substantially harder, however, when $e > 2$.

(iii) It is perhaps more speaking to write, as it is often the case, the $p$-adic $L$-function in terms of a variable $s \in \mathbb{Z}_p$ instead of the variable $\sigma \in \mathcal{W}(\mathbb{C}_p)$. That is to say, if for $s \in \mathbb{Z}_p$, one defines $N_s : \mathbb{Z}_p^* \to \mathbb{C}_p^*$ as the character that sends $z$ to $\langle z \rangle^s$, where $\langle z \rangle$ is the component of $z$ in $1 + q\mathbb{Z}_p$, then one sets $L_p(f_\beta, s) := L_p(f_\beta, N_s)$ (cf. e.g. [MTT, page 19]), and one sets $L_{p,\text{Katz}}(\tau, s) = L_{p,\text{Katz}}(\tau(N_s^{-1})\chi)$ (up to a constant scalar that doesn’t matter for us, cf. [Gr, (2.4) page 90]). With those (standard) definitions, (2) takes the more engaging form:

$$L_p(f_\beta, s) = s(s-1)\ldots(s-k)L_{p,\text{Katz}}(\chi, s)$$

The author would like to thank G. Chenevier, H. Darmon, R. Pollack, and G. Stevens for many useful conversations.
2. Proof of Theorem 2

2.1. Reminder of some results of [B2]. Let $H$ be the commutative algebra over $\mathbb{Z}$ generated by the symbols $T_l$ (for $l$ prime to $Np$), $U_p$, and $\langle a \rangle$ for $a \in (\mathbb{Z}/N\mathbb{Z})^\times$.

In general, notations of [B2] remain in force. In particular, if $M$ is some module over $H$, and $f$ in an $H$-eigenvector (in $M$ or some other $H$-module $N$), then we note $M[f]$ for the common eigenspace of all the elements of $H$ acting on $M$, with the eigenvalues they have on $f$, and $M(f)$ for the corresponding generalized eigenspace.

Let $W$ denote the weight space, $C$ the eigencurve of tame level $N$, and $\kappa$ the locally finite weight map $C \to W$. One has a natural ring map $H \to \mathcal{O}(C)$.

The refined form $f_\beta$ corresponds to a unique point $x \in C(L)$ where $L = L(x)$ is the field of definition of $x$, a finite extension of $\mathbb{Q}_p$. In what follows, we will replace $L$ by a finite extension, each time it is necessary, without further notice.

Let us call $e$ the degree of $\kappa$ at $x$.

We shall call $T_{\kappa(x),x}$ the $L$-algebra corresponding to the connected component at $x$ of the schematic fiber of $\kappa$ at $\kappa(x)$. By definition this algebra has length (that is, dimension over $L$) $e$. There is a natural ring map $H \to T_{\kappa(x),x}$ whose image generates $T_{\kappa(x),x}$ as an $L$-vector space.

Following Stevens ([PS1],[PS2],[S],[B2],[B3]), we define $A$ the Banach space of continuous functions $\mathbb{Z}_p \to L$ that extend to analytic function on each ball in $C_p$ of center in $\mathbb{Z}_p$ and radius $p^{-1/(p-1)}$, and $D$ its topological dual. For every $w \in W(L)$, in particular for every $k \in \mathbb{Z}$, these spaces have a natural action of $\Gamma_0(p)$ and we denote them with an index $w$ (resp. $k$) when they are provided with that action.

When $k \in \mathbb{N}$, $D_k$ has an equivariant finite-dimensional quotient $V_k$, defined as the dual of the polynomials of degree at most $k$ over $L$. It is possible to define an $L$-Banach space $\text{Symb}^+ (D_k)$, and a finite dimensional vector space $\text{Symb}^+_1 (V_k)$, on which the Hecke algebra $H$ acts naturally by bounded operators, and also an involution $\iota$ which commutes with $H$.

We recall from [B2]:

**Proposition 2.1.**

(i) The eigencurve $C$ is smooth at $x$.

(ii) The algebra $T_{\kappa(x),x}$ is isomorphic to $L[t]/t^e$ for some $t \in T_{\kappa(x),x}$.

(iii) The Hecke action of $H$ on $\text{Symb}^+_1 (D_k)$ (resp. $\text{Symb}^+_1 (D_k)$, resp. $M_{k+2}^+ (\Gamma)$) factors through $T_{\kappa(x),x}$, and $\text{Symb}^+_1 (D_k)$ (resp. $\text{Symb}^+_1 (D_k)$, resp. $M_{k+2}^+ (\Gamma)$) is free of rank one over $T_{\kappa(x),x}$.

**Proof** — Assertion (i) is [B2, Theorem 2.16] and due to Chenevier. Assertion (ii), and assertion (iii) for $\text{Symb}^+_1 (D_k)$ are [B2, Theorem 4.7]. Assertion (iii) for $M_{k+2}^+ (\Gamma)$ can be proved exactly in the same way. □

We also recall that as a corollary, the eigenspaces $\text{Symb}^+_1 (D_k)$ have dimension 1, that we call $\Phi^+_1 (f_\beta)$ generators of these spaces, and that the $L$-functions $L^\pm (f_\beta, \cdot)$ are defined the usual way (see §2.4 below) from the modular symbols $\Phi^+_1$. 
2.2. The companion form. Let us recall that we call $y$ the point of $\mathcal{C}(L)$ corresponding to $g$ (the companion point of $x$). We write $T^\pm_g$ for the subalgebra generated by $\mathcal{H}$ of End$_L(Symb^\pm_1(D_{-k-2})(g))$.

