UNITARY EIGENVARIETIES AT ISOBARIC POINTS

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Abstract. In this article (a sequel of [B1]), we study the geometry of the eigen-varieties of unitary groups at points corresponding to tempered non-stable representations with an anti-ordinary (a.k.a evil) refinement. We prove that, except in the case the Galois representation attached to the automorphic form is a sum of characters, the eigenvariety is non-smooth at such a point, and that (under some additional hypotheses) its tangent space is big enough to account for all the relevant Selmer group. We also study the local reducibility locus at those points, proving that in general, in contrast with the case of the eigencurve, it is a proper subscheme of the fiber of the eigenvariety over the weight space.

1. Introduction

in [BC2], the author of this note and Chenevier studied the geometry of the eigenvariety of a unitary group at a point corresponding to a non-tempered automorphic representation \( \pi \), with an anti-ordinary refinement, and used this result to prove new cases of the lower bound on the Selmer group in the Bloch-Kato conjecture for an essentially self-dual Galois representation \( \rho \) of arbitrary dimension. While the methods developed in the book were more general, the case needed for the application to the Bloch-Kato conjecture was the one of an automorphic representation \( \pi \) with attached Galois representation \( \rho_\pi = 1 \oplus \omega \oplus \rho \), where 1 is the trivial character (of motivic weight 0), \( \omega \) is the cyclotomic character (of motivic weight \(-2\) according to the most used conventions, that we shall also use) and \( \rho \) is some irreducible, essentially self-dual Galois representation of motivic weight \(-1\). Observe that \( \rho_\pi \) is not isobaric, that is that the motivic weights of the components of \( \rho_\pi \) are distinct. This corresponds, in accordance to Arthur’s philosophy, to the fact that the automorphic representation \( \pi \) is not tempered.

Let us recall that the lower bound in the Bloch-Kato conjecture is the prediction that the Bloch-Kato Selmer group \( H^1_f(G_K, \rho) \) has dimension at least the order of vanishing of the \( L \)-function \( L(\rho^*(1), s) \) at \( s = 0 \). When the motivic weight of \( \rho \) is not \(-1\), and \( \rho \) is automorphic, then using the functional equation, and the Hadamard-De-La-Vallée-Poussin theorem of Jacquet-Shalika, that order of vanishing \( \text{ord}_{s=0} L(\rho^*(1), s) \) can be expressed purely in terms of local invariants of \( \rho \), and moreover this lower bound on \( \text{dim} H^1_f(G_K, \rho) \) can be proved by relatively elementary means (essentially by Poitou-Tate duality – cf. [B2, Corollary 4.1, page 49]). Hence working with non-tempered automorphic representations is essential to new applications to the Bloch-Kato conjecture. I conjecture that the geometry of the eigenvariety at \( \pi \) is closely related to the \( p \)-adic \( L \)-function of \( \rho \), in such a way that

During the elaboration and writing of this paper, Joël Bellaïche was supported by the NSF grant DMS 08-01023.
the methods of [BC2], or variants, will imply the lower bound in the Bloch-Kato conjecture for $\rho$. There is an ongoing program to prove this conjecture.

In this article, one studies the geometry of the eigenvariety at a tempered automorphic representation (still with anti-ordinary refinement) with two aims in mind: to provide support for the aforementioned conjecture, and to test tools to solve it. The temperedness of $\pi$ means that the components of the Galois representation $\rho_\pi$ all have the same motivic weight, hence whatever lower bound in the Bloch-Kato conjecture one might deduce from our results will concern representations of motivic weight 0, and thus be already known by elementary methods. Yet, the fact that, as we shall show, the geometry of the eigenvariety is rich enough to allow us to construct the whole Selmer group of a self-dual weight 0 Galois representation $\rho$ is a strong encouragement to try to do so for weight $-1$ Galois representations.

This study is a sequel of my paper [B1], which deals with the case of the unitary groups with three variables $U(3)$, and endoscopic forms of type $(2,1)$. In the present paper, we work out the general case. While the technics are similar in nature with the ones used in [B1] (and thus relying heavily on the work of [BC2]), the algebraic and combinatorial arguments needed to solve the general case are much harder.

1.1. Reminder on eigenvarieties, Galois representations, and refinements. Notations introduced here will stay in force during all this paper.

1.1.1. Unitary groups. We fix a prime $p$.

Let $K$ be a quadratic imaginary field, in which $p$ is split. We denote by $G_K$ the absolute Galois group of $K$, and for every place $v$ of $K$, $G_v$ the absolute Galois group of the completion $K_v$ of $K$ at a place $v$, that we see as a decomposition subgroup of $G_K$. We fix a prime $p$ that is split in $K$. Let $U(d)$ be the unitary group over $\mathbb{Q}$ attached to an hermitian form over $K$ in $d$ variables, which is definite over $\mathbb{R}$.

1.1.2. Eigenvarieties. The $p$-adic eigenvariety $X$ for $U(d)$ (and a fixed level $U = K_p U^p$, where $U^p$ is an open compact subgroup of $U(d)(\mathbb{A}_f^p)$ and $K_p$ is a maximal compact subgroup of $U(d)(\mathbb{Q}_p)$) has been constructed independently by Chenevier ([C]) and Emerton ([E]). The two constructions have been shown in [BC2] to lead to the same eigenvariety, which indeed can be characterized by some simple natural properties. The eigenvariety $X$ is a reduced rigid analytic space over $\mathbb{Q}_p$, equidimensional of dimension $d-1$ (it would be $d$ if we were allowing central twists), provided with a Zariski-dense set of so-called classical points, which corresponds to automorphic eigenforms for $U(d)$ of level $U$ and of various weights.

1.1.3. Galois representations attached to automorphic representations. Another recent progress that is of importance to us is the construction, for any automorphic forms $\pi$ for $U(d)$, of an attached Galois representation $\rho_\pi : G_K \to \text{GL}_d(\mathbb{Q}_p)$, whose restriction to $G_v$ for any place $v$ above a prime $l$ such that $\pi_l$ is unramified has the expected properties (namely corresponds to the base change of $\pi_l$ to $K_v$ by

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1The methods of this paper could in principle be applied to any CM field $K$. 

