

## Math 111a, Fall 2005, Homework # 5

### Measurable Functions and Measure Spaces

Unless otherwise specified, in all the problems below  $(X, \mathcal{A}, \mu)$  is a measure space, and subsets belonging to  $\mathcal{A}$  are referred to as measurable.

1. Lang, Problem 3 on p. 172.
2. Lang, Problem 9 on p. 174.
3. Generalize **M8** as follows: a map of  $X$  into a separable metric space is measurable if and only if it is a pointwise limit of simple maps.
4. Define the *upper limit* of a sequence of subsets  $A_n$  of  $X$  by

$$\overline{\lim} A_n = \{x \in X \mid x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

- (a) Prove that  $\overline{\lim} A_n$  is measurable if every  $A_n$  is measurable.
  - (b) [The Borel-Cantelli Lemma] Let  $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ ; prove that  $\mu(\overline{\lim} A_n) = 0$ .
5. Let  $E$  be a normed space with norm  $\|\cdot\|$ . Say that a sequence of measurable maps  $f_n : X \mapsto E$  converges to  $f : X \mapsto E$  in measure (notation:  $f_n \xrightarrow[\mu]{} f$ ) if

$$\forall \varepsilon > 0 \quad \lim_{n \rightarrow \infty} \mu(\{x \in X \mid \|f_n(x) - f(x)\| > \varepsilon\}) = 0.$$

Prove that

- (a)  $f_n \xrightarrow[\mu]{} f \Rightarrow$  there exists a subsequence  $f_{n_k}$  such that  $f_{n_k} \xrightarrow[\text{a.e.}]{} f$ ;
- (b)  $\mu(X) < \infty$  and  $f_n \xrightarrow[\text{a.e.}]{} f \Rightarrow f_n \xrightarrow[\mu]{} f$ .
- (c) Provide examples showing that the conclusion of (a) cannot be strengthened to  $f_n \xrightarrow[\text{a.e.}]{} f$ , and the assumption  $\mu(X) < \infty$  in (b) cannot be removed.