**Proposition 2.2.** The spaces End$_L(Symb^\pm_1(D_{-k-2})(g))$ have dimension $e - 1$, and are free of rank one over the algebra $T^\pm_g$, which is isomorphic to $L[t]/t^{e-1}$ for some $t \in T^\pm_g$.

**Proof** — We recall from [PS2] the exact sequence of $\mathcal{H}$-modules

$$0 \to Symb^{\pm(-1)^{k+1}}_1(D_{-2-k})(g)(k+1) \xrightarrow{\Theta} Symb^\pm_1(D_k)(f_\beta) \to Symb^\pm_1(V_k)(f_\beta) \to 0,$$

Here $\Theta$ is the map induced by functoriality of $Symb_1(-)$, from the map

$$\left(\frac{d^{k+1}}{dz^{k+1}}\right)^*: D_{-2-k} \to D_k,$$

which is the dual of the $(k+1)$-th derivative on analytic functions on $\mathbb{Z}_p$. The $(k+1)$ after the first term indicates a twist on the $\mathcal{H}$-action, the action of $T_l$ (resp. $U_p$) being multiplied by $t^{k+1}$ (resp. $p^{k+1}$), and also of the action of the $\iota$-distribution, multiplied by $(-1)^{k+1}$.

Since the last space of the above exact sequence has dimension 1, the dimension of $Symb^{\pm(-1)^{k+1}}_1(D_{-2-k})(g)$ is $e - 1$. Since this space is an $\mathcal{H}$-stable subspace of $Symb^{\pm}_1(D_k)(f_\beta)$, $T^\pm_g$ is a quotient of the sub-algebra generated by $\mathcal{H}$ in $Symb^{\pm}_1(D_k)(f_\beta)$, which is isomorphic to $L[t]/t^{e}$ by Prop. 2.1. To determine which quotient it is, suppose that for some for some $i$, $t^i$ act by 0 on $Symb^{\pm(-1)^{k+1}}_1(D_{-2-k})(g)$. Then $t^i$ would act by 0 on a subspace of codimension 1 of $Symb^\pm_1(D_k)(f_\beta)$; that is of dimension $e - 1$. But by Prop 2.1, the kernel of $t^i$ on $Symb^\pm_1(D_k)(f_\beta)$ has dimension $i$, so we get $i \geq e - 1$. Since by definition, $T^\pm_g$ acts faithfully on $Symb^{\pm(-1)^{k+1}}_1(D_{-2-k})(g)$, the dimension of $T^\pm_g$ is at most $e - 1$. So finally $T^\pm_g \simeq L[t]/t^{e-1}$, and the freeness assertion follows.

**Corollary 2.3.** For every choice of a sign $\pm$, the eigenspace $Symb^\pm_1(D_{-k-2})(g)$ has dimension 1. If one calls $\Phi^\pm_g$ a generator of this space, one has $\Theta \Phi^\pm_g = \Phi^\pm(-1)^k$ up to a scalar.

Let us recall how one attaches (following Stevens) a $p$-adic $L$-function $L^\pm_{\Phi}(\sigma)$ to a rigid analytic modular symbol $\Phi$ in $Symb^\pm_1(D_w)$, where $w \in \mathcal{W}(\mathbb{C}_p)$.

First we set $\mu_{\Phi} = \Phi((\infty) - \{0\})$; that’s an element of $D$, that is a locally analytic distribution on $\mathbb{Z}_p$. Second, we take the Mellin transform of $\mu_{\Phi}$:

$$L_{\Phi}(\sigma) := \mu_{\Phi}(\sigma) := \int_{\mathbb{Z}_p^*} \sigma \, d\mu_{\Phi}.$$

Note that $L_{\Phi}(\sigma)$ is zero whenever $\sigma$ is not of parity $\pm$, so $L^\pm_{\Phi}$ can be seen as an analytic function on $\mathcal{W}^\pm$. 
For example, by definition of the $p$-adic $L$-function $L^\pm(f_\beta, \sigma)$, one has
\[ L(f_\beta, \sigma) = L_{\Phi_f}^\pm(\sigma) = \mu_{\Phi_f}^\pm(\sigma) \]
where $\sigma \in W(\mathbb{C}_p)$ and the sign $\pm$ is determined by the parity of $\sigma$.

Our next aim is to prove the following:

**Proposition 2.4.** For each choice of the sign $\pm$, we have, up to a multiplicative constant,
\[ L_{\Phi_f}^\pm(\sigma) = L_{p,Katz}(\chi\omega_p^{-k-1}\sigma_K^{-1}), \]
for each $\sigma \in W(\mathbb{C}_p)$ such that $\sigma(-1) = \pm 1$.

The proof, which needs some preparation, will be done in the next subsection (§2.5).

**2.3. Proof of Proposition 2.4.**

2.3.1. A two-variables $p$-adic $L$-function. To prove Prop. 2.4, we need to construct a two-variables $p$-adic $L$-function on a neighborhood of the point $y$ corresponding to $g$ in $\mathcal{C}$:

**Proposition 2.5.** There exists an affinoid admissible neighborhood $U \in \mathcal{C}$ of $y$, and a two-variable analytic $p$-adic $L$-function $L_p : U \times W(\mathbb{C}_p) \to \mathbb{C}_p$ such that for every point $z \in U$ of weight $\kappa(z) \in W(\mathbb{C}_p)$, and every choice of a sign $\pm$,

(i) there exists a unique (up to a scalar) non-zero modular symbol
\[ \Phi^\pm_x(\sigma) \in Symb_\Gamma(D[\kappa(z)]^\pm)[z]; \]

(ii) one has $L_{\Phi_x}^\pm(\sigma) = L_p(z, \sigma)$ for all character $\sigma$ such that $\sigma(-1) = \pm 1$.