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Local Langlands if \( l \neq p \), and is crystalline with expected Hodge-Tate weights and Frobenius eigenvalues if \( l = p \). Those \( \rho_\pi \) are polarized of weight \( d - 1 \), that is

\[
\rho^\perp \simeq \rho \omega^{d-1}
\]

where for a representation \( \rho : G_K \to \text{GL}_n(\bar{\mathbb{Q}}_p) \) the representation \( \rho^\perp \) is defined by, if \( c \) is any lift of the nontrivial elements of \( \text{Gal}(K/\mathbb{Q}) \) in \( G_\mathbb{Q} \) by

\[
\rho^\perp(g) = t \rho(gc^{-1})^{-1},
\]

and \( \omega : G_K \to \mathbb{Z}_p^* \subset (\bar{\mathbb{Q}}_p)^* \) is the cyclotomic character. This construction of \( \rho_\pi \) is due to Sophie Morel [Mor], Shin [S] and the authors of the book project [GRFAbook] lead by Harris, all those people relying on works done by many authors since Langlands, in particular fundamental contributions by Shelstad, Kottwitz, Arthur, Clozel, Waldspurger, Laumon and Ngo.

When \( \pi \) is stable tempered, that is when the base change \( \pi_K \) of \( \pi \) to \( \text{GL}_n(A_K) \) is cuspidal, it is expected that \( \rho_\pi \) is irreducible. This is known in many cases, though apparently not yet in full generality. For instance, the irreducibility of \( \rho_\pi \) is known if \( d \leq 3 \) (cf. [BlRo]), and if \( d \leq 5 \) and \( \pi_K \) comes by base change from an automorphic representation for \( \text{GL}_n(A_\mathbb{Q}) \) (cf. [CG]); in any dimension, it is known that \( \rho_\pi \) is irreducible when \( \pi \) is square-integrable at some place ([TY]), or for any given \( \pi \) for a positive density of primes \( p \) ([PT]), or for a generic \( \pi \) in suitable \( p \)-adic families ([BC2]).

1.1.4. Families of Galois representations. By interpolating the Galois representations attached to classical automorphic forms, a simple argument due to Chenevier gives a pseudocharacter \( T : G_K \to \mathcal{O}(X) \) of dimension \( d \). If \( x \in X(\bar{\mathbb{Q}}_p) \), the post-composition of \( T \) with the “evaluation at \( x \)” map \( \mathcal{O}(X) \to \bar{\mathbb{Q}}_p \) gives a pseudocharacter \( T_x : G_K \to \bar{\mathbb{Q}}_p \) which is the trace of a unique semi-simple representation \( \rho_x : G_K \to \text{GL}_n(\bar{\mathbb{Q}}_p) \) up to isomorphism. When \( x \) is attached to a classical automorphic representation \( \pi \), we have \( \rho_x \simeq \rho_\pi \).

1.1.5. Tempered and isobaric automorphic representations. If \( \pi \) is an automorphic representation for \( U(d) \), then we can write \( \rho \) as a sum of irreducible representations

\[
\rho_\pi \simeq \rho_1 \oplus \cdots \oplus \rho_r,
\]

for some \( r \) between 1 and \( d \), and the \( \rho_i \)'s are pairwise non isomorphic, since \( \rho_\pi \) has distinct Hodge-Tate weights. The type of \( \rho_\pi \) is the \( r \)-tuple \( (d_1, \ldots, d_r) \) where \( d_i = \dim \rho_i \). Since the map \( \rho \mapsto \rho^\perp(d-1) \) is an involution, the set \( \{1, \ldots, r\} \) can be partitioned into singletons \( \{i\} \) such that \( \rho_i \simeq \rho_i^\perp(d-1) \) and pairs \( \{i,j\} \) such that \( \rho_i \simeq \rho_j^\perp(d-1) \).

**Definition 1.1.** If for all \( i = 1, \ldots, r \), we have \( \rho_i \simeq \rho_i^\perp(d-1) \), we say that \( \rho_\pi \) (and \( \pi \)) is isobaric.

**Remark 1.2.** Conjecturally, the isobaric representations \( \pi \) should be exactly the tempered representations, and a \( \rho_\pi \) should be isobaric reducible if and only if \( \pi \) is endoscopic tempered (see e.g. the review of Arthur’s conjecture given in [BC2, Appendix].)
More precisely, if \( \pi_1, \ldots, \pi_r \) are stable tempered representations of unitary groups \( U(d_1), \ldots, U(d_r) \), with \( d = d_1 + \cdots + d_r \), there should exist a tempered representation \( \pi \) of \( U(d) \), called the endoscopic transfer of \( (\pi_1, \ldots, \pi_r) \) such that \( \rho_\pi = \rho_{\pi_1} \oplus \cdots \oplus \rho_{\pi_r} \), provided that the Hodge-Tate weights of \( \rho_{\pi_1}, \ldots, \rho_{\pi_r} \) are distinct. Conversely, all tempered representation \( \pi \) should be \( L \)-equivalent to one arising this way. Those results, which have long been conjectured, are now almost known in full generality. The analog for quasi-split unitary groups of the construction of the endoscopic transfer from stable tempered representations \( \pi_1, \ldots, \pi_r \) has been recently completed by Mok [Mok] extending to unitary groups earlier results of Arthur for symplectic and orthogonal groups. The transfer back and forth between definite at infinity and quasi-split unitary groups seems a comparatively easy task, even if, to our knowledge, it has been only written down for \( d \leq 3 \) ([R]).

Thus a representation \( \pi \) obtained by endoscopic transfer from stable tempered representations \( \pi_1, \ldots, \pi_r \) would be isobaric of type \( (d_1, \ldots, d_r) \) provided the representations \( \rho_{\pi_1}, \ldots, \rho_{\pi_r} \) are irreducible, as they are expected to be. In view of the discussion at the end of §1.1.3, this provides a large supply of examples of isobaric representations.

