Extending tests for convergence of number series

Elijah Liflyand, Sergey Tikhonov, and Maria Zeltser

September, 2012

Goals and History

Main Goal:

Relax the monotonicity assumption for the sequence of terms of the series.

Goals and History

Main Goal:

Relax the monotonicity assumption for the sequence of terms of the series.

- In various well-known tests for convergence/divergence of number series

$$\sum_{k=1}^{\infty} a_k, \tag{1}$$

with positive a_k, monotonicity of the sequence of these a_k is the basic assumption.

Goals and History

Main Goal:

Relax the monotonicity assumption for the sequence of terms of the series.

- In various well-known tests for convergence/divergence of number series

$$\sum_{k=1}^{\infty} a_k, \tag{1}$$

with positive a_k, monotonicity of the sequence of these a_k is the basic assumption.

- Such series are frequently called monotone series.

Goals and History

Main Goal:

Relax the monotonicity assumption for the sequence of terms of the series.

- In various well-known tests for convergence/divergence of number series

$$\sum_{k=1}^{\infty} a_k, \tag{1}$$

with positive a_k, monotonicity of the sequence of these a_k is the basic assumption.

- Such series are frequently called monotone series.
- Tests by Abel, Cauchy, de la Vallee Poussin, Dedekind, Dirichlet, du Bois Reymond, Ermakov, Leibniz, Maclaurin, Olivier, Sapogov, Schlömilch are related to monotonicity.

References

Books:

References

Books:

- T.J.I'a Bromwich, *Introduction to the Theory of Infinite Series*, MacMillan, New York, 1965.

References

Books:

- T.J.I'a Bromwich, *Introduction to the Theory of Infinite Series*, MacMillan, New York, 1965.
- K. Knopp, *Theory and Application of Infinite Series*, Blackie& Son Ltd., London-Glasgow, 1928.

References

Books:

- T.J.I'a Bromwich, *Introduction to the Theory of Infinite Series*, MacMillan, New York, 1965.
- K. Knopp, *Theory and Application of Infinite Series*, Blackie& Son Ltd., London-Glasgow, 1928.
- G.M. Fichtengolz, *Infinite Series: Rudiments*, Gordon and Breach, New York, 1970.

References

Books:

- T.J.I'a Bromwich, *Introduction to the Theory of Infinite Series*, MacMillan, New York, 1965.
- K. Knopp, *Theory and Application of Infinite Series*, Blackie& Son Ltd., London-Glasgow, 1928.
- G.M. Fichtengolz, *Infinite Series: Rudiments*, Gordon and Breach, New York, 1970.
- D.D. Bonar and M.J. Khoury, *Real Infinite Series*, MAA, Washington, DC, 2006.

Tests

In its initial form the Maclaurin-Cauchy integral test reads as follows:

Consider a non-negative monotone decreasing function f defined on $[1, \infty)$. Then the series

$$\sum_{k=1}^{\infty} f(k) \tag{2}$$

converges if and only if the integral

$$\int_1^{\infty} f(t)\, dt$$

is finite. In particular, if the integral diverges, then the series diverges.

Tests

Ermakov's test.

Tests

Ermakov's test.

- In its simplest form, it is given as follows.

Tests

Ermakov's test.

- In its simplest form, it is given as follows.
- Let f be a continuous (this is not necessary) non-negative monotone decreasing function for $t > 1$.

Tests

Ermakov's test.

- In its simplest form, it is given as follows.

- Let f be a continuous (this is not necessary) non-negative monotone decreasing function for $t > 1$.
- If for t large enough $f(e^t)e^t/f(t) \le q < 1$, then series (2) converges,

Tests

Ermakov's test.

- In its simplest form, it is given as follows.

- Let f be a continuous (this is not necessary) non-negative monotone decreasing function for $t > 1$.
- If for t large enough $f(e^t)e^t/f(t) \leq q < 1$, then series (2) converges,
- while if $f(e^t)e^t/f(t) \geq 1$, then series (2) diverges.

Tests

Ermakov's test.

- In its simplest form, it is given as follows.

- Let f be a continuous (this is not necessary) non-negative monotone decreasing function for $t > 1$.
- If for t large enough $f(e^t)e^t/f(t) \leq q < 1$, then series (2) converges,
- while if $f(e^t)e^t/f(t) \geq 1$, then series (2) diverges.
- In fact, it is known that one can get a family of tests by replacing e^t with a positive increasing function $\varphi(t)$ satisfying certain properties.

Tests

Ermakov's test.

- In its simplest form, it is given as follows.