2.3.2. Proof of proposition 2.5. Since the $p$-adic modular form $g$ is cuspidal, the point $y$ belongs to the cuspidal part of the eigencurve $\mathcal{C}$, and it follows from [B2, Theorem 3.22] that locally around $y$, the Coleman-Mazur-Buzzard eigencurve $\mathcal{C}$ and the eigencurves constructed with modular symbols of either sign $\pm$ (see [B2, §3.4]) are isomorphic. This means that by construction, a basis of neighborhood of $y$ in $\mathcal{C}$ can be constructed as follows:

Take $W = \text{Sp}R$ a sufficiently small affinoid of $W$ containing the point $-2 - k$. Let us call $D[R]$ the $R$-module of families of distributions over $R$ (see [B2, §3] or [B3] for the precise definition: this module is a proper submodule of $D \otimes R$), with its natural action of the $\Gamma_0(p)$. Then set $M' = Symb_1^\pm(D[R])^0$, where $^0$ means the ordinary part. Then according to [B2], $M'$ is a finite projective module over $R$ with an action of $\mathcal{H}$ and $\iota$. Let $\mathbb{T}'$ be the sub-algebra of $\text{End}_R(M')$ generated by the Hecke operators, which is a finite flat affinoid $R$-algebras, and let $U' = \text{Sp}\mathbb{T}'$. Then by construction, $U'$ is a neighborhood of $y$ in $\mathcal{C}'$, the restriction of the weight map $\kappa$ from $U'$ to $W$ being given by the inclusion $R \to R$. However $U'$ may be have several connected component, so take $U$ to be the connected component of $y$ in $U'$. This connected component correspond to an idempotent $\epsilon$ of $\mathbb{T}'$ and if we set
\( T = \epsilon T' \) and \( M' = \epsilon M \), then \( U = \text{Sp} T \) and \( T \) is the sub-algebra generated by the Hecke operators on \( M \). As an \( R \)-module, \( M \) is finite projective, so can be assumed free up to shrinking \( U \). For \( U \) sufficiently small, \( M \) enjoys the following property: for every \( w \in W(L) \) we have a canonical Hecke-compatible isomorphism

\[(5) \quad M_w = \epsilon \text{Sym}^\pm_T (D_w)^0 = \oplus_{z \in U, \kappa(z) = w} \text{Sym}^\pm_T (D_w)(z)\]

where \( M_w \) is defined as \( M \otimes_R L \), the morphism \( R \to L \) being the one defining the point \( w \), and \( \text{Sym}^\pm_T (D_w)(z) \) is the generalized eigenspace in \( \text{Sym}^\pm_T (D_w) \) for the Hecke operators with system of eigenvalues the values at \( z \) of those operators seen as functions on the eigencurve. Shirking \( U \) again if necessary, we can assume that \( y \) is the only point of \( U \) above \( -k - 2 \), so (5) becomes:

\[(6) \quad M_{-k-2} = \text{Sym}^\pm_T (D_{-k-2})(g)\]

We may also assume that the weight map \( \kappa : U \to W \) is étale at every point \( x \in U \) except \( y \).

**Lemma 2.6.** The \( U \)'s as above form a basis of neighborhood of \( y \) in \( C \).

**Proof** — This follows from the construction of the eigencurve by an easy argument given in [BC1, Lemma 1]. \qed

**Lemma 2.7.** If \( U \) is as above and sufficiently small, then \( U \) has degree \( e - 1 \) over \( W \).

**Proof** — By (6) and Prop 2.2, \( M_{-k-2} \) has dimension \( e - 1 \), so \( M \) is free of rank \( e - 1 \) over \( R \). Let us choose a integer \( k' \in W \cap \mathbb{N} \). Enlarge \( L \) such that every point in \( U \) above \( k' \) is define over \( L \).

Every \( z \in U(L) \) such that \( \kappa(z) = k' \) is attached by Hida’s or Coleman’s control’s theorem (e.g. [Co]) to a classical modular form \( f_z \). If \( U \) is sufficiently small, then all the \( f_z \) is new at \( N \), since the closed subvariety of \( C \) parametrizing old forms at \( N \) does not contain \( y \), as it follows from looking at the ramification of the Galois representations. Hence the spaces \( \text{Sym}^\pm_T (D_{k'})(z) = \text{Sym}^\pm_T (V_{k'})(z) = (M_{k'+2}^\dagger(f_z))' \) (the first equality is Stevens’ control theorem ([S]), the second the Eichler-Shimura-Manin-Shokurov isomorphism, see [PS2] or [B2] or [B3]) have dimension 1. By (5), the number of \( z \in U(L) \) such that \( \kappa(z) = k' \) is therefore \( e - 1 \). But since \( \kappa \) is étale above \( k' \), \( e - 1 \) is the degree of \( U \) over \( W \). \qed

**Lemma 2.8.** The eigencurve \( C \) is Gorenstein at \( y \) and if \( U \) is as above and sufficiently small, then \( M \) is free of rank one over \( T \), and so is its dual \( M^* = \text{Hom}_R(M, R) \).

**Proof** — Let \( T_{-k-2,y} \) be the local Artinian algebra of the connected component at \( y \) of the fiber of \( \kappa \) at \( -k - 2 \). By the definitions and [C1], we have a natural
surjective map $T_{-k-2,y} \rightarrow T_{y}^{\pm}$. Since those two algebras have the same dimension $e - 1$ (by Lemma 2.7) and Prop. 2.2), this map is an isomorphism. In particular $T_{-k-2,y} = L[t]/(t^{e-1})$ which is Gorenstein.