1.1.6. Anti-ordinary refinements. If a classical automorphic form \( \pi \) of level \( U \) is given, it does not define yet a point in the eigenvariety \( X \). For this, we need to choose some combinatorial data, called either a \( p \)-stabilization or a refinement \( R \) of \( \pi_p \). For the general definition of a refinement, we refer the reader to [BC2, §6.4]. We will place ourselves in the case where the eigenvalues of the crystalline Frobenius on \( D_{\text{crys}}(\rho_{\pi_v}) \) (here \( v \) is one of the two places of \( K \) above \( p \)) are distinct.

In that case, a refinement is simply defined by an ordering of those eigenvalues. Thus there are at most \( d! \) refinements. When \( \rho_{\pi} \) is isobaric, which we shall assume, every ordering of the eigenvalues of the crystalline Frobenius defines a refinement (cf. [BC2, 6.4.5]), so there are exactly \( d! \) refinements.

Each pair \( (\pi, R) \) defines a classical point on the eigenvariety \( X \). For a given \( \pi \), those \( d! \) different points \( x_1, \ldots, x_{d!} \) have in general very different Galois-theoretic behavior: even if the representation \( \rho_{x_i} \) is the same at the different \( x_i \), namely \( \rho_{\pi} \), its deformations to a small neighborhood of \( x_i \) are very different.

Among the various refinements of \( \pi \) we will consider a special kind, that we call the anti-ordinary refinements. They are the generalizations of the evil refinements of Eisenstein series in the theory of \( p \)-adic modular forms. They are also critical refinements, though not all critical refinements are anti-ordinary. The definition of anti-ordinary, and some properties, are given in §2.

1.2. Results.

1.2.1. Non-smoothness of the eigenvariety at isobaric anti-ordinary points. We are almost ready to state our non-smoothness result. As we import results from [BC2], we are also obliged to import as well some technical hypotheses from there. Those technical hypotheses, called (REG) (for regular), (MF') (for multiplicity free) and (NGD) (for no geometric deformation), are as follows:

- (REG)
- (MF')
- (NGD)
Let $\pi$ be an automorphic representation for $U(n)$, unramified at $p$, of type $(d_1, \ldots, d_r)$ with $\rho_\pi = \rho_1 \oplus \cdots \oplus \rho_r$.

(REG) For every integer $a$, with $1 \leq a \leq d - 1$, the eigenvalues of the crystalline Frobenius on $D_{\text{crys}}(\Lambda^a \rho_\pi)$ have multiplicity 1.

(MF') For every family of integers $(a_i)_{i=1}^r$ with $1 \leq a_i \leq d_i$, the representation $\rho(a_i) := \bigotimes_{i=1}^r \Lambda^{a_i} \rho_i$ is absolutely irreducible. Moreover, if $(a_i)$ and $(a'_i)$ are two distinct sequences as above with $\sum_{i=1}^r a_i = \sum_{i=1}^r a'_i$, then $\rho(a_i) \not\cong \rho(a'_i)$.

(NGD) For $i = 1, 2, \ldots, r$, $H^1_g(G_{K,S}, \rho_i) = 0$.

Both assumptions (REG) and (MF') are multiplicity one statements, which are only necessary due to our imperfect knowledge on the trianguline nature of the Galois representation on the eigenvarieties. It should be possible to remove them in some near future.

The hypothesis (NGD) asserts that any de Rham (at every place of $K$ dividing $p$) deformation of $\rho_i$ to $\overline{Q}_p[\varepsilon]/(\varepsilon)^2$ is trivial. This is a standard infinitesimal version of the Fontaine-Mazur conjecture, which states that de Rham representations of $G_{K,S}$ are geometric, and therefore form a countable set and cannot have non-trivial families. From another point of view, the hypothesis (NGD) is a special case of the assertion that for any representation $\rho'$ of non-negative motivic weight, $H^1_g(G_{K,S}, \rho') = 0$ – namely the case $\rho' = \text{ad} \rho_i$. This assertion is the $p$-adic avatar of the famous ”Yoga of Weights” of Grothendieck. It is also a part of Bloch-Kato’s conjecture. At any rate, the assertion (NGD) is known already in a significant number of cases, for example for one-dimensional representations and for many 2-dimensional representations $\rho_i$ coming from $G_{\mathbb{Q}}$ (by results of Weston [W] and Kisin [Kis]). It is also provable for all representations satisfying (1) whose residual representations satisfy the hypotheses of the fast growing set of theorems on potential automorphy: see forthcoming work of Davide Reduzzi.

The following result is analogous to, but much harder than, the main theorem of [B1]:

**Theorem 1.** Let $\pi$ be an automorphic representation of $U(d)$, unramified at $p$, isobaric of type $(d_1, \ldots, d_r)$ (with $r \geq 2$, that is $\rho_\pi$ reducible) Assume that $\rho_\pi$ satisfies (REG), (MF'), (NGD). Let $\mathcal{R}$ be an anti-ordinary refinement of $\pi$, and let $x \in X(\overline{Q}_p)$ be the point corresponding to $(\pi, \mathcal{R})$ on the eigenvariety. Assume furthermore that $(d_1, \ldots, d_r) \neq (1, \ldots, 1)$. Then $X$ is non-smooth at $x$. Even more, the local ring at $x$ of every irreducible component of $X$ through $x$ is not factorial.

The condition that the type of $\pi$ is not $(1, \ldots, 1)$ is necessary.

1.2.2. The case of the type $(1, n)$. We shall now explain two results which provide examples of two important phenomena: a case where the method of Ribet on eigenvarieties can produce not only one, but all the $n$ independent extensions that are supposed to exist between two Galois representations, for $n$ arbitrary large; and a classical anti-ordinary point of the eigenvariety where the local reducibility locus of the Galois representation, and the schematic fiber over weight space at this point do not coincide (in contrast with the case of the eigencurve, where they
always coincide). As we are interested here in constructing examples (or counter-examples), we don’t mind adding restrictive hypotheses (cf. (IRR) and (WEI) below) which simplify the argument, since they still are satisfied by a large set of representations.

