Math 111a, Fall 2008, Homework # 4

Banach and Hilbert Spaces

1. (Cf. Lang, Problem 4 on p. 91) Let F be a closed subspace of a Banach space E. For each coset $x+F$ of F, define $\|x+F\| \stackrel{\text{def}}{=} \inf_{y\in F} \|x+y\|$. Show that (a) this defines a norm on the quotient space E/F and (b) makes the latter a Banach space as well. [Hint: for (b) it may be convenient to use Problem 4 from Homework 2.]

2. For a closed subspace F of a Banach space E, define

$$F^{\perp} \stackrel{\text{def}}{=} \{\lambda \in E' : \lambda(x) = 0 \ \ \forall\, x \in F\}\,.$$

Prove that $F' = E'/F^{\perp}$ and $(E/F)' = F^{\perp}$.

3. Say that a closed subspace F of a Banach space E is *complementable* if there exists a closed subspace G of E such that $F+G=E$ and $F\cap G=\{0\}$. Show that:

(a) (Lang, Problem 6(b) on p. 91) $\dim(F)<\infty \Rightarrow F$ is complementable;

(b) $\dim(E/F)<\infty \Rightarrow F$ is complementable.

4. Let E be a Hilbert space and S a subset of E. Prove that $S^{\perp} = \left(\overline{\mathrm{Span}(S)}\right)^{\perp}$ and $(S^{\perp})^{\perp} = \overline{\mathrm{Span}(S)}$.

5. Prove a Hilbert space version of the Hahn-Banach Theorem: let F be a subspace of a Hilbert space E and $\lambda \in F'$; then there exists a <u>unique</u> extension $\bar{\lambda} \in E'$ of λ such that $\|\bar{\lambda}\|$ is not bigger than (in fact, equal to) $\|\lambda\|$.

6. Let E be a separable infinite-dimensional Hilbert space, let $\{v_n : n \in \mathbb{N}\}$ be its orthonormal basis, and let $S = \{x\in E : \|x\| = 1\}$ be the unit sphere in E.

(a) (Lang, Problem 1 on p. 107) Show that the sequence $\{v_n\}$ converges weakly to 0 (and hence 0 is in the weak closure of S);

(b) (Lang, Problem 2 on p. 107) Show that for any $x \in E$ with $\|x\| \le 1$ there exists a sequence $\{u_n\}$ of elements of S which converges weakly to x (and hence the weak closure of S coincides with the unit ball in E).