- Let f be a continuous (this is not necessary) non-negative monotone decreasing function for $t > 1$.
- If for t large enough $f(e^t)e^t/f(t) \leq q < 1$, then series (2) converges,
- while if $f(e^t)e^t/f(t) \geq 1$, then series (2) diverges.
- In fact, it is known that one can get a family of tests by replacing e^t with a positive increasing function $\varphi(t)$ satisfying certain properties.
- We will extend such a more general assertion.

Tests

The well known Cauchy condensation test states that

Tests

The well known Cauchy condensation test states that

- *If* $\{a_k\}$ *is a positive monotone decreasing null sequence, then the series* (1) *and*

$$\sum_{k=1}^{\infty} 2^k a_{2^k}$$

converge or diverge simultaneously.

Tests

The well known Cauchy condensation test states that

- *If* $\{a_k\}$ *is a positive monotone decreasing null sequence, then the series* (1) *and*

$$\sum_{k=1}^{\infty} 2^k a_{2^k}$$

converge or diverge simultaneously.

- This assertion is a partial case of the following classical result due to Schlömilch.

Tests

The well known Cauchy condensation test states that

- *If* $\{a_k\}$ *is a positive monotone decreasing null sequence, then the series* (1) *and*

$$\sum_{k=1}^{\infty} 2^k a_{2^k}$$

converge or diverge simultaneously.

- This assertion is a partial case of the following classical result due to Schlömilch.
- *Let* (1) *be a series whose terms are positive and non-increasing,*

Tests

and let

Tests

and let

- $u_0 < u_1 < u_2 < \ldots$ *be a sequence of positive integers such that*

$$\frac{\Delta u_k}{\Delta u_{k-1}} \le C.$$

Then series (1) *converges if and only if the series*

$$\sum_{k=1}^{\infty} \Delta u_k a_{u_k} = \sum_{k=1}^{\infty} (u_{k+1} - u_k) a_{u_k}$$

converges.

Tests

and let

- $u_0 < u_1 < u_2 < \dots$ *be a sequence of positive integers such that*

$$\frac{\Delta u_k}{\Delta u_{k-1}} \le C.$$

Then series (1) *converges if and only if the series*

$$\sum_{k=1}^{\infty} \Delta u_k a_{u_k} = \sum_{k=1}^{\infty} (u_{k+1} - u_k) a_{u_k}$$

converges.

- In the theory of monotone series there are statements on the behavior of its terms. Such is Abel–Olivier's kth term test:

and let

- $u_0 < u_1 < u_2 < \dots$ *be a sequence of positive integers such that*

$$\frac{\Delta u_k}{\Delta u_{k-1}} \le C.$$

Then series (1) *converges if and only if the series*

$$\sum_{k=1}^{\infty} \Delta u_k a_{u_k} = \sum_{k=1}^{\infty} (u_{k+1} - u_k) a_{u_k}$$

converges.

- In the theory of monotone series there are statements on the behavior of its terms. Such is Abel–Olivier's kth term test:
- *Let* $\{a_k\}$ *be a positive monotone null sequence. If series* (1) *is convergent, then* ka_k *is a null sequence.*

Tests

Sapogov's test reads as follows.

If $\{b_k\}$ *is a positive monotone increasing sequence, then the series*

$$\sum_{k=1}^{\infty}(1 - \frac{b_k}{b_{k+1}})$$

as well as

$$\sum_{k=1}^{\infty}(\frac{b_{k+1}}{b_k} - 1)$$

converges if the sequence $\{b_k\}$ *is bounded and diverges otherwise.*

Tests

The tests of Dedekind and of du Bois Reymond are united as the next assertion.

Tests

The tests of Dedekind and of du Bois Reymond are united as the next assertion.

- Sequence of bounded variation – $\{a_k\} \in BV$: $\quad \sum |a_{k+1} - a_k| < \infty$.

Tests

The tests of Dedekind and of du Bois Reymond are united as the next assertion.

- Sequence of bounded variation – $\{a_k\} \in BV$: $\quad \sum |a_{k+1} - a_k| < \infty$.
- *Let $\{a_k\}$ and $\{b_k\}$ be two sequences.*
  - (i) *If $\{a_k\} \in BV$, $\{a_k\}$ is a null sequence, and the sequence of partial sums of $\sum\limits_k b_k$ is bounded, then the series $\sum\limits_{k=1}^{\infty} a_k b_k$ is convergent.*
  - (ii) *If $\{a_k\} \in BV$ and $\sum\limits_k b_k$ is convergent, then the series $\sum\limits_{k=1}^{\infty} a_k b_k$ is convergent.*

Tests

The tests of Dedekind and of du Bois Reymond are united as the next assertion.