We keep the assumptions of Theorem 1, but we assume in addition that \( \pi \) is of type \((1,n)\), where \( n = d - 1 \) is an integer \( \geq 2 \). In this case, \( \rho_\pi = \rho_1 \oplus \rho_2 \), with \( \dim \rho_1 = 1 \). We set
\[
\rho = \rho_1^{-1} \otimes \rho_2,
\]
which is a representation of dimension \( n \) and motivic weight 0, satisfying \( \rho = \rho_1^{\perp} \).

The hypotheses on \( \pi \) are equivalent to the following hypotheses on \( \rho \):

(REG) For every integer \( a \), with \( 1 \leq a \leq n - 1 \), the eigenvalues of the crystalline Frobenius on \( D_{\text{crys}}(\Lambda^a \rho) \) have multiplicity 1.

(MF') For every integer \( a \), with \( 1 \leq a \leq n - 1 \), the representation \( \Lambda^a \rho \) is absolutely irreducible.

(NGD) \( H^1_{\partial}(G_K, \text{ad}\rho) = 0 \).

Let us add another hypothesis:

(IRR) The restriction \( \rho|_{G_v} \) is irreducible.

Let us call \( v \) and \( \bar{v} \) the two places of \( K \) above \( p \). By proposition 2.3 below, applicable because of (IRR), \( \pi \) has an anti-ordinary refinement if and only if the Hodge-Tate weights of \( \rho \) at \( v \) are all negative, or all positive\(^2\). Those two case are symmetric (just exchange \( v \) and \( \bar{v} \)), so let us assume that the Hodge-Tate weights of \( \rho \) at \( v \) are all negative, and a little more to simplify some arguments:

(WEI) The Hodge-Tate weights of \( \rho|_{G_v} \) are all \( \leq -2 \).

Note that the Hodge-Tate weights of \( \rho|_{G_{\bar{v}}} \) are therefore all \( \geq 2 \). We can choose an anti-ordinary refinement of \( \pi \) and call it \( \mathcal{R} \). Let \( x \) be the point of \( X \) corresponding to \((\pi, \mathcal{R})\).

The extension of Ribet’s method which is developed in [BC2] uses the families of Galois representations carried by the eigenvariety \( X \) around the point \( x \) (where the Galois representation is \( \rho_x = \rho_1 \oplus \rho_2 \)) to construct extensions of \( \rho_1 \) by \( \rho_2 \). The space of extensions we can construct by this method is a subspace of \( H^1(G_K, \rho) = \text{Ext}^1_{G_K}(\mathbb{Q}_p, \rho) \simeq \text{Ext}^1_{G_K}(\rho_1, \rho_2) \) that we shall call \( H^1_x(G_K, \rho) \). The Bloch-Kato conjecture implies\(^3\) \( \dim H^1_x(G_K, \rho) = n \), and it is an easy consequence of Poitou-Tate duality that \( \dim H^1(G_K, \rho) \geq n \).

The following result is analogous to, but harder than, [B1, Theorem 2].

**Theorem 2.** With the above hypotheses, one has \( \dim H^1_x(G_K, \rho) \geq n \).

\(^2\)Our convention regarding Hodge-Tate weight is that the cyclotomic character of \( G_v \) has Hodge-Tate weight +1.

\(^3\)Indeed, an extension of 1 by \( \rho \) has automatically good reduction in the sense of Bloch-Kato at all places \( w \) not dividing \( p \) and also at the place \( \bar{v} \) above \( p \). This is true because \( \rho \) is of weight 0, hence \( \rho|_{G_v} \) does not contain \( \mathbb{Q}_p(1) \), and at \( \bar{v} \) because of hypotheses (WEI). Then we can argue as in [B1, Proposition 12].
The significance of this result is that Ribet’s method on eigenvarieties can construct all the \( n \) independent extensions in \( H^1(G_K, \rho) \) that are known to exist. This is encouraging for the ability of the Ribet’s method on eigenvarieties to construct all extensions that are supposed (but not necessarily known) to exist in other cases (for example when \( \rho \) has motivic weight \(-1\)).

Keep the assumptions of the preceding theorem and let \( R_x \) be the local reducibility locus at \( x \), that is the largest local closed subscheme of \( X \) containing \( x \) on which the family of Galois representations carried by \( X \), restricted to the decomposition group \( D_v \), is reducible. Let \( \kappa : X \to W \) be the weight map from the eigenvariety to the weight space \( W \) (a union of \( n \)-dimensional rigid open balls), and let \( F_x \) be the connected component at \( x \) of the schematic fiber of \( \kappa \) at \( \kappa(x) \). Since \( \kappa \) is locally finite, \( F_x \) is a finite local closed subscheme of \( X \). One of the main result of [BC2] (Theorem 4.4.6) is that we have an inclusion of closed subschemes of \( X \):

\[
R_x \subset F_x.
\]

In particular, \( R_x \) is a finite local scheme. In the case of the eigencurve, two proofs are given in [BC1] that \( R_x = F_x \). Surprisingly (for the author), this is not the case in higher dimension – or the Bloch-Kato’s conjecture is false:

**Theorem 3.** Keep the hypotheses of the above theorem, and assume that the Bloch-Kato conjecture holds for \( \rho \) and all its twists \( \rho \tau \) where \( \tau \) is a character of \( G_K \) satisfying \( \tau = \tau^\perp \). (That is to say, since \( \rho \tau \) has motivic weight 0, assume \( H^1_f(G_K, \rho \tau) = 0 \).)

Then there are infinitely many points \( x \in X \) such that the inclusion \( R_x \subset F_x \) is strict, that is such that the local reducibility locus is strictly smaller than the fiber over the weight space.