Since $T$ is finite free over $R$ which is a regular dimension 1, it suffices to prove that it is Gorenstein (after if necessary shrinking $U$ again), it suffices to prove that its fiber at the closed point $-k - 2$ is. But this fiber is just $T_{-k-2,y}$ by definition, which we have seen is Gorenstein.

Finally to prove that $M$ is free over $T$ after shrinking $U$ it suffices by Nakayama to observe that its fiber at $-2 - k$ (that is $\text{Symb}_T^\pm(D_{-k-2}(y))$ is free over the fiber of $T$ (that is $T_{-2-k,y}$ and we have already proved this.

Since $T$ is Gorenstein and $M$ is free of rank one, we know that $M^*$ is finite projective of rank one. Shrinking $U$ again makes it free.

We can now proceed to the construction of the two-variables L-function on $U \times W(C_p)$, for $U$ satisfying all the lemmas above.

By definition of $M$, we have a canonical $R$-linear map

$$L : M \hookrightarrow \text{Symb}_T(D[R]) \text{ eval. at } \{\infty\} - \{0\} \hookrightarrow D[R] \hookrightarrow \hat{D} \hat{\otimes} R$$

This is the same as an element

$$L \in M^* \otimes_R (\hat{D} \hat{\otimes} R) = M^* \hat{\otimes} D \simeq T \hat{\otimes} D$$

where the last isomorphism comes from the fact that $M^*$ is free of rank one over $T$. Since this isomorphism is only well-defined up to multiplication by an element of $T^*$, so is our element $L \in T \hat{\otimes} D$. Taking the Mellin transform in family (see [B2]) defines a two-variable $p$-adic L-function $L_p(z,\sigma)$ on $U \times W(C_p)$ that we were looking for.

If $z \in U(L)$, and $z \neq y$, then since $U$ is étale at $z$ over $W$ the generalized space $\text{Symb}_T^\pm(D_{\kappa(z)})_{(z)}$ has dimension 1. Therefore the eigenspace $\text{Symb}_T^\pm(D_{\kappa(z)})_{[z]}$ has also dimension 1. The same result also holds for $z = y$ by Prop. 2.2. This shows (i) in Prop 2.5.

Calling $\Phi_z^\pm$ a generator of $\text{Symb}_T^\pm(D_{\kappa(z)})_{[z]}$, let us see it by biduality as a linear $\Phi_z^\pm : (\text{Symb}_T^\pm(D_{\kappa(z)})^* \rightarrow L$, that is $\Phi_z^\pm : M_{\kappa(z)}^* \rightarrow L$. Applying this linear form to our two-variables L-function $L \in M^* \hat{\otimes} D$ gives an element $L_z$ of $D$, which by construction is just $\Phi_z^\pm(\{\infty\} - \{0\})$. On the other hand, in our identification $M^* \simeq T$, inducing an identification $M_{\kappa(z)} \simeq T_{\kappa(z)}$, the (a non-zero multiple of the) map $\Phi_z^\pm : M_{\kappa(z)} \rightarrow L$ identifies with the morphism of algebra $T_{\kappa(z)} \rightarrow L$ corresponding to $z$ (this is because $\Phi_z^\pm$ was an eigenvector to begin with). Therefore $L_z$ is also the specialization at $z$ of the element $L \in T \hat{\otimes} D$. Point (ii) of Prop 2.5 follows, just using the (trivial) compatibility between Mellin transform in family and punctual Mellin transform (see [B2]).

2.3.3. Proof that Prop. 2.5 implies Prop. 2.4. Let $f'$ be any classical modular eigenform with CM by $K$, such that $\rho_{f'} = \text{Ind}_K^Q \chi'$ where $\chi'$ is a character $G_K \rightarrow \hat{Q}_p$. 

Let \( f'_\alpha \) be its ordinary \( p \)-refinement. We recall that we have, up to a constant, the following relation between the \( p \)-adic \( L \)-function of \( f'_\alpha \) and Katz \( p \)-adic \( L \)-function:

\[
L_p(f'_\alpha, \sigma) = L_{p, \text{Katz}}(\chi'_\sigma^{-1})
\]

To see this, it is sufficient to check that the formula holds for all character \( \sigma \) of finite order and conductor \( p^\nu (\nu = 0, 1, 2, 3, \ldots) \) since both terms are analytic bounded functions on \( \sigma \). For those \( \sigma \), it is sufficient to compare the interpolation formula defining the \( p \)-adic \( L \)-function of \( f'_\alpha \) (cf. [MTT]) and the Katz \( p \)-adic \( L \)-function (see [Gr, (2.4) and Theorem 2.3]).

Let \( C_K \) be the family of forms having CM by \( K \) through the form \( g \), whose existence is a well-known by Hida theory. The curve \( C_K \) is an irreducible component of \( C \) through \( y \), and if \( z \in C_K \), the Galois representation \( \rho_z \) carried by the eigencurve at \( z \) can be written \( \text{Ind}_{K}^Q \chi'_{z} \), where \( \chi'_{z} \) is some character of \( G_K \). We claim that for \( z \in \text{Im}C_K \cap U \), we have

\[
L_p(z, \sigma) = L_{p, \text{Katz}}(\chi'_z \sigma^{-1}).
\]

Indeed, we regard the equality to prove as an equality between analytic function on \( z \in U \cap C_K, \sigma \) being fixed, and we now by (8) and Prop. 2.5 that this equality holds when \( z \) correspond to a classical CM form, that is when \( \kappa(z) \) is a positive integer by Coleman’s control theorem. As those points are Zariski-dense in \( U \cap C_K \), (8) follows.

To get Proposition 2.4, we apply (8) to \( z = y \). We observe that \( \rho_y = \rho_g = \rho_f \omega_p^{-k-1} = \rho_f \) by looking at the eigenvalues for \( T_l (l \text{ prime to } Np) \) of \( f \) and \( g \). It follows that \( \chi'_y = \chi'_{p^{-k-1}} \), and (8) gives us the desired equality.