- Sequence of bounded variation – $\{a_k\} \in BV$: $\quad \sum |a_{k+1} - a_k| < \infty$.
- *Let $\{a_k\}$ and $\{b_k\}$ be two sequences.*
  - (i) *If $\{a_k\} \in BV$, $\{a_k\}$ is a null sequence, and the sequence of partial sums of $\sum_k b_k$ is bounded, then the series $\sum_{k=1}^{\infty} a_k b_k$ is convergent.*
  - (ii) *If $\{a_k\} \in BV$ and $\sum_k b_k$ is convergent, then the series $\sum_{k=1}^{\infty} a_k b_k$ is convergent.*
- Two well-known and widely used corollaries of this test -

Tests

The tests of Dedekind and of du Bois Reymond are united as the next assertion.

- Sequence of bounded variation – $\{a_k\} \in BV$: $\sum |a_{k+1} - a_k| < \infty$.
- *Let* $\{a_k\}$ *and* $\{b_k\}$ *be two sequences.*
  - (i) *If* $\{a_k\} \in BV$, $\{a_k\}$ *is a null sequence, and the sequence of partial sums of* $\sum_k b_k$ *is bounded, then the series* $\sum_{k=1}^{\infty} a_k b_k$ *is convergent.*
  - (ii) *If* $\{a_k\} \in BV$ *and* $\sum_k b_k$ *is convergent, then the series* $\sum_{k=1}^{\infty} a_k b_k$ *is convergent.*
- Two well-known and widely used corollaries of this test -
- Dirichlet's and Abel's tests - involve monotone sequences.

Tests

The first one is as follows.

Tests

The first one is as follows.

- *Let* $\{a_k\}$ *be a* *monotone* *null sequence and* $\{b_k\}$ *be a sequence such that the sequence of its partial sums is bounded. Then the series* $\sum_{k=1}^{\infty} a_k b_k$ *is convergent.*

Tests

The first one is as follows.

- *Let* $\{a_k\}$ *be a monotone null sequence and* $\{b_k\}$ *be a sequence such that the sequence of its partial sums is bounded. Then the series* $\sum_{k=1}^{\infty} a_k b_k$ *is convergent.*
- One of corollaries of this test is the celebrated Leibniz test:
  Let $\{a_k\}$ *be a monotone null sequence. Then the series* $\sum_{k=1}^{\infty} (-1)^k a_k$ *is convergent.*

Tests

The first one is as follows.

- *Let* $\{a_k\}$ *be a monotone null sequence and* $\{b_k\}$ *be a sequence such that the sequence of its partial sums is bounded. Then the series* $\sum_{k=1}^{\infty} a_k b_k$ *is convergent.*
- One of corollaries of this test is the celebrated Leibniz test: *Let* $\{a_k\}$ *be a monotone null sequence. Then the series* $\sum_{k=1}^{\infty} (-1)^k a_k$ *is convergent.*
- Further, Abel's test reads as follows. *Let* $\{a_k\}$ *be a bounded monotone sequence and* $\sum_k b_k$ *a convergent series. Then the series* $\sum_{k=1}^{\infty} a_k b_k$ *is convergent.*

Classes of sequences and functions

Let us introduce certain classes of functions and sequences more general than monotone.

Classes of sequences and functions

Let us introduce certain classes of functions and sequences more general than monotone.

-

Definition

We call a non-negative null (that is, tending to zero at infinity) sequence $\{a_k\}$ *weak monotone*, written WMS, if for some positive absolute constant C it satisfies

$$a_k \le C a_n \quad \text{for any} \quad k \in [n, 2n].$$

Classes of sequences and functions

Let us introduce certain classes of functions and sequences more general than monotone.

-

Definition

We call a non-negative null (that is, tending to zero at infinity) sequence $\{a_k\}$ *weak monotone*, written WMS, if for some positive absolute constant C it satisfies

$$a_k \le C a_n \quad \text{for any} \quad k \in [n, 2n].$$

- To introduce a counterpart for functions, we will assume all functions to be defined on $(0, \infty)$, locally of bounded variation, and vanishing at infinity.