This result raises the question of what exactly is the reducibility locus \( R_x \). This question is important if one wishes to extend the results of [BC2, Chapter 9] relating the absolute geometry of the eigenvariety \( X \) at non-ordinary points to the rank of suitable Selmer groups, in a way which take into account the finer relative (that is, as a variety over the weight space \( W \)) geometry of \( X \). One reason one might wish to do that is because it is this relative geometry of \( X \) that seems more directly related to the vanishing of a suitable \( p \)-adic \( L \)-function (cf. [Kim],[B4, Chapter V]).

1.3. **Thanks and apologies.** In four talks at seminars and conferences in north-America during the second half of 2008, I explained Theorem 1 and Proposition 2, with ideas of their proof. I want to thank the audiences of those talks for their questions and comments. In two of those talks, I also mentioned a third result, which was that the map (3), page 13, was not injective. As should be clear from the reading of this paper, this result is false. The proof I had in mind used the erroneous fact that \( R_x = F_x \), which I then mistakenly believed to have proved along the lines of [BC1, théorème 2]. I realized when writing the first version of this paper at the end of 2008 that that result was false (or at least, in contradiction with the Bloch-Kato’s conjecture), and it took me a few weeks to identify with certainty
the guilty lemma (namely that $R_x = F_x$). In doing so, I was helped by an email of R. Greenberg, whom I want to thank here. I also want to offer my apologies to the audience of the two seminars where I announced results whose proofs were insufficiently checked.

The author wishes to thank the anonymous referee for his careful reading and his advices.

2. Anti-ordinary refinements

We use the same notations as in the introduction: $K$ is a quadratic imaginary field, $p$ a prime that splits in $K$, $v$ a place of $K$ above $p$, and “$\rho$ is crystalline” (or “De Rham”, or “Hodge-Tate”) means the same thing for the restriction $\rho$ to the local Galois group $G_v$.

Let $\rho$ be a representation of $G_{K,S}$ which is crystalline. Recall that a refinement $R$ of $\rho$ is an ordering of the eigenvalues of the crystalline Frobenius $\phi$ on $D_{\text{crys}}(\rho)$ (we assume that those eigenvalues are distinct). If $\rho'$ is a subrepresentation of $\rho$, a refinement of $\rho$ induces, by restriction, a refinement of $\rho'$.

We assume that $\rho \simeq \rho_1 \oplus \cdots \oplus \rho_r$ where the $\rho_i$ are irreducible of dimension $d_i$, and that the Hodge-Tate weights of $\rho$ are distinct.

Recall that a refinement $R$ defines a maximal flag in $D_{\text{crys}}(\rho)$ (namely the flag whose space of dimension $k$ is generated by the eigenvectors corresponding to the first $k$ eigenvalues of the refinement) and that $R$ is said to be non-critical if this flag is in general position with respect to the Hodge filtration in $D_{\text{crys}}(\rho)$. Otherwise it is said to be critical. It is clear from the definition that there always exists at least one non-critical refinement.

To a refinement $F$ as above, we can attach a permutation $\sigma \in S_d$ by the following recipe: let $k_1, \ldots, k_d$ be the Hodge-Tate weights of $\rho$ in increasing order. Let $\phi_1, \ldots, \phi_d$ be the eigenvalues of $\phi$ on $D_{\text{crys}}(\rho)$ in the order given by $R$. For $i = 1, \ldots, r$, let $W_i$ be the set of indices $l$ such that $k_l$ is a Hodge-Tate weight of $\rho_i$, and $R_i$ be the set of indices $l$ such that $\phi_l$ is an eigenvalue of $\phi$ on $D_{\text{crys}}(\rho_i)$. The $R_i$’s, and the $W_i$’s, are two partitions of $\{1, \ldots, d\}$ into $r$ parts, with $|W_i| = |R_i| = d_i$. We define $\sigma$ as the unique permutation of $\{1, \ldots, d\}$ that maps $R_i$ to $W_i$ and is increasing on $R_i$ for $i = 1, \ldots, r$.

**Definition 2.1.** A refinement $R$ of $\rho$ is said to be anti-ordinary if

(INT) The $R_i$ are intervals of $\{1, \ldots, n\}$

(TR) The permutation $\sigma$ attached to $F$ and $\rho$ is transitive.

(NCR) The restriction of $F$ to $\rho_i$, $i = 1, \ldots, r$, is non-critical.

**Remark 2.2.**

(i) It should be noted that the anti-ordinariness of $(\rho, F)$ is a global property: it depends on $\rho$ and not only on its restriction to $G_v$.

(ii) If $\rho$ is non-critical, then the permutation $\sigma$ is the identity. (We leave this as an exercise for the reader.) Thus, in some sense, the anti-ordinary refinements are highly critical.
(iii) If $\rho$ is irreducible, then $\sigma = 1$ and no refinement is anti-ordinary. If $\rho$ is a sum of characters, then conditions (NCR) and (INT) are automatic and is easy to see that the number of anti-ordinary refinements is $(d - 1)!$.

(iv) For another case of existence of an anti-ordinary refinement, see [BC2, Lemma 9.3.4].

**Proposition 2.3.** Assume $d_1 = 1$ and $d_2 = d - 1$. Then $\rho$ admits an anti-ordinary refinement if and only of the Hodge-Tate weights of $\rho_2$ are all bigger, or all smaller, than the Hodge-Tate weight of $\rho_1$.

*Proof —* Assume $\rho$ admits an anti-ordinary refinement. By hypothesis (INT), either $R_1 = \{1\}$ or $R_1 = \{d\}$. Assume first that $R_1 = \{1\}$. If $W_1 = \{i\}$, with $i < d$, then by definition $\sigma$ maps the integer $j > i$ to themselves, contradicting (TR). Thus $W_1 = \{d\}$ which means that the weights of $\rho_2$ are all smaller than the weight of $\rho_1$. Similarly if $R_1 = \{d\}$, then the weight of $\rho_1$ is smaller than all the weights of $\rho_2$.