### 2.4. End of the proof of Theorem 2

Let us choose a sign \( \pm \). The image by \( \Theta_{k+1} \) of the modular symbol \( \Phi_{g^{\mp(-1)^{k+1}}} \) is a non-zero eigenvector with the same eigenvalues as \( f_\beta \) in \( \text{Symb}^{\mp}_V(D_k) \). Hence it is \( \Phi_{f_\beta}^{\pm} \), up to a scalar. We thus have

\[
\mu_{\Phi_{f_\beta}^{\pm}} = \left( \frac{d^{k+1}}{dz^{k+1}} \right) \mu_{\Phi_{g^{\mp(-1)^{k+1}}}} \quad \text{(up to a multiplicative constant)}.
\]

**Lemma 2.9.** If \( \mu \) and \( \mu' \) are two distributions (in \( D[0] \)), such that \( \mu' = \left( \frac{d^{k+1}}{dz^{k+1}} \right) \mu \),

then

\[
Mел\mu'(\sigma) = \log_p^{[k+1]}(\sigma) Mел\mu(\sigma/z^{k+1})
\]

for \( \sigma \in \mathcal{W}(\mathbb{C}_p) \), up to a multiplicative constant.
Proof — By induction we may assume $k = 0$. One then has

$$\text{Mel}_{\mu'}(\sigma) = \text{Mel}_{\mu} \left( \frac{d\sigma(z)}{dz} \right)$$

$$= \text{Mel}_{\mu} \left( \frac{d \left( \psi(\langle z \rangle) \chi_u \left( \frac{z}{\langle z \rangle} \right) \right)}{dz} \right)$$

$$= \text{Mel}_{\mu} \left( \psi(\langle z \rangle) \frac{d\chi_u}{dz} \left( \frac{z}{\langle z \rangle} \right) \right)$$ since $\langle z \rangle$ is locally constant.

But for $z \in 1 + p\mathbb{Z}_p$, $\chi_u(z) = \exp(\log(z) \log(s)/\log(\gamma))$, so $\frac{d\chi_u}{dz}(z) = \log(\frac{u}{z}) \chi_u(z)$.

Therefore

$$\text{Mel}_{\mu'}(\sigma) = \text{Mel}_{\mu} \left( \psi(\langle z \rangle) \frac{\log(u)}{\langle z \rangle} \frac{\log(\gamma)}{\log(\gamma)} \chi_u(z) \right)$$

$$= \log_p(u)/\log_p(\gamma) \text{Mel}_{\mu}(\sigma/z).$$

□

Therefore, if we observe that the parity of the character $\sigma/z^{k+1}$ is $(-1)^{k+1}$ of the parity of $\sigma$, Theorem 2 follows immediately, using the lemma, from Prop 2.4.

3. Proof of Proposition 1

We have already proved the assertions of the first two paragraphs of Prop. 1 (cf. Proposition 2.1 and 2.2). It remains to prove the equivalence between the properties (i) to (v) of the third paragraph, and the fact that Jannsen’s conjecture implies those properties. We shall also give a different proof, Galois-theoretic, that $e \geq 2$.

3.1. Proof of the equivalence of (i), (ii), (iii) of (iv). Since $y$ corresponds to a an (overconvergent $p$-adic) ordinary modular form with complex multiplication by $K$, there exists by Hida’s theory an irreducible component $C_K$ of the eigencurve through $y$, whose all classical points correspond to modular form with multiplication by $K$, and such that the restriction of $\kappa$ to $C_K$ is an isomorphism. Obviously, if $C$ is smooth at $y$, then it is irreducible near $y$ and $C = C_K$ in a neighborhood of $y$, so the degree of $\kappa$ at $y$ is 1. If the degree of $\kappa$ at $y$ is 1, then $C = C_K$ locally near $y$ and $C$ is smooth at $y$ since $C_K$ is, being isomorphic to $W$. This proves the equivalence between (i), (ii), (iii) and (iv)

3.2. Reducibility loci, and equivalence of (iv) and (v). In this §, we recall the definition of and study the reducibility loci at $x$ of the restriction of the family of Galois representations carried by $C$ to either $G_K$ or $G_{\mathbb{Q}_p}$. This will allow us to prove the equivalence between (iv) and (v), and to give a new proof, independent of the theory of the $\theta$-operator so more elementary, that $e \geq 2$.

First we need to fix carefully some notations:

We call $v$ and $\bar{v}$ the two places of $K$ above $p$. We call $D_v$ one of the decomposition subgroup of $v$ in $G_K$. It is well defined up to conjugacy in $G_K$. We also call
$D_p$ a decomposition group of $p$ in $G\mathbb{Q}$, chosen so that its image by the restriction map $G\mathbb{Q} \to G_K$ is $D_v$. Note that $D_p$ is well-defined up to conjugacy by an element of $G_K$ (not $G\mathbb{Q}$, since conjugating $D_p$ by an element of $G_K - G\mathbb{Q}$ gives a decomposition subgroup of $p$ in $G\mathbb{Q}$ whose image in $G_K$ is a decomposition subgroup of $\bar{v}$). Note that $D_v$ and $D_p$ are canonically isomorphic, and that we have a canonical isomorphism $(\rho_f)|_{D_p} \simeq ((\rho_f)|_{G_K})|_{D_v}$. Using (1), one gets

$$(\rho_f)|_{D_p} = (\chi)|_{D_v} \oplus (\chi^c)|_{D_v}.$$  

Therefore, one sees that $\chi$ and $\chi^c$ are crystalline at $v$, of weight 0 and $-1 - k$ in some order. Since $\chi$ and $\chi^c$ play symmetric role, we can and shall assume that $\chi$ is of weight 0 and $\chi^c$ is of weight $-1 - k$. From this, one deduces immediately that $\chi$ and $\chi^c$ are also crystalline at $\bar{v}$, of weight $-1 - k$ and 0.

