

Math 111a, Fall 2008, Homework # 6

Integration

Unless otherwise specified, in all the problems below (X, \mathcal{A}, μ) is a measure space, μ is complete and σ -finite, functions are measurable, and E (the target space of functions) is a separable Banach space with norm $|\cdot|$.

1. (Lang, Problem 2 on p. 172) Egoroff's Theorem: Let $\mu(X) < \infty$ and let $f_n \rightarrow f$ almost everywhere. Prove that for any $\varepsilon > 0$ there exists a set Z with $\mu(Z) < \varepsilon$ such that $f_n \rightarrow f$ uniformly on the complement of Z . (See Lang for a hint.) Is it still true when μ is infinite and σ -finite?

2. (Lang, Problem 13 on p. 175) Let T be a metric space and let $f : X \times T \rightarrow E$ be such that for each $t \in T$ the map $x \mapsto f(x, t)$ is in L^1 , and for each $x \in X$ the map $t \mapsto f(x, t)$ is continuous. Also assume that there exists $g \in L^1(X, \mathbb{R})$ such that $|f(x, t)| \leq g(x)$ for all x, t . Show that the function $t \mapsto \int_X f(x, t) d\mu(x)$ is continuous.

3. (Lang, Problem 21 on p. 177) Let $\mu(X) < \infty$ and $f \in L^1$; compute the limit

$$\lim_{n \rightarrow \infty} \int_X |f(x)|^{1/n} d\mu.$$

[Note: offered at an analysis qualifying exam at Yale in 1988.]

4. Recall a definition of *convergence in measure* from the last homework: $f_n \xrightarrow[\mu]{} f$ if $\forall \varepsilon > 0, \mu(\{x \in X \mid |f_n(x) - f(x)| > \varepsilon\}) \rightarrow 0$ as $n \rightarrow \infty$.

(a) Prove that $f_n \rightarrow f$ in $L^1 \Rightarrow f_n \xrightarrow[\mu]{} f$.

(b) Find a counterexample to the converse implication.

5. Show that a nonnegative function f belongs to $L^1(X, \mu)$ if and only if:

(a)

$$\sup \left\{ \int_X g d\mu \mid g \text{ a simple function such that } g(x) \leq f(x) \text{ a.e.} \right\} < \infty;$$

(b) $f|_A \in L^1(A, \mu)$ for any $A \in \mathcal{A}$ with $\mu(A) < \infty$, and

$$\sup \left\{ \int_A f d\mu \mid A \in \mathcal{A}, \mu(A) < \infty, f \text{ is bounded on } A \right\} < \infty;$$

(c) assuming that $\mu(X) < \infty$:

$$\sum_{n=0}^{\infty} 2^n \mu(\{x \in X \mid f(x) \geq 2^n\}) < \infty;$$

(d) assuming that $\mu(X) = \infty$ and f is bounded:

$$\sum_{n=0}^{\infty} 2^{-n} \mu(\{x \in X \mid f(x) \geq 2^{-n}\}) < \infty.$$