Math 111a, Fall 2008, Homework # 2

Metric Spaces, Completeness

1. (Lang, Problem 5(c) on p. 45) Given a metric space (X, d), consider a map $x \mapsto g_x$ from X to the normed space of bounded continuous functions on X with the supremum norm, given by $g_x(y) = d(x, y) - d(a, y)$, where a is some fixed point of X. Show that this map is distance-preserving. [Quoting Lang: *thus one need not fuss too much with abstract metric spaces.*]

2. (a) Define a new metric on $\mathbb{R}$ by $d(x, y) = |e^x - e^y|$. Is it complete? if yes, prove it; if no, describe its completion.

(b) Do the same for $d(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$.

3. (a) Show that a metric space is complete $\Leftrightarrow$ any nested sequence of closed balls with radii tending to 0 has nonempty intersection. [You only need to prove the '$\Leftarrow$' part; the other direction was proved in class.]

(b) Is it possible to omit 'with radii tending to 0' in part (a)? [If you answer is 'yes', you need to prove the '$\Rightarrow$' part in (a) without assuming that radii must tend to 0; if no, you need to find an example of a complete metric space and a nested sequence of closed balls B_i with $\cap_{i=1}^{\infty} B_i = \varnothing$.]

4. Prove that a normed vector space is complete if and only if any absolutely convergent series of its elements converges. [That is, given a sequence $\{x_n\}$ of elements of the space, the sequence of partial sums $\sum_{k=1}^{n} x_k$ converges if $\sum_{n=1}^{\infty} \|x_n\| < \infty$.]

5. Let $\{f_n\}$ be a sequence of continuous real-valued functions on a complete metric space X such that the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every $x \in X$. Prove that given $\varepsilon > 0$, there exists a nonempty open subset U of X and $N \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \varepsilon$ for all $n \geq N$ and $x \in U$.

6. Fix a prime number p, and define a new metric d_p on $\mathbb{Z}$ by

$$d_p(m, n) = \begin{cases} 0 & \text{if} \quad m = n \\ p^{-k} & \text{if} \quad m - n = p^k l \quad \text{where } l \text{ is not divisible by } p \end{cases}$$

(a) Prove that $(\mathbb{Z}, d_p)$ is a metric space, and that the triangle inequality can be strengthened to the *ultrametric* inequality $d_p(m, q) \leq \max\big(d_p(m, n), d_p(n, q)\big)$.

(b) Show that $(\mathbb{Z}, d_p)$ is not complete. [Hint: elements of the completion are in one-to-one correspondence with series $\sum_{i=0}^{\infty} a_i p^i$, where $0 \leq a_i \leq p - 1$.]