Math 111a, Fall 2015, Homework # 1

Topological and Metric Spaces

1. (Lang, p. 22.) If A is a subset of a topological space, define the *closure* $\overline{A}$ of A to be the intersection of all the closed sets containing A. Prove that for any two sets A, B in a topological space,

$$\overline{A \cup B} = \overline{A} \cup \overline{B} \text{ and } \overline{A \cap B} \subset \overline{A} \cap \overline{B}\,.$$

Find an example showing that equality does not necessarily hold in the second formula.

2. Define also the *interior* A^o of A to be the union of all the open sets contained in A, and the *boundary* ∂A of A to be the "closure minus interior", $\partial A = \overline{A} \setminus A^o$. Prove that for any subset A in a topological space, $\partial(\overline{A}) \subset \partial A$ and $\partial(A^o) \subset \partial A$. Give an example where all these three sets are different.

3. (Lang, Problem 5(c) on p. 45) Given a metric space (X, d), consider a map $x \mapsto g_x$ from X to the normed space of bounded continuous functions on X with the supremum norm, given by $g_x(y) = d(x, y) - d(a, y)$, where a is some fixed point of X. Show that this map is distance-preserving. [Quoting Lang: *thus one need not fuss too much with abstract metric spaces.*]

4. (Lang, Problem 6 on p. 45) Let A be a subset of a metric space X. For each $x \in X$ define $d(x, A) = \inf_{y \in A} d(x, y)$ (the distance from x to A). Show that $d(\cdot, A)$ is continuous, and $d(x, A) = 0$ if and only if $x \in \overline{A}$. Then use these functions to prove that metric spaces are normal.

5. (Lang, Problem 13(a) on p. 47) Show that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\} \subset X \times X$ is closed in $X \times X$.

6. (Lang, Problem 14(c) on p. 47) Let X, Y be topological spaces, $A \subset X$, $B \subset Y$. Show that $\partial(A \times B) = (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B)$.