Conversely, assume that the weights of $\rho_2$ are all smaller than the weight of $\rho_1$. Then construct a refinement $\mathcal{R}$ by taking $\phi_d$ the eigenvalue of the crystalline Frobenius on $\rho_1$, and $\phi_1, \ldots, \phi_{d-1}$ the eigenvalues on $\rho_2$ in whatever order defines a non-critical refinement of $\rho_2$ (such an order always exists: see [BC2, §2.4.6]). Then (NCR) is satisfied by construction, and we have $R_1 = \{d\}$ and $R_2 = \{1, 2, \ldots, d - 1\}$ so (INT) is satisfied. We also have $W_1 = \{1\}$ and $W_2 = \{2, \ldots, d\}$ by assumption, so $\sigma(d) = 1$ and $\sigma(i) = i + 1$ for $1 \leq i \leq d - 1$, and (TR) is also satisfied. \qed

### 3. Eigenvarieties at reducible, isobaric, anti-ordinary points

In this section we prove the Theorem 1. We use the notations introduced in the statement of that theorem: $\pi$ is an automorphic forms for $U(n)$, unramified at $p$, which is isobaric and such that $\rho_\pi = \rho_1 \oplus \cdots \oplus \rho_r$, with the $\rho_i$ irreducible of dimension $d_i$. The representation $\rho$ satisfies the hypotheses (REG), (NF'), (NGD) and $\mathcal{R}$ is an anti-ordinary refinement of $\pi$. The eigenvariety for $U(n)$ (and some suitable level) is denoted by $X$ and $x \in X(\overline{\mathbb{Q}}_p)$ is the point corresponding to $(\pi, \mathcal{R})$.

#### 3.1. Reminder of results of [BC2]

Let $\mathcal{O}_x$ be the local ring of $X$ at $x$, and let $A$ be the quotient of $\mathcal{O}_x$ by a minimal prime ideal (or in other words, the local ring at $x$ of an irreducible component of $X$ through $x$). The ring $A$ is a local domain whose maximal ideal will be denoted by $m$, and whose residue field is $\overline{\mathbb{Q}}_p$, and it carries a pseudo-character $T : G_K \to A$ of dimension $d$ which residually is $T \otimes_A m = \text{tr} \rho_x = \text{tr} \rho_\pi = \text{tr} \rho_1 + \cdots + \text{tr} \rho_r$. In particular, $T$ is a residually multiplicity free pseudocharacter. Those pseudocharacters are subject to a close analysis in [BC2, chapter 1]. In particular, attached to $T$, there exist a family $(A_{i,j})_{i,j=1,\ldots,r}$ of fractional ideals of $A$ (that is, finite type $A$-submodules of $L = \text{frac}(A)$) such that

(a) $A_{i,j} A_{j,i} \subset m$ for every two distinct elements $i, j$ in $\{1, \ldots, r\}$.

(b) $A_{i,j} A_{j,k} \subset A_{i,k}$ for every three distinct elements $i, j$ in $\{1, \ldots, r\}$.
(c) \( \sum_{i \neq j} A_{i,j} A_{j,i} \) is the total reducibility ideal of \( T \), that is the smallest ideal \( I \) of \( A \) such that \( T \otimes A/I \) is a sum of \( r \) pseudocharacters.

A key fact, which uses many of the main results of the book [BC2], is that the total reducibility ideal of \( T \) at \( x \) is \( m \), which intuitively means that on any closed subscheme of \( \text{Spec } A \) that is strictly larger than the reduced closed point \( \{ x \} \) (for example, on the closed subscheme of \( X \) of ring \( \mathbb{Q}_p[e]/(e^2) \) that is defined by a non-zero tangent vector of \( \text{Spec } A \) at \( x \)), then \( T \) is less reducible that it is at \( \{ x \} \), namely is not the sum of as many as \( r \) pseudocharacters. This result relies on the anti-ordinarity of \( \rho \), and the hypotheses (REG), (MF'), (NGD). For a proof, see [BC2, Proposition 9.3.7] (see also [B1, Proposition 7]).

Therefore, we can restate (c) as

\[ (c') \sum_{i \neq j} A_{i,j} A_{j,i} = m \]

We need one more fact. Since \( T \) is defined by interpolating representations satisfying \( \rho^{-1}(1 - d) \simeq \rho \), it satisfies \( T(cg^{-1})T(g)^{1-d} = T(g) \) for every \( g \in G_{K,S} \), and the same relation is also satisfied by each of the \( \text{tr } p_i \) since \( \pi \) is isobaric. We thus have, according to [BC2, Lemma 1.8.5],

\[ (d) \text{ For every } i \neq j, A_{i,j} \text{ and } A_{j,i} \text{ are isomorphic as } A \text{-modules.} \]

### 3.2. An algebraico-combinatorial result.

We now prove a purely algebraico-combinatorial result.

**Proposition 3.1.** Let \( A \) be a local noetherian domain of maximal ideal \( m \), residue field \( A/m = F \), with a family of fractional ideals \( \{ A_{i,j} \}_{i,j=1,...,r} \) that satisfy (a), (b), (c') and (d) above. Then if \( A \) is factorial, we have \( r > \dim A \).

**Proof** — If \( A \) is an UFD, there exist elements \( x_1, \ldots, x_r \) in \( L^* \), such that for every \( i \neq j, i x_j^{-1} A_{i,j} \) is a true ideal of \( A \) (see the proof of [BC2, Prop 1.6.1]). Changing the \( A_{i,j} \) by \( i x_j^{-1} A_{i,j} \) obviously preserves conditions (a), (b), (c') and (d). So we can assume that the fractional ideals \( A_{i,j} \) are true ideals of \( A \).

