Extending tests for convergence of number series

Elijah Liflyand, Sergey Tikhonov, and Maria Zeltser

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Relax the monotonicity assumption for the sequence of terms of the series.
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- Such series are frequently called monotone series.

- Tests by Abel, Cauchy, de la Vallee Poussin, Dedekind, Dirichlet, du Bois Reymond, Ermakov, Leibniz, Maclaurin, Olivier, Sapogov, Schlömilch are related to monotonicity.
References

Books:

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In its initial form the Maclaurin-Cauchy integral test reads as follows:

Consider a non-negative monotone decreasing function $f$ defined on $[1, \infty)$. Then the series

$$\sum_{k=1}^{\infty} f(k)$$

converges if and only if the integral

$$\int_{1}^{\infty} f(t) \, dt$$

is finite. In particular, if the integral diverges, then the series diverges.
Ermakov’s test.

In its simplest form, it is given as follows. Let $f$ be a continuous (this is not necessary) non-negative monotone decreasing function for $t > 1$. If for $t$ large enough

$$f(e^t) e^t / f(t) \leq q < 1,$$

then series (2) converges, while if

$$f(e^t) e^t / f(t) \geq 1,$$

then series (2) diverges.

In fact, it is known that one can get a family of tests by replacing $e^t$ with a positive increasing function $\phi(t)$ satisfying certain properties.

We will extend such a more general assertion.
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- If \( \{a_k\} \) is a positive monotone decreasing null sequence, then the series (1) and

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\sum_{k=1}^{\infty} 2^k a_{2^k}
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converge or diverge simultaneously.
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- This assertion is a partial case of the following classical result due to Schlömilch.
- Let (1) be a series whose terms are positive and non-increasing,
and let

Tests

Let $u_0 < 1 < 2 < \ldots$ be a sequence of positive integers such that $\Delta u_k - 1 \leq C$. Then series (1) converges if and only if the series $\sum_{k=1}^{\infty} (u_k + 1 - u_k) a_u k$ converges.

In the theory of monotone series there are statements on the behavior of its terms. Such is Abel–Olivier's $k$th term test: Let ${a_k}$ be a positive monotone null sequence. If series (1) is convergent, then $ka_k$ is a null sequence.

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\[ u_0 < u_1 < u_2 < \ldots \] be a sequence of positive integers such that

\[ \frac{\Delta u_k}{\Delta u_{k-1}} \leq C. \]

Then series (1) converges if and only if the series

\[ \sum_{k=1}^{\infty} \Delta u_k a_{u_k} = \sum_{k=1}^{\infty} (u_{k+1} - u_k) a_{u_k} \]

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- $w_0 < w_1 < w_2 < \ldots$ be a sequence of positive integers such that

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In the theory of monotone series there are statements on the behavior of its terms. Such is Abel–Olivier’s $k$th term test:

- Let $\{a_k\}$ be a positive \textit{monotone null sequence}. If series (1) is convergent, then $ka_k$ is a null sequence.
Sapogov’s test reads as follows.

If \( \{b_k\} \) is a positive monotone increasing sequence, then the series

\[
\sum_{k=1}^{\infty} \left( 1 - \frac{b_k}{b_{k+1}} \right)
\]

as well as

\[
\sum_{k=1}^{\infty} \left( \frac{b_{k+1}}{b_k} - 1 \right)
\]

converges if the sequence \( \{b_k\} \) is bounded and diverges otherwise.
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- Let \( \{a_k\} \) and \( \{b_k\} \) be two sequences.
  
  (i) If \( \{a_k\} \in BV \), \( \{a_k\} \) is a null sequence, and the sequence of partial sums of \( \sum b_k \) is bounded, then the series \( \sum_{k=1}^{\infty} a_k b_k \) is convergent.

  (ii) If \( \{a_k\} \in BV \) and \( \sum b_k \) is convergent, then the series \( \sum_{k=1}^{\infty} a_k b_k \) is convergent.
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- **Two well-known and widely used corollaries of this test -**
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  - Dirichlet’s and Abel’s tests - involve **monotone** sequences.
Tests

The first one is as follows.

Let \( \{a_k\} \) be a monotone null sequence and \( \{b_k\} \) be a sequence such that the sequence of its partial sums is bounded. Then the series \( \sum_{k=1}^{\infty} a_k b_k \) is convergent.

