

## Math 111a, Fall 2005, Homework # 6

### Integration

Unless otherwise specified, in all the problems below  $(X, \mathcal{A}, \mu)$  is a measure space,  $\mu$  is complete and  $\sigma$ -finite, functions  $f$  are measurable, and  $E$  (the target space of functions) is a separable Banach space.

1. Lang, Problem 2 on p. 172.
2. Lang, Problem 13 on p. 175.
3. Lang, Problem 21 on p. 177. [Note: it was offered at an analysis qualifying exam at Yale in 1988.]
4. Recall a definition of *convergence in measure* from the last homework:  $f_n \xrightarrow{\mu} f$  if  $\forall \varepsilon > 0$ ,  $\mu(\{x \in X \mid \|f_n(x) - f(x)\| > \varepsilon\}) \rightarrow 0$  as  $n \rightarrow \infty$ .
  - (a) Prove that  $f_n \rightarrow f$  in  $L^1 \Rightarrow f_n \xrightarrow{\mu} f$ .
  - (b) Find a counterexample for the converse implication.
5. Show that a nonnegative function  $f$  belongs to  $L^1(X, \mu)$  if and only if:
  - (a)

$$\sup \left\{ \int_X g \, d\mu : g \text{ a simple function such that } g(x) \leq f(x) \text{ a.e.} \right\} < \infty;$$

- (b)  $f|_A \in L^1(A, \mu)$  for any  $A \in \mathcal{A}$  with  $\mu(A) < \infty$ , and

$$\sup \left\{ \int_A f \, d\mu : A \in \mathcal{A}, \mu(A) < \infty, f \text{ is bounded on } A \right\} < \infty;$$

- (c) assuming  $\mu(X) < \infty$ :

$$\sum_{n=0}^{\infty} 2^n \mu(\{x \in X : f(x) \geq 2^n\}) < \infty;$$

- (d) assuming  $\mu(X) = \infty$  and  $f$  is bounded:

$$\sum_{n=0}^{\infty} 2^{-n} \mu(\{x \in X : f(x) \geq 2^{-n}\}) < \infty.$$