The two eigenvalues of the crystalline Frobenius $\phi$ on $D_{crys}(\chi|_{D_v})$ are the one $\alpha$ of $\phi$ on $D_{crys}(\chi|_{D_v})$ and the one $\beta$ of $D_{crys}(\chi^c|_{D_v})$. One has $v_p(\alpha) = 0$ and $v_p(\beta) = k + 1$.

The eigencurve carries a pseudocharacter $T : G\mathbb{Q} \to \mathcal{O}(C)$ of dimension 2, whose evaluation at $x$ is $\rho_f$. Since $(\rho_f)|_{G_K} = \chi \oplus \chi^c$ and $(\rho_f)|_{D_v} = \chi|_{D_v} \oplus \chi^c|_{D_v}$, one can define the reducibility locus $R_K$ of $T|_{G_K}$ at $x$ as the maximal local closed subscheme of $C$ with closed point $x$ on which $T|_{G_K}$ is still the sum of two characters, and similarly the reducibility locus $R_p$ of $T|_{G_p}$ at $x$ as maximal local closed subscheme of $C$ with closed point $x$ on which $T|_{G_p}$ is still the sum of two characters. In our situation, the existence of those irreducibility loci is obvious, since the local closed subschemes of $C$ at $x$ form a totally ordered set (since the local ring of $C$ at $x$ is a discrete valuation ring). In general, existence and nice properties of reducibility loci depend on the condition of residual multiplicity freeness, which here means that $\chi \neq \chi^c$ and $\chi|_{D_v} \neq \chi^c|_{D_v}$ for $R_K$ and $R_p$ respectively and is satisfied (think of the Hodge-Tate weights). See [BC1] for a quick exposition, and [BC2, Chapter 1] for more results. Since $D_p = D_v$ is a subgroup of $G_K$, $R_K$ is a closed subscheme of $R_p$.

**Theorem 3.**

(i) The closed subscheme $R_p$ is the connected component at $x$ of the schematic fiber of $\kappa$ at $\kappa(x)$. Therefore, its algebra is $\mathbb{T}_{x,\kappa(x)}$ and its length is $e$.

(ii) The closed subscheme $R_K$ has length 2.

**Proof —** The proof of (i) goes exactly as in [BC1]. More precisely, the inclusion $R_p \subset$ the schematic fiber of $\kappa$ at $\kappa(x)$, is proved exactly like [BC1, §5.4] (the main tool being the theorem of Kisin on extension of crystalline periods in family and the critical nature of the refinement attached to $f_\beta$), while the other inclusion is proved exactly like the second proof of [BC1, Lemme 7]

Let us prove (ii). Let $A$ be the local ring of $C$ at $x$ (which is an Henselian local $\mathbb{Q}_p$-algebra), $m$ its maximal ideal, so $A/m = \mathbb{Q}_p$, and $I \subset m$ be the ideal of $A$ corresponding to the reducibility locus $R_K$. We want to prove that $I = m^2$. 


Since the pseudo-character \( T : G_\mathbb{Q} \to A \) is residually absolutely irreducible (since the residual representation \( \rho_f \) is absolutely irreducible as a representation of \( G_\mathbb{Q} \)), the theorem of Rouquier-Nyssen states that there exists a representation \( \tilde{\rho} : G_\mathbb{Q} \to \text{GL}_2(A) \) of trace \( T \), unique up to isomorphism.

Let us choose an \( s \in G_K \) such that \( \chi(s) \neq \chi^c(s) \). The matrix \( \tilde{\rho}(s) \) has two distinct eigenvalues in \( A \) by Hensel’s lemma, so there exists a basis of \( A^2 \) such that in the new basis, \( \tilde{\rho}(s) \) is diagonal, with entries reducing modulo \( m \) to \( \chi(s) \) and \( \chi^c(s) \), in that order. (This is an adapted basis in the sense of [BC1]).

In that basis, we have for all \( g \in G_K \), \( \tilde{\rho}(g) \equiv \begin{pmatrix} \chi(g) & 0 \\ 0 & \chi'(g) \end{pmatrix} \pmod{m} \). Indeed, \( \tilde{\rho}|_{G_K} \pmod{m} \approx \chi \oplus \chi^c \) has two stable lines, which are in particular stable by \( \tilde{\rho}(s) \pmod{m} \), but since \( \chi(s) \neq \chi^c(s) \) those two lines have to be the lines generated by the two vectors of our adapted basis.

Now let \( B \) and \( C \) be the ideals of \( A \) generated respectively by the elements \( b(g) \) and \( c(g) \) for \( g \in G_K \), where we write \( \rho(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix} \). It is well-known (since Mazur-Wiles, see [BC1]) and elementary that \( I = BC \). By the above, we have \( b(g) \in m \), and \( c(g) \in m \) for all \( g \in G_K \), so \( B \subseteq m \) and \( C \subseteq m \), from which \( I = BC \subseteq m^2 \).

We now want to prove that \( m^2 \subseteq I \). First, a definition: we shall say that an element \( x \) of \( A/I \) is constant if it is equal to its image \( \bar{x} \) in \( A/m = \mathbb{Q}_p \), seen as an element of \( A/I \) via the canonical section \( A/m = \mathbb{Q}_p \to A/I \). A function with codomain \( A/I \) is said constant if all elements in its range are constant.