Classes of sequences and functions

Definition

We say that a non-negative function f defined on $(0, \infty)$, is *weak monotone*, written WM, if

$$f(t) \le C f(x) \quad \text{for any} \quad t \in [x, 2x].$$

Classes of sequences and functions

Definition

We say that a non-negative function f defined on $(0, \infty)$, is *weak monotone*, written WM, if

$$f(t) \leq Cf(x) \quad \text{for any} \quad t \in [x, 2x].$$

- Setting $a_k = f(k)$ in this case, we obtain $\{a_k\} \in WMS$.

Classes of sequences and functions

Definition

We say that a non-negative function f defined on $(0, \infty)$, is *weak monotone*, written WM, if

$$f(t) \le Cf(x) \quad \text{for any} \quad t \in [x, 2x].$$

- Setting $a_k = f(k)$ in this case, we obtain $\{a_k\} \in WMS$.
- Clearly, in these definitions $2n$ and $2x$ can be replaced by $[cn]$ (where $[a]$ denotes the integer part of a) and cx, respectively, with some $c > 1$ and another constant C.

Classes of sequences and functions

Definition

We say that a non-negative function f defined on $(0, \infty)$, is *weak monotone*, written WM, if

$$f(t) \leq Cf(x) \quad \text{for any} \quad t \in [x, 2x].$$

- Setting $a_k = f(k)$ in this case, we obtain $\{a_k\} \in WMS$.
- Clearly, in these definitions $2n$ and $2x$ can be replaced by $[cn]$ (where $[a]$ denotes the integer part of a) and cx, respectively, with some $c > 1$ and another constant C.
- In some problems one should consider a smaller class than WMS.

Classes of sequences and functions

Denote $\Delta a_k = a_{k+1} - a_k$.

Classes of sequences and functions

Denote $\Delta a_k = a_{k+1} - a_k$.

- **Definition.** A positive null sequence $\{a_k\}$ is called general monotone if it satisfies

$$\sum_{k=n}^{2n} |\Delta a_k| \le C a_n,$$

for any n and some absolute constant C.

Classes of sequences and functions

Denote $\Delta a_k = a_{k+1} - a_k$.

- **Definition.** A positive null sequence $\{a_k\}$ is called general monotone if it satisfies

$$\sum_{k=n}^{2n} |\Delta a_k| \leq C a_n,$$

  for any n and some absolute constant C.
- The class of quasi-monotone sequences,

Denote $\Delta a_k = a_{k+1} - a_k$.

- **Definition.** A positive null sequence $\{a_k\}$ is called general monotone if it satisfies

$$\sum_{k=n}^{2n} |\Delta a_k| \le C a_n,$$

for any n and some absolute constant C.

- The class of quasi-monotone sequences,
- that is, $\{a_k\}$ such that there exists $\tau > 0$ so that $k^{-\tau} a_k \downarrow$,

Classes of sequences and functions

Denote $\Delta a_k = a_{k+1} - a_k$.

- **Definition.** A positive null sequence $\{a_k\}$ is called general monotone if it satisfies

$$\sum_{k=n}^{2n} |\Delta a_k| \le C a_n,$$

  for any n and some absolute constant C.
- The class of quasi-monotone sequences,
- that is, $\{a_k\}$ such that there exists $\tau > 0$ so that $k^{-\tau} a_k \downarrow$,
- is a proper subclass of GMS.

Classes of sequences and functions

Denote $\Delta a_k = a_{k+1} - a_k$.

- **Definition.** A positive null sequence $\{a_k\}$ is called general monotone if it satisfies

$$\sum_{k=n}^{2n} |\Delta a_k| \le C a_n,$$

for any n and some absolute constant C.

- The class of quasi-monotone sequences,
- that is, $\{a_k\}$ such that there exists $\tau > 0$ so that $k^{-\tau} a_k \downarrow$,
- is a proper subclass of GMS.
- One of the simple basic properties of GMS is $GMS \subsetneq WMS$.

A similar function class:

Definition

We say that a non-negative function f is general monotone, GM, if for all $x \in (0, \infty)$

$$\int_x^{2x} |df(t)| \le Cf(x).$$

Classes of sequences and functions

A similar function class:

Definition

We say that a non-negative function f is general monotone, GM, if for all $x \in (0, \infty)$

$$\int_x^{2x} |df(t)| \le Cf(x).$$

- We just remark that if $f(\cdot)$ is a GM function,

Classes of sequences and functions

A similar function class:

Definition

We say that a non-negative function f is general monotone, GM, if for all $x \in (0, \infty)$

$$\int_x^{2x} |df(t)| \le Cf(x).$$

- We just remark that if $f(\cdot)$ is a GM function,
- then for $a_k = f(k)$ there holds $\{a_k\} \in GMS$.