One of corollaries of this test is the celebrated Leibniz test:

Let \( \{a_k\} \) be a monotone null sequence. Then the series \( \sum_{k=1}^{\infty} (-1)^k a_k \) is convergent.

Further, Abel's test reads as follows.

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Let us introduce certain classes of functions and sequences more general than monotone.

**Definition**

We call a non-negative null (that is, tending to zero at infinity) sequence \( \{a_k\} \) weak monotone, written WMS, if for some positive absolute constant \( C \) it satisfies

\[ a_k \leq Ca_n \]

for any \( k \in [n, 2n] \).

To introduce a counterpart for functions, we will assume all functions to be defined on \((0, \infty)\), locally of bounded variation, and vanishing at infinity.
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Setting $a_k = f(k)$ in this case, we obtain $\{a_k\} \in WMS$. 

Clearly, in these definitions $2n$ and $2x$ can be replaced by $\lfloor cn \rfloor$ (where $\lfloor a \rfloor$ denotes the integer part of $a$) and $cx$, respectively, with some $c > 1$ and another constant $C$. In some problems one should consider a smaller class than $WMS$. 

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- The class of **quasi-monotone** sequences,
- that is, $\{a_k\}$ such that there exists $\tau > 0$ so that $k^{-\tau} a_k \downarrow$,
- is a proper subclass of $GMS$.
- One of the simple basic properties of $GMS$ is $GMS \subsetneq WMS$. 

Classes of sequences and functions
We say that a non-negative function \( f \) is general monotone, \( GM \), if for all \( x \in (0, \infty) \)

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\int_x^{2x} |df(t)| \leq Cf(x).
\]
Classes of sequences and functions

A similar function class:

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And of course $GM \subset W$. 

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- And of course $GM \subset WM$. 

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An extension of the Maclaurin-Cauchy integral test reads as follows:

**Theorem**

Let $f$ be a WM function. Then the series

$$\sum_{k=1}^{\infty} f(k)$$

and integral

$$\int_1^{\infty} f(t) \, dt$$

converge or diverge simultaneously.
An extension of Ermakov’s test:

**Theorem**

Let $f$ be a WM function and let $\varphi(t)$ be a monotone increasing, positive function having a continuous derivative and satisfying $\varphi(t) > t$ for all $t$ large enough. If for $t$ large enough

$$\frac{f(\varphi(t))}{f(t)} \frac{\varphi'(t)}{f(t)} \leq q < 1,$$

then series (2) converges, while if

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  \]
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Extension of ”monotone” tests

An extension of Abel–Olivier’s $k$th term test:

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- Analyzing one Dvoretzky’s result and its proof,
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An extension of Abel–Olivier’s $k$th term test:

**Theorem**

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- Analyzing one Dvoretzky’s result and its proof,
- we obtain the following statement.

**Theorem**

If $\{a_k\}, \{b_k\} \in WMS$, and the series are convergent and divergent, respectively, then for every $M > 1$ there exist infinitely many $R_j$, $R_j \to \infty$, such that for all $k$ with $R_j \leq k \leq MR_j$, we have $a_k < b_k$. 
Extension of ”monotone” tests

Our next result is ”dual” to the above Schlömilch-type extensions.
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- We recall that the increasing sequence \( \{u_k\} \) is called **lacunary** if \( u_{k+1}/u_k \geq q > 1 \).
Extension of "monotone" tests

Our next result is "dual" to the above Schlömilch-type extensions.

- We recall that the increasing sequence $\{u_k\}$ is called lacunary if $u_{k+1}/u_k \geq q > 1$.
- A more general class of sequences is the one in which each sequence can be split into finitely-many lacunary sequences. In the latter case we will write $\{u_k\} \in \Lambda$.
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- This is true if and only if

$$\sum_{j=1}^{k} u_j \leq C u_k.$$
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  \]

- In different terms, \( \{u_k\} \in \Lambda \) is true if and only if there exists \( r \in \mathbb{N} \) such that

  \[
  \frac{u_{k+r}}{u_k} \geq q > 1, \quad k \in \mathbb{N}.
  \]
We denote \( \overline{\Delta} a_{u_k} := a_{u_k} - a_{u_{k+1}} \).

**Proposition.** Let \( \{ a_k \} \) be a non-negative WMS, and let a sequence \( \{ u_k \} \) be such that \( \{ u_k \} \in \Lambda \) and \( u_{k+1} = O(u_k) \).