**Claim 1:** The pseudocharacter \( T|_{G_K} \otimes A/I \) is constant. By definition, \( T \otimes A/I = \bar{\chi} + \bar{\chi}^c \), where \( \bar{\chi} \) and \( \bar{\chi}^c \) are continuous characters \( G_K \to (A/I)^* \). Since \( R_K \subseteq R_p \), we know by (i) that the Hodge-Tate-Sen weights of \( \bar{\chi} \) and \( \bar{\chi}^c \) are constant on \( A/I \). This implies that \( \bar{\chi} \) and \( \bar{\chi}^c \) are themselves constant, and so is their sum \( T \otimes A/I \).

**Claim 2:** For all \( g \in G_\mathbb{Q} \), setting \( t = (T \otimes A/I)(g) \), we have \( (t - \bar{t})^2 = 0 \). If \( g \in G_K \), we know that \( t \) is constant by Claim 1, so \( t - \bar{t} = 0 \) and \( (t - \bar{t})^2 = 0 \).

So assume that \( g \in G_\mathbb{Q} - G_K \), and consider the matrix \( \rho_f(g) \in \text{GL}_2(A/m) \) in the basis chosen above. This matrix normalizes \( \rho_f(G_K) \), so \( \rho_f(g) \) is either diagonal or anti-diagonal. If it was diagonal, then \( \rho_f(G_\mathbb{Q}) \) would be diagonalizable, but we know that \( \rho_f \) is irreducible, so \( \rho_f(g) \) is anti-diagonal.

Now write \( (\tilde{\rho} \otimes A/I)(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(A/I) \), so that \( t = a + d \). Since this matrix reduces to \( \rho_f(g) \) modulo \( m \), we have \( a, d \in m \), and thus \( b, c \notin m \).

One has \( \tilde{\rho}_f(g^2) \otimes A/I = \begin{pmatrix} a^2 + bc & (a + d)b \\ (a + d)c & d^2 + bc \end{pmatrix} = \begin{pmatrix} a^2 + bc & bt \\ ct & d^2 + bc \end{pmatrix} \). Since \( g^2 \in G_K \), we deduce that \( bt \in B \) and \( ct \in C \), so \( bct^2 \in BC = I \). In other words, \( bct^2 = 0 \in A/I \). But \( b \) and \( c \) are invertible in \( A/I \) (since they are not in the unique
maximal ideal \( m \), so \( t^2 = 0 \) in \( A/I \). It follows that \( \bar{t}^2 = 0 \) in the field \( A/m \), so \( \bar{t} = 0 \), and finally \( (t - \bar{t})^2 = t^2 = 0 \) in \( A/I \), as claimed.

**Claim 3:** The \( \bar{Q}_p \)-algebra \( A/I \) is generated by elements \( x \) such that \( (x - \bar{x})^2 = 0 \). Indeed, we know that \( A/I \) is generated by the weight \( \kappa \) and the Hecke operator \( U_p \) and \( T_l \) for \( l \not| N_p \) (all those being functions on \( C \) restricted to the closed subscheme \( R_K = \text{Spec} A/I \)). Now \( \kappa \) is constant in \( A/I \) by (i), \( U_p \) is constant by Kisin’s theorem and the \( T_l = \text{tr} \bar{\rho}(\text{Frob}_l) \) satisfy \( (T_l - \bar{T}_l)^2 = 0 \) by Claim 2.

**Conclusion:** Since \( A/I \) is local, and generated by a families of element \( x \) such that \( (x - \bar{x})^2 = 0 \), there is an \( x \) that family that is in the maximal ideal but not in the square of the maximal ideal, and this \( x \) satisfies \( x^2 = 0 \) (since \( \bar{x} = 0 \)). Now this \( x \) generates the maximal ideal of \( A/I \) by Nakayama and the fact that \( A \) is a d.v.r. So the square of the maximal ideal is 0 in \( A/I \). In other words, \( I \subset m^2 \).

**Corollary 3.1.** One has \( e \geq 2 \). One has \( e = 2 \) if and only if the unique non trivial extension (in the category of \( G_K \)-representations) of \( \chi \) by \( \chi^c \) becomes split after restriction to \( D_{\bar{\theta}} \).

Proof — By the theorem, the length of \( R_K \) is 2, the length of \( R_p \) is \( e \). Since \( R_K \subset R_p \) as closed subschemes of \( C \), \( e \geq 2 \) follows.

Proving the second assertion is essentially a variation on Ribet’s lemma. We keep the notations \( B, C, I = BC \) (all ideals of the local ring \( A \) of \( C \) at \( x \)) of the proof of the preceding theorem, and we introduce \( B_p \) (the ideal of \( A \) generated by the \( b(g) \) with \( g \in D_v \) – instead of \( g \in G_K \)), \( C_p \) (similarly defined) and \( I_p = B_p C_p \) the ideal corresponding to \( R_p \). We have by definition \( B_p \subset B \) and \( C_p \subset C \). Ribet’s lemma, as generalized by Mazur and Wiles, and reformulated (see[BC1]), provides us with four injective linear applications (the horizontal maps in the following diagrams)

\[
\begin{align*}
(B/mB)^* & \longrightarrow \text{Ext}^1_{G_K}(\chi^c, \chi) \\
\downarrow & \\
(B_p/mB_p)^* & \longrightarrow \text{Ext}^1_{D_v}(\chi^c, \chi)
\end{align*}
\]

and similarly

\[
\begin{align*}
(C/mC)^* & \longrightarrow \text{Ext}^1_{G_K}(\chi, \chi^c) \\
\downarrow & \\
(C_p/mC_p)^* & \longrightarrow \text{Ext}^1_{D_v}(\chi, \chi^c)
\end{align*}
\]