Classes of sequences and functions

A similar function class:

Definition

We say that a non-negative function f is general monotone, GM, if for all $x \in (0, \infty)$

$$\int_x^{2x} |df(t)| \leq Cf(x).$$

- We just remark that if $f(\cdot)$ is a GM function,
- then for $a_k = f(k)$ there holds $\{a_k\} \in GMS$.
- And of course $GM \subsetneq WM$.

Extension of "monotone" tests

An extension of the Maclaurin-Cauchy integral test reads as follows:

Theorem

Let f be a WM function. Then the series

$$\sum_{k=1}^{\infty} f(k)$$

and integral

$$\int_1^{\infty} f(t)\,dt$$

converge or diverge simultaneously.

Extension of "monotone" tests

An extension of Ermakov's test:

Theorem

Let f be a WM function and let $\varphi(t)$ be a monotone increasing, positive function having a continuous derivative and satisfying $\varphi(t) > t$ for all t large enough. If for t large enough

$$\frac{f(\varphi(t))\varphi'(t)}{f(t)} \leq q < 1,$$

then series (2) converges, while if

$$\frac{f(\varphi(t))\varphi'(t)}{f(t)} \geq 1,$$

then series (2) diverges.

Extension of "monotone" tests

Theorem

Let $\{u_k\}$ *be an increasing sequence of positive numbers such that* $u_{k+1} = O(u_k)$ *and* $u_k \to \infty$.

Extension of "monotone" tests

Theorem

Let $\{u_k\}$ *be an increasing sequence of positive numbers such that* $u_{k+1} = O(u_k)$ *and* $u_k \to \infty$.

- Let f be a WM function. Then both series

$$\sum_{k=1}^{\infty} f(u_k)\Delta u_k \quad and \quad \sum_{k=1}^{\infty} f(u_{k+1})\Delta u_k$$

converge (or diverge) with $\int_1^\infty f(t)\,dt$.

Extension of "monotone" tests

Theorem

Let $\{u_k\}$ be an increasing sequence of positive numbers such that $u_{k+1} = O(u_k)$ and $u_k \to \infty$.

- Let f be a WM function. Then both series

$$\sum_{k=1}^{\infty} f(u_k)\Delta u_k \quad \text{and} \quad \sum_{k=1}^{\infty} f(u_{k+1})\Delta u_k$$

converge (or diverge) with $\int_1^{\infty} f(t)\,dt$.

- Let $\{a_k\}$ be a WMS. Then (1) converges if and only if the series

$$\sum_{k=1}^{\infty} \Delta u_k a_{u_k} = \sum_{k=1}^{\infty} (u_{k+1} - u_k) a_{u_k}$$

converges.

Extension of "monotone" tests

An extension of Abel–Olivier's kth term test:

Theorem

Let $\{a_k\}$ be a WMS. If series (1) converges, then ka_k is a null sequence.

Extension of "monotone" tests

An extension of Abel–Olivier's kth term test:

Theorem

Let $\{a_k\}$ be a WMS. If series (1) converges, then ka_k is a null sequence.

- Analyzing one Dvoretzky's result and its proof,

Extension of "monotone" tests

An extension of Abel–Olivier's kth term test:

Theorem

Let $\{a_k\}$ be a WMS. If series (1) converges, then ka_k is a null sequence.

- Analyzing one Dvoretzky's result and its proof,
- we obtain the following statement.

Theorem

If $\{a_k\}, \{b_k\} \in WMS$, and the series are convergent and divergent, respectively, then for every $M > 1$ there exist infinitely many R_j, $R_j \to \infty$, such that for all k with $R_j \le k \le MR_j$, we have $a_k < b_k$.

Extension of "monotone" tests

Our next result is "dual" to the above Schlömilch-type extensions.

Extension of "monotone" tests

Our next result is "dual" to the above Schlömilch-type extensions.

- We recall that the increasing sequence $\{u_k\}$ is called lacunary if $u_{k+1}/u_k \geq q > 1$.

Extension of "monotone" tests

Our next result is "dual" to the above Schlömilch-type extensions.

- We recall that the increasing sequence $\{u_k\}$ is called lacunary if $u_{k+1}/u_k \geq q > 1$.
- A more general class of sequences is the one in which each sequence can be split into finitely-many lacunary sequences. In the latter case we will write $\{u_k\} \in \Lambda$.

Extension of "monotone" tests

Our next result is "dual" to the above Schlömilch-type extensions.

