

Math 111a, Fall 2008, Homework # 1

Topological and Metric Spaces

1. (Lang, p. 22.) Prove that for any two sets A, B in a topological space,

$$\overline{A \cup B} = \overline{A} \cup \overline{B} \text{ and } \overline{A \cap B} \subset \overline{A} \cap \overline{B}.$$

Find an example showing that equality does not necessarily hold in the second formula.

2. Prove that for any subset A in a topological space, $\partial(\overline{A}) \subset \partial A$ and $\partial(\text{Int } A) \subset \partial A$. Give an example where all these three sets are different.

3. Prove that the set of 2008-th powers of rational numbers is dense in the set of all non-negative real numbers.

4. (Lang, Problem 6 on p. 45) Let A be a subset of a metric space X . For each $x \in X$ define $d(x, A) = \inf_{y \in A} d(x, y)$ (the distance from x to A). Show that $d(\cdot, A)$ is continuous, and $d(x, A) = 0$ if and only if $x \in \overline{A}$.

5. (Lang, Problem 13(a) on p. 47) Show that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\} \subset X \times X$ is closed in $X \times X$.

6. (Lang, Problem 14(c) on p. 47) Let X, Y be topological spaces, $A \subset X$, $B \subset Y$. Show that $\partial(A \times B) = (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B)$.

7. (Lang, Problem 17(a) on p. 47) Prove that any open covering of a separable topological space X (that is, a family of open subsets U_i , $i \in I$, of X such that $\cup_{i \in I} U_i = X$) has a countable subcovering (that is, there exists a countable subset J of I such that $\cup_{i \in J} U_i = X$).