Then the series (1),

\[
\sum_{k=1}^{\infty} u_k \overline{\Delta} a_{u_k},
\]

and

\[
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\]

converge or diverge simultaneously.
General monotonicity

It is also possible to get equiconvergence results for an important case $u_n = n$, where the lacunarity can no more help.
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- We proceed to a smaller class than $WMS$. Indeed, assuming general monotonicity of the sequences, we prove the following result.
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- We proceed to a smaller class than $WMS$. Indeed, assuming general monotonicity of the sequences, we prove the following result.
- **Proposition.** Let $\{a_k\}$ be a GMS. Then series (1) and

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\sum_k k|\Delta a_k|
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**Proposition.** Let $\{a_k\}$ be a GMS. Then series $\sum k|\Delta a_k|$ converge or diverge simultaneously.

**A similar result for functions:**

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It is also possible to get equiconvergence results for an important case \( u_n = n \), where the lacunarity can no more help.

- We proceed to a smaller class than \( WMS \). Indeed, assuming general monotonicity of the sequences, we prove the following result.

- **Proposition.** Let \( \{a_k\} \) be a GMS. Then series (1) and

\[
\sum_k k|\Delta a_k|
\]

converge or diverge simultaneously.

- A similar result for functions:

**Proposition.** Let \( f \) be a \( GM \) function. Then the integrals

\[
\int_1^\infty f(t) \, dt \quad \text{and} \quad \int_1^\infty t|df(t)|
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Negative type results

We now proceed to a group of tests where extending monotonicity to weak monotonicity fails in that or another sense. Sapogov type test cannot be true if \( \{b_k\} \) (as well as \( \{1/b_k\} \)) is WMS. A counterexample can be constructed against generalization of Abel's test. As for extending the Leibniz test, it cannot hold without additional assumption of the boundedness of variation: just take \( a_k = 1/\ln k \) everywhere except \( n = 2k \) where \( a_n = 2/\ln n \).
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Wider classes

We shall show that WMS is, in a sense, the widest class for which such tests are still valid.
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  \[ a_k \leq C \sum_{n=k/2}^{k} \frac{a_n}{n}, \]

- or a bit more general

  \[ a_k \leq C' \sum_{n=[k/c]}^{[ck]} \frac{a_n}{n} \]

  for some $c > 1$.  

The principle difference between \textit{WMS} and these classes is that the latter two allow certain amount of zero members, unlike \textit{WMS} that forbid even a single zero, i.e., $a_{n_0} = 0$ implies $a_n = 0$ for $n \geq n_0$. 

Putting zeros on certain positions, say $k = 2^n$, we easily construct a counterexample to show that the Cauchy condensation test cannot be valid for these classes nor its extensions. In a similar way, just letting a function $f$ to take non-zero values only close to integer points, one sees that the Maclaurin-Cauchy integral test may fail as well. 

In conclusion, note that \textit{WMS} is a subclass of the broadly used $\Delta_2$-class, 

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$$a_k = \begin{cases} 2^{-k}, & k \neq 2^n; \\ n^{-2}, & k = 2^n. \end{cases}$$
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- Obviously,

\[
\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n^2}
\]

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The main results may be summarized as the following statement.

\begin{align*}
\int_{1}^{\infty} f(t) \, dt; \\
\sum_{k} f(k) \\
\sum_{k} (u_{k+1} - u_{k}) f(u_{k}) \\
\sum_{k} u_{k} f(u_{k}) \\
\sum_{k} |f(u_{k+1}) - f(u_{k})| \\
\int_{1}^{\infty} \frac{|f(t)|}{t} \, df(t)
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- **Theorem.** Let $f(\cdot) \in WM$. Then the following series and integrals converge or diverge simultaneously:

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Example

**Besov space:**

For $k > r$, $\theta, r > 0$ and $p \geq 1$

$$\|f\|_{B^{r,p,\theta}} = \left(\frac{1}{t^{r\theta + 1}} \int_0^t \|\omega^{r\theta \omega^k(f; t)}\|_{L^p} dt \right)^{1/\theta}.$$ 

Equivalent form:

$$\|f\|_{B^{r,p,\theta}} = \left(\int_1^{\infty} \frac{1}{t^{r\theta + 1}} \|\omega^{r\theta \omega^k(f; 1/t)}\|_{L^p} dt \right)^{1/\theta}.$$
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Besov space:

- For \( k > r, \theta, r > 0 \) and \( p \geq 1 \)

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- convenient equivalent form:

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