- We recall that the increasing sequence $\{u_k\}$ is called lacunary if $u_{k+1}/u_k \ge q > 1$.
- A more general class of sequences is the one in which each sequence can be split into finitely-many lacunary sequences. In the latter case we will write $\{u_k\} \in \Lambda$.
- This is true if and only if

$$\sum_{j=1}^{k} u_j \le C u_k.$$

Extension of "monotone" tests

Our next result is "dual" to the above Schlömilch-type extensions.

- We recall that the increasing sequence $\{u_k\}$ is called lacunary if $u_{k+1}/u_k \geq q > 1$.
- A more general class of sequences is the one in which each sequence can be split into finitely-many lacunary sequences. In the latter case we will write $\{u_k\} \in \Lambda$.
- This is true if and only if

$$\sum_{j=1}^{k} u_j \leq C u_k.$$

- In different terms, $\{u_k\} \in \Lambda$ is true if and only if there exists $r \in \mathbb{N}$ such that

$$\frac{u_{k+r}}{u_k} \geq q > 1, \qquad k \in \mathbb{N}.$$

Extension of "monotone" tests

We denote $\bar\Delta a_{u_k} := a_{u_k} - a_{u_{k+1}}$.

Proposition. *Let* $\{a_k\}$ *be a non-negative WMS, and let a sequence* $\{u_k\}$ *be such that* $\{u_k\} \in \Lambda$ *and* $u_{k+1} = O(u_k)$.
Then the series (1),

$$\sum_{k=1}^{\infty} u_k |\bar\Delta a_{u_k}|,$$

and

$$\sum_{k=1}^{\infty} u_k a_{u_k}$$

converge or diverge simultaneously.

General monotonicity

It is also possible to get equiconvergence results for an important case $u_n = n$, where the lacunarity can no more help.

General monotonicity

It is also possible to get equiconvergence results for an important case $u_n = n$, where the lacunarity can no more help.

- We proceed to a smaller class than WMS. Indeed, assuming general monotonicity of the sequences, we prove the following result.

General monotonicity

It is also possible to get equiconvergence results for an important case $u_n = n$, where the lacunarity can no more help.

- We proceed to a smaller class than WMS. Indeed, assuming general monotonicity of the sequences, we prove the following result.
- **Proposition.** Let $\{a_k\}$ be a GMS. Then series (1) and

$$\sum_k k|\Delta a_k|$$

converge or diverge simultaneously.

General monotonicity

It is also possible to get equiconvergence results for an important case $u_n = n$, where the lacunarity can no more help.

- We proceed to a smaller class than WMS. Indeed, assuming general monotonicity of the sequences, we prove the following result.
- **Proposition.** Let $\{a_k\}$ be a GMS. Then series (1) and

$$\sum_k k|\Delta a_k|$$

  converge or diverge simultaneously.
- A similar result for functions:

General monotonicity

It is also possible to get equiconvergence results for an important case $u_n = n$, where the lacunarity can no more help.

- We proceed to a smaller class than WMS. Indeed, assuming general monotonicity of the sequences, we prove the following result.
- **Proposition.** Let $\{a_k\}$ be a GMS. Then series (1) and

$$\sum_k k|\Delta a_k|$$

  converge or diverge simultaneously.
- A similar result for functions:
- **Proposition.** Let f be a GM function. Then the integrals $\int_1^\infty f(t)\,dt$ and $\int_1^\infty t|df(t)|$ converge or diverge simultaneously.

Negative type results

We now proceed to a group of tests where

Negative type results

We now proceed to a group of tests where

- extending monotonicity to weak monotonicity fails in that or another sense.

Negative type results

We now proceed to a group of tests where

- extending monotonicity to weak monotonicity fails in that or another sense.
- Sapogov type test cannot be true if $\{b_k\}$ (as well as $\{1/b_k\}$) is WMS.

Negative type results

We now proceed to a group of tests where

- extending monotonicity to weak monotonicity fails in that or another sense.
- Sapogov type test cannot be true if $\{b_k\}$ (as well as $\{1/b_k\}$) is WMS.
- A counterexample can be constructed against generalization of Abel's test.

Negative type results

We now proceed to a group of tests where

- extending monotonicity to weak monotonicity fails in that or another sense.
- Sapogov type test cannot be true if $\{b_k\}$ (as well as $\{1/b_k\}$) is WMS.
- A counterexample can be constructed against generalization of Abel's test.
- As for extending the Leibniz test, it cannot hold without additional assumption of the boundedness of variation:

Negative type results

We now proceed to a group of tests where

- extending monotonicity to weak monotonicity fails in that or another sense.
- Sapogov type test cannot be true if $\{b_k\}$ (as well as $\{1/b_k\}$) is WMS.
- A counterexample can be constructed against generalization of Abel's test.
- As for extending the Leibniz test, it cannot hold without additional assumption of the boundedness of variation:
- just take $a_k = 1/\ln k$ everywhere except $n = 2^k$ where $a_n = 2/\ln n$.