Let us call a pair \( \{ i,j \} \subset \{ 1, \ldots, r \} \) bad if \( A_{i,j} A_{j,i} \subset m^2 \), good otherwise. By Nakayama’s lemma, (c') can be re-written as

\[ (e'') \sum_{\{ i,j \} \text{ good pair } \subset \{ 1, \ldots, r \}} A_{i,j} A_{j,i} = m. \]

Let \( \{ i,j \} \) be a good pair. The two ideals \( A_{i,j} \) and \( A_{j,i} \) are not both contained in \( m \), otherwise \( A_{i,j} A_{j,i} \) would be contained in \( m^2 \). But since \( A \) is local, the only ideal not contained in \( m \) is \( A \). Therefore, one of the ideals \( A_{i,j} \) or \( A_{j,i} \) is \( A \), and in particular is free of rank one as an \( A \)-module, and by (d), so is the other, which is therefore principal. Call \( X_{\{ i,j \}} \) a generator of that ideal. By (a), we have \( X_{\{ i,j \}} \in m \), but since \( \{ i,j \} \) is a good pair, \( X_{\{ i,j \}} \notin m^2 \). In particular, \( X_{\{ i,j \}} \) is irreducible. Note that this analysis allows to give a natural orientation to a good pair \( \{ i,j \} \). We orient \( \{ i,j \} \) as \( (i,j) \) if \( A_{i,j} = AX_{\{ i,j \}} \), and as \( (j,i) \) if \( A_{j,i} = AX_{\{ i,j \}} \).

Now reduce \( (e'') \) modulo \( m^2 \). We get

\[ \sum_{\{ i,j \} \text{ good pair } \subset \{ 1, \ldots, r \}} X_{\{ i,j \}} F = m/m^2, \]
where $\tilde{X}_{\{i,j\}}$ is the image of $X_{\{i,j\}}$ in $m/m^2$. In other words, the family of $X_{\{i,j\}}$'s for $\{i, j\}$ running among good pairs, is a generating family of the cotangent space $m/m^2$ of $A$. Let $t$ be the dimension (over the residue field $F$) of this space. We can choose $t$ good pairs such that the corresponding $X_{\{i,j\}}$ are a basis over $F$ of $m/m^2$. Call the chosen pairs very good. So we have $t$ very good pairs, and as is well known (by the Hauptidealsatz), $t \geq \dim A$.

Consider the non-oriented graph $\Gamma$ whose vertices are the elements in $\{1, \ldots, r\}$, and whose edges are the very good pairs $\{i, j\}$. We claim that this graph is simply connected (that is, is a forest). For if there is a cycle $(i_0, i_1, \ldots, i_k)$, with $i_k = i_0$, in this graph, we can assume (changing the cycle to $(i_k, i_{k-1}, \ldots, i_0)$ if necessary) that $(i_1, i_0)$ is the natural orientation of the very good pair $(\{i_0, i_1\})$, so that $A_{i_1, i_0} = X_{\{i_0, i_1\}}A$. But by (b), we have

$$A_{i_1, i_2}A_{i_2, i_3} \cdots A_{i_{k-1}, i_0} \subset A_{i_1, i_0}.$$

The $A_{i_l, i_{l+1}}$, for $l = 1, \ldots, k - 1$ are either $A$ or of the form $X_{\{i,j\}}A$ for a very good pair $\{i, j\}$. So we obtain that $X_{\{i_0, i_1\}}$ divides in $A$ a product of $X_{\{i,j\}}$'s for other very good pairs $\{i, j\}$. Since the $X_{\{i,j\}}$ are irreducible, $X_{\{i_0, i_1\}}$ has to be equal, up to a unit in $A$, to $X_{\{i,j\}}$ for $\{i, j\}$ another very good pair. But this contradicts the fact that the family of $X_{\{i,j\}}$'s for $\{i, j\}$ very good is a basis of $m/m^2$, hence a linearly independent set.

Since $\Gamma$ is simply connected, a well-known and elementary result of graph theory asserts that its number of edges is strictly less that its number of vertices. So $t < r$, and therefore $\dim A < r$. $\square$

3.3. End of the proof of Theorem 1. We have now proved Theorem 1. Indeed, $A$ is the local ring at $x$ of an irreducible component of $X$ through $x$, so what the theorem states in its precise form is that if $\pi$ is not of type $(1, \ldots, 1)$, then $A$ is not factorial. Since $X$ is equidimensional of dimension $d - 1$, $\dim A = d - 1$. If $A$ was factorial, we would have $r > \dim A = d - 1$, so $r \geq d$. On the other hand, since $r$ is the number of irreducible factors of $\rho_\pi$ which has dimension $d$, we have $r \leq d$ with equality if and only if all factors are characters, that is $\pi$ is of type $(1, \ldots, 1)$. In all other cases, we get a contradiction proving that $A$ can not be factorial.

To prove the first assertion of the theorem, namely that $X$ is non smooth at $x$ if $\pi$ is not of type $(1, \ldots, 1)$, we use the fact that $X$ is smooth at $x$ if and only if it is irreducible in a neighborhood of $x$, and $O_x = A$ is regular, together with the Auslander-Buchsbaum theorem which states that every regular noetherian local ring is factorial.

4. Finer study of eigenvarieties at isobaric endoscopic anti-ordinary points of type $(n, 1)$

Let $\pi$ be as in Theorem 2. By assumption, $\rho_\pi$ satisfies all hypotheses (MF'), (NGD) and (REG) of Theorem 1. Therefore we can apply this theorem and all the tools developed for its proof. We adopt the same notations. As we recalled in §3.1, the residually multiplicity free pseudo-character $T$ define fractional ideals $A_{1,2}$
and $A_{2.1}$. To simplify notations, we set $B = A_{1.2}$ and $C = A_{2.1}$. We restate the properties (a) to (d) above, and add another property (e), which is [BC2, Theorem I.5.5]

(a) $BC \subset m$.
(b) Void here since $r = 2$
(c),(c') The ideal $BC$ is the total reducibility ideal of $T$, that is the smallest ideal $I$ of $A$ such that $T \otimes A/I$ is a sum of 2 pseudocharacters, and this ideal is $m$.
(d) $B$ and $C$ are isomorphic as $A$-modules.
(e) There is a natural injective linear map

$$
\iota_B : (B \otimes A/m)^* \to \Ext^1_{G_K}(\rho_1, \rho_2) \simeq H^1(G_K, \rho).
$$

There is a similar map

$$(C \otimes A/m)^* \to \Ext^1_{G_K}(\rho_2, \rho_1) = H^1(G_K, \rho^*)$$

Note that the idea of (e) (in this case $r = 2$) is due to Mazur and Wiles (see [MaW], and also [HP]). The subspace of $H^1(G_K, \rho)$ we denote by $H^1_2(G_K, \rho)$ is by definition the image of $\iota_B$.

