

## Math 111a, Fall 2008, Homework # 5

### Measurable Functions and Measure Spaces

Unless otherwise specified, in all the problems below  $(X, \mathcal{A}, \mu)$  is a measure space, and subsets belonging to  $\mathcal{A}$  are referred to as measurable.

1. (Lang, Problem 3 on p. 172) Let  $\{f_n\}$  be a sequence of real-valued measurable functions on  $X$ . Show that the set of those  $x$  such that  $\{f_n(x)\}$  converges is a measurable set.

2. (Lang, Problem 9 on p. 174) Let  $E$  be a separable Hilbert space. Say that a map  $f : X \rightarrow E$  is *weakly measurable* if  $\lambda \circ f$  is measurable for every  $\lambda \in E'$ . Suppose that  $f, g : X \rightarrow E$  are weakly measurable. Show that the map  $x \mapsto \langle f(x), g(x) \rangle$  is measurable. (See Lang's book for a hint.)

3. Generalize **M8** as follows: a map of  $X$  into a separable metric space is measurable if and only if it is a pointwise limit of simple maps.

4. Define the *upper limit* of a sequence of subsets  $A_n$  of  $X$  by

$$\overline{\lim} A_n = \{x \in X \mid x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

(a) Prove that  $\overline{\lim} A_n$  is measurable if every  $A_n$  is measurable.

(b) [The Borel-Cantelli Lemma] Let  $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ ; prove that  $\mu(\overline{\lim} A_n) = 0$ .

5. Let  $E$  be a normed space with norm  $\|\cdot\|$ . Say that a sequence of measurable maps  $f_n : X \rightarrow E$  converges to  $f : X \rightarrow E$  in measure (notation:  $f_n \xrightarrow[\mu]{} f$ ) if

$$\forall \varepsilon > 0 \quad \lim_{n \rightarrow \infty} \mu(\{x \in X \mid \|f_n(x) - f(x)\| > \varepsilon\}) = 0.$$

Prove that

(a)  $f_n \xrightarrow[\mu]{} f \Rightarrow$  there exists a subsequence  $f_{n_k}$  such that  $f_{n_k} \xrightarrow[\text{a.e.}]{} f$ ;

(b)  $\mu(X) < \infty$  and  $f_n \xrightarrow[\text{a.e.}]{} f \Rightarrow f_n \xrightarrow[\mu]{} f$ .

(c) Provide examples showing that the conclusion of (a) cannot be strengthened to  $f_n \xrightarrow[\text{a.e.}]{} f$ , and the assumption  $\mu(X) < \infty$  in (b) cannot be removed.