We claim that all the eight spaces in the diagrams above have dimension 1, and that all horizontal maps are isomorphism. Indeed, for the space on the left lines, the one-dimensionality follows immediately from Nakayama’s lemma and the fact that \( B, C, B_p, C_p \) are non-zero principal ideals. For the local (that is, on \( D_v \)) \( \text{Ext}^1 \) spaces, this is a trivial computation using Tate’s local Euler characteristic formula.
and duality in Galois cohomology. While for \( \text{Ext}^1_{G_K}(\chi^c, \chi) \), we note that by Rubin a proof of the main conjecture for \( K \), we know that the subspace \( \text{Ext}^1_{G_K,f}(\chi^c, \chi) \) that parametrizes extensions that are crystalline at \( v \) and \( \bar{v} \) have dimension 0. Therefore, we have an injection \( \text{Ext}^1_{G_K}(\chi^c, \chi) \hookrightarrow \text{Ext}^1_{D_v,f}(\chi^c, \chi) \oplus \text{Ext}^1_{D_{\bar{v}},f}(\chi^c, \chi) \), where the /\( f \) indicates that we take the quotient of the extension spaces by the subspaces of crystalline extensions. Now a non-trivial extension, in the category of representations of \( D_v \), of \( \chi^c \) (which has weight \( -k-1 \) at \( v \)) by \( \chi \) (which has weight 0 at \( v \)) is never crystalline, since the weights are in the wrong direction (cf. the dimension formulas in [BK]), so \( \dim \text{Ext}^1_{D_v,f}(\chi^c, \chi) = 1 \), while at \( \bar{v} \) such an extension is always crystalline, so \( \dim \text{Ext}^1_{D_{\bar{v}},f}(\chi^c, \chi) = 0 \). Therefore \( \text{Ext}^1_{G_K}(\chi^c, \chi) \) has dimension at most 1, but since the one-dimensional space \( (B/mB)^* \) injects in it, it has dimension 1 and the injection is an isomorphism. A similar argument works for \( \text{Ext}^1_{G_K}(\chi, \chi^c) \).

Now we look at the vertical maps in the above diagram. We claim that the right vertical map in the \( B \)-diagram is an isomorphism. For if it was not, the non-trivial extension (in the category of \( G_K \)-representations) of \( \chi \) by \( \chi^c \) would be trivial, hence crystalline, at \( v \), but is automatically crystalline, as we have seen, at \( \bar{v} \), contradicting the fact used above that \( \text{Ext}^1_{G_K,f}(\chi^c, \chi) = 0 \). Therefore \( \text{Ext}^1_{G_K}(\chi^c, \chi) \) has dimension at most 1, but since the one-dimensional space \( (B/mB)^* \) injects in it, it has dimension 1 and the injection is an isomorphism. A similar argument works for \( \text{Ext}^1_{G_K}(\chi, \chi^c) \).

We therefore have a chain of trivial equivalences: \( e > 2 \) if and only if the inclusion \( I_p \subset I \) is proper if and only if the inclusion \( C_p \subset C \) is proper (remember that \( I = BC \) and \( I_p = B_pC_p \)) if and only if the left vertical map in the \( C \)-diagram is 0 (by Nakayama), if and only if the right vertical map in that diagram is 0. The last assertion reads: The non-trivial extension of \( \chi \) by \( \chi^c \) is trivial restricted to \( D_v \). \( \square \)

The non-trivial extension of \( \chi \) by \( \chi^c \) is automatically crystalline at \( v \), so it is not crystalline at \( \bar{v} \) (otherwise it would be in the \( \text{Ext}^1_{G_K,f} \) which is 0), hence it is non-trivial at \( \bar{v} \). Therefore, the negation of the assertion that "the non-trivial extension of \( \chi \) by \( \chi^c \) is trivial restricted to \( D_v \)" is "the non-trivial extension of \( \chi \) by \( \chi^c \) is non-trivial restricted at both \( D_v \) and \( D_{\bar{v}} \)", which is exactly condition (v) of Theorem 1. Thus the corollary above implies the equivalence between (iv) and (v).

3.3. Relation with Jannsen’s conjecture. Let \( M \) be a number field, and \( \Sigma \) a finite set of finite places of \( M \) containing the set \( \Sigma_p \) of places above \( p \). We denote by \( G_{M,\Sigma} \) the Galois group of the maximal extension of \( M \) unramified outside \( \Sigma \) and the archimedean places. For \( v \) a place of \( M \), we call \( G_v \) the decomposition group at \( v \).

We call Jannsen’s conjecture the following statement:

**Conjecture 3.2.** Let \( \rho \) be and geometric Galois representation \( \rho : G_{M,\Sigma} \to GL_n(\bar{Q}_p) \) pure of motivic weight \( w \neq -1 \). Assume that \( \rho_{G_v} \) for \( v \in \Sigma - \Sigma_p \) does not admit
\( \mathbb{Q}_p(1) \) has a quotient. The map
\[
H^1(G_{M, \Sigma}, \rho) \to \prod_{v \in \Sigma} H^1(G_v, \rho)
\]
is injective.

We observe that this statement (or something equivalent) is called a question in Jannsen’s paper, which reserves the term conjecture to a weakest statement. However, it is safe to promote it a conjecture (cf [B1] for a more detailed discussion.) Applying this to \( M = K, V = \chi/\chi^c \) (which has motivic weight 0), we see that Jannsen conjecture implies that \( H^1(G_K, \chi/\chi^c) \to H^1(G_v, \chi/\chi^c) \times H^1(G_{\bar{v}}, \chi/\chi^c) \) is injective, which implies condition (e) of Prop. 1

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