Let us introduce the following notation: for any finite-type $A$-module $M$, the minimal number of elements of a family of generators of $M$ is equal, by Nakayama’s lemma, to the dimension of $M \otimes A/m$ over the field $A/m = \bar{\mathbb{Q}}_p$. We shall denote this number by $\text{gen}(M)$. Therefore, by (e),

$$
\dim H^1_2(G_K, \rho) = \text{gen}(B).
$$

**Proposition 4.1.** We have $\text{gen}(B) \geq n$ and $\text{gen}(C) \geq n$.

**Proof** — By (d), there exists an $x \in L^*$ such that $C = xB$. Let $L' = L[u]/(u^2 - x)$ if $x$ has no square root in $L$. Otherwise let $L' = L$ and denote by $u$ a square root of $x$ in $L$. Let $A'$ be the integral closure of $A$ in $L$. Since $A$ is excellent, $A'$ is of finite type over $A$, and the morphism $\text{Spec } A' \to \text{Spec } A$ is finite surjective. In particular, $\dim A' = \dim A = n$.

Now let $B' = BA'$, $C' = CA'$ and $m' = mA'$. From $BC = m$ it follows immediately that $B'C' = m'$, and from $C = xB$ that $C' = xB'$. Set $D = uB'$.

Then $D^2 = u^2B'B' = xB'B'C' = C'B' = m'$.

The fractional ideal $D$ is therefore a true ideal (since the square of any of its elements is in $m'$, so in $A'$, and $A'$ is normal).

The closed subschemes of $\text{Spec } A'$ defined by $D$ and by $m'$ have the same underlying topological space, and therefore the same dimension, which is 0 since the closed subscheme attached to $m'$ is the fiber of the closed point of $\text{Spec } A$, and therefore is finite. Since $A'$ has dimension $n$, the hauptidealsatz implies that $\text{gen}(D) \geq n$.

But clearly, $\text{gen}(C) = \text{gen}(B) \geq \text{gen}(B') = \text{gen}(D)$ and the proposition follows. \(\square\)

We deduce that $\dim H^1_2(G_K, \rho) \geq n$ which is Theorem 2 from the introduction.

We now turn to the proof of Theorem 3. We focus on the restriction $T|_{G_v}$ of the pseudocharacter $T$ to the local Galois group $G_v$. Its residual pseudocharacter
$T_{G_v} \otimes A/m$ is $(\rho_1)_{G_v} + \text{tr}(\rho_2)_{G_v}$. By our hypothesis (IRR) that $\rho_{|G_v}$ is irreducible, where $\rho = \rho_1^{-1} \otimes \rho_2$, this is a residually multiplicity-free pseudocharacter with two residual factors. Therefore the same analysis as before applies: there exists two fractional ideals $B_v$, $C_v$ of $A$ such that

1. $B_v \subset B$ and $C_v \subset C$. In particular $B_vC_v \subset m$.
2. $B_vC_v$ is the reducibility ideal of $T_{G_v}$, namely the smallest ideal $I$ of $A$ such that $T_{G_v} \otimes A/I$ is the sum of 2 pseudocharacters. In other words, $R_x = \text{Spec } A/BC$.
3. By Nakayama’s lemma, this means that at least one of the linear maps $B_v \otimes A/m$ has to be strict.

Indeed an extension of 1 by $\rho$ has good reduction (in the sense of Bloch-Kato) everywhere but perhaps at $v$ (cf. footnote 3), and if it lies in the kernel of the first map, it is trivial (and in particular has good reduction) at $v$; so such an extension lies in $H^1_{f}(G_K, \rho)$ which is zero according to Bloch-Kato’s conjecture.
We thus deduce that the second restriction map, (3), is not injective. This does not, by itself, lead to a contradiction. However, applying this reasoning to $\rho$ and many of its twists, we will quickly derive a contradiction. We argue as follows:

Let $\tau : G_K \to 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^*$ be a Hodge-Tate character of $G_K$ satisfying $\tau^\perp = \tau$, whose Hodge-Tate weight at $v$ is negative. Let $k \in \mathbb{Z}$. By [H], there exists an automorphic representation $\pi_k$ for $U(d)$ such that $\rho_{\pi_k} = \rho_1 \oplus \rho_2 \tau^k$. When $k \geq 0$, it is obvious that $\pi_k$ satisfies all conditions of Theorem 2. (When $k < 0$, the hypothesis (WEI) may and will eventually fail). Hence the same reasoning as above tells us that for $k \geq 0$, the map (3) for $\rho$ replaced by $\rho \tau^k$ is not injective. That is: for $k \geq 0$, the map

$$H^1(G_K, \rho^* \tau^{-k}) \to H^1(G_v, \rho^* \tau^{-k})$$

is not injective.

However, the non-injectivity of a map like (4) for all $k \geq 0$ implies its non-injectivity for all $k \in \mathbb{Z}$ (and even, all $k \in \mathbb{Z}_p$). This is because the function

$$d(k) = \dim \ker(H^1(G_K, \rho^* \tau^{-k}) \to H^1(G_v, \rho^* \tau^{-k}))$$

is lower semi-continuous (this is easy, but see e.g. [B3, page 15] for a proof). Hence for all $k \in \mathbb{Z}$, (4) is not injective.

But if we take $k \ll 0$, the Hodge-Tate weights of $\rho \tau^{-k}$ at $\bar{v}$ will be all $\geq 2$, hence any extension of 1 by $\rho \tau^{-k}$ will have good reduction at $\bar{v}$, as well as at all places not dividing $p$. If such an extension lies in the kernel of (4), it has good reduction everywhere, hence must be 0 by Bloch-Kato conjecture (remember that $\rho \tau^{-k}$ also has motivic weight 0). This contradicts the non-injectivity of (4) and concludes the proof of Theorem 3.

References


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