Wider classes

We shall show that WMS is, in a sense, the widest class for which such tests are still valid.

Wider classes

We shall show that WMS is, in a sense, the widest class for which such tests are still valid.

- An immediate natural extension of WMS is the class defined by

$$a_k \leq C \sum_{n=k/2}^{k} \frac{a_n}{n},$$

Wider classes

We shall show that WMS is, in a sense, the widest class for which such tests are still valid.

- An immediate natural extension of WMS is the class defined by

$$a_k \leq C \sum_{n=k/2}^{k} \frac{a_n}{n},$$

- or a bit more general

$$a_k \leq C \sum_{n=[k/c]}^{[ck]} \frac{a_n}{n}$$

  for some $c > 1$.

Wider classes

The principle difference between WMS and these classes is that the latter two allow certain amount of zero members, unlike WMS that forbid even a single zero, i.e., $a_{n_0} = 0$ implies $a_n = 0$ for $n \geq n_0$.

Wider classes

The principle difference between WMS and these classes is that the latter two allow certain amount of zero members, unlike WMS that forbid even a single zero, i.e., $a_{n_0} = 0$ implies $a_n = 0$ for $n \geq n_0$.

- Putting zeros on certain positions, say $k = 2^n$, we easily construct a counterexample to show that the Cauchy condensation test cannot be valid for these classes nor its extensions.

Wider classes

The principle difference between WMS and these classes is that the latter two allow certain amount of zero members, unlike WMS that forbid even a single zero, i.e., $a_{n_0} = 0$ implies $a_n = 0$ for $n \geq n_0$.

- Putting zeros on certain positions, say $k = 2^n$, we easily construct a counterexample to show that the Cauchy condensation test cannot be valid for these classes nor its extensions.
- In a similar way, just letting a function f to take non-zero values only close to integer points, one sees that the Maclaurin-Cauchy integral test may fail as well.

Wider classes

The principle difference between WMS and these classes is that the latter two allow certain amount of zero members, unlike WMS that forbid even a single zero, i.e., $a_{n_0} = 0$ implies $a_n = 0$ for $n \geq n_0$.

- Putting zeros on certain positions, say $k = 2^n$, we easily construct a counterexample to show that the Cauchy condensation test cannot be valid for these classes nor its extensions.
- In a similar way, just letting a function f to take non-zero values only close to integer points, one sees that the Maclaurin-Cauchy integral test may fail as well.
- In conclusion, note that WMS is a subclass of the broadly used Δ_2-class,

Wider classes

that is, the one

Wider classes

that is, the one

- for which the doubling condition $a_{2k} \le C a_k$ holds for each $k \in \mathbb{N}$.

Wider classes

that is, the one

- for which the doubling condition $a_{2k} \leq Ca_k$ holds for each $k \in \mathbb{N}$.
- We observe that assuming the doubling condition by no means can guarantee the above tests to be extended.

Wider classes

that is, the one

- for which the doubling condition $a_{2k} \le Ca_k$ holds for each $k \in \mathbb{N}$.
- We observe that assuming the doubling condition by no means can guarantee the above tests to be extended.
- To illustrate this, the easiest way is to consider Cauchy's condensation test.

Wider classes

that is, the one

- for which the doubling condition $a_{2k} \le C a_k$ holds for each $k \in \mathbb{N}$.
- We observe that assuming the doubling condition by no means can guarantee the above tests to be extended.
- To illustrate this, the easiest way is to consider Cauchy's condensation test.
- Take $\{a_k\}$ such that

$$a_k = \begin{cases} 2^{-k}, & k \neq 2^n; \\ n^{-2}, & k = 2^n. \end{cases}$$

Wider classes

that is, the one

- for which the doubling condition $a_{2k} \leq Ca_k$ holds for each $k \in \mathbb{N}$.
- We observe that assuming the doubling condition by no means can guarantee the above tests to be extended.
- To illustrate this, the easiest way is to consider Cauchy's condensation test.
- Take $\{a_k\}$ such that

$$a_k = \begin{cases} 2^{-k}, & k \neq 2^n; \\ n^{-2}, & k = 2^n. \end{cases}$$

- This sequence is doubling but not WMS.

Wider classes

that is, the one

- for which the doubling condition $a_{2k} \le C a_k$ holds for each $k \in \mathbb{N}$.
- We observe that assuming the doubling condition by no means can guarantee the above tests to be extended.
- To illustrate this, the easiest way is to consider Cauchy's condensation test.
- Take $\{a_k\}$ such that

$$a_k = \begin{cases} 2^{-k}, & k \neq 2^n; \\ n^{-2}, & k = 2^n. \end{cases}$$

- This sequence is doubling but not WMS.
- Obviously,

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n^2}$$

diverges

Overview

The main results may be summarized as the following statement.

Overview

The main results may be summarized as the following statement.

- **Theorem.** Let $f(\cdot) \in WM$. Then the following series and integrals converge or diverge simultaneously:

Overview

The main results may be summarized as the following statement.

- **Theorem.** Let $f(\cdot) \in WM$. Then the following series and integrals converge or diverge simultaneously:
-

$$\int_1^\infty f(t)\,dt; \qquad \sum_k f(k);$$

Overview

The main results may be summarized as the following statement.

- **Theorem.** Let $f(\cdot) \in WM$. Then the following series and integrals converge or diverge simultaneously:
- $$\int_1^\infty f(t)\,dt; \qquad\qquad \sum_k f(k);$$
- $$\sum_k (u_{k+1} - u_k) f(u_k), \qquad \text{provided} \quad u_k \uparrow, \ \ u_{k+1} = O(u_k);$$

Overview

The main results may be summarized as the following statement.

- **Theorem.** Let $f(\cdot) \in WM$. Then the following series and integrals converge or diverge simultaneously:
- $$\int_1^\infty f(t)\,dt; \qquad \sum_k f(k);$$
- $$\sum_k (u_{k+1} - u_k) f(u_k), \qquad \text{provided} \quad u_k \uparrow, \;\; u_{k+1} = O(u_k);$$
- $$\sum_k u_k f(u_k), \sum_k u_k |f(u_{k+1}) - f(u_k)|, \text{ with } u_k \text{ lacunary, } u_{k+1} = O(u_k);$$

Overview

The main results may be summarized as the following statement.

- **Theorem.** Let $f(\cdot) \in WM$. Then the following series and integrals converge or diverge simultaneously:
- $$\int_1^\infty f(t)\,dt; \qquad \sum_k f(k);$$
- $$\sum_k (u_{k+1} - u_k) f(u_k), \qquad \text{provided} \quad u_k \uparrow, \ \ u_{k+1} = O(u_k);$$
- $$\sum_k u_k f(u_k), \sum_k u_k |f(u_{k+1}) - f(u_k)|, \text{ with } u_k \text{ lacunary, } u_{k+1} = O(u_k);$$
- $$\sum_k k|f(k+1) - f(k)|, \qquad \int_1^\infty t\,|df(t)|, \qquad \text{provided} \quad f(\cdot) \ \text{ is } GM.$$

Example

Besov space:

Example

Besov space:

- For $k > r$, $\theta, r > 0$ and $p \geq 1$

$$\|f\|_{B^r_{p,\theta}} = \left(\int\limits_0^1 t^{-r\theta-1} \omega_k^\theta(f;t)_p \frac{dt}{t} \right)^{1/\theta}.$$

Example

Besov space:

- For $k > r$, $\theta, r > 0$ and $p \geq 1$

$$\|f\|_{B^r_{p,\theta}} = \left(\int_0^1 t^{-r\theta-1} \omega_k^\theta(f;t)_p \frac{dt}{t} \right)^{1/\theta}.$$

- Equivalent form:

$$\|f\|_{B^r_{p,\theta}} = \left(\int_1^\infty t^{r\theta+1} \omega_k^\theta(f;1/t)_p \frac{dt}{t} \right)^{1/\theta}.$$

Example

Since $\omega_k(f;t)_p \ \uparrow$, the function under the integral sign is WM.

Example

Since $\omega_k(f;t)_p \uparrow$, the function under the integral sign is WM.

- Applying generalized Maclaurin-Cauchy and Cauchy condensation tests, we obtain

Example

Since $\omega_k(f;t)_p \ \uparrow$, the function under the integral sign is WM.

- Applying generalized Maclaurin-Cauchy and Cauchy condensation tests, we obtain
- convenient equivalent form:

$$\left(\sum_{\nu=0}^{\infty}\left[2^{r\nu}\omega_k(f;2^{-\nu})_p\right]^{\theta}\right)^{1/\theta}.$$