Math 111a, Fall 2008, Homework # 7

Extension of measures; Lebesgue measure on $\mathbb{R}$; L^1 and L^∞

In the first three problems, $\mathcal{A}$ is an algebra of subsets of X, μ a positive measure on $\mathcal{A}$, $X \in \mathcal{A}$, $\mu(X) < \infty$, and μ^* is the outer measure as defined in class (or in Lemma 7.2 of Lang's book).

1. For $A, B \subset X$, say that $A \sim B$ if $\mu^*(A \triangle B) = 0$. Prove that it is an equivalence relation, and $d(A, B) \stackrel{\text{def}}{=} \mu^*(A \triangle B)$ defines a metric on the set of equivalence classes of subsets of X.

2. Prove that the above metric space is complete. (Thus the σ-algebra of μ^*-measurable sets (modulo equivalence) can be identified with the completion of $\mathcal{A}$.)

3. Prove that $A \subset X$ is μ^*-measurable iff $\mu^*(A) + \mu^*(A^c) = \mu(X)$.

In the next three problems μ is the Lebesgue measure on $\mathbb{R}$, and "measurable" means "Lebesgue measurable".

4. Prove Luzin's Theorem: for a function $f : \mathbb{R} \mapsto \mathbb{R}$, the following are equivalent:

(1) f is measurable;
(2) for any $a < b$ and any $\varepsilon > 0$ there exists $g \in C[a, b]$ such that

$$\mu\big(\{x \in [a, b] \mid f(x) \neq g(x)\}\big) < \varepsilon\,.$$

[In other words, f can be made continuous by changing it on a set of arbitrary small measure. Hint: you can use the density of $C[a, b]$ in $L^1([a, b])$ plus Egoroff's Theorem (Problem 1 from the previous homework).]

5. [A textbook example which we cannot avoid: Lang, Problem 20 on p. 177] In each coset $x + \mathbb{Q}$ of $\mathbb{R}$ select an element $y \in [0, 1]$. Show that the set consisting of all such elements cannot be Lebesgue measurable.

6. [Lang, Problem 10 on p. 220] Let $1 \leq p < \infty$ and let $f \in L^p(\mathbb{R})$. For $a \in \mathbb{R}$ define T_a to be the translation by a, that is, $(T_a f)(x) \stackrel{\text{def}}{=} f(x - a)$. Prove that $T_a f$ converges to f in L^p as $a \to 0$. Is the conclusion still true if $p = \infty$?

Finally, two problems about spaces L^1 and L^∞ for arbitrary measures μ:

7. [Lang, Problem 3 on p. 218] Let $X = \{x, y\}$ and define a measure μ on X by $\mu(\{x\}) = 1$ and $\mu(\{y\}) = \infty$. What is the dual space of $L^1(X, \mu)$?

8. [Lang, Problem 11 on p. 220] Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and suppose $L^1(X) \subset L^\infty(X)$. Show that there exists $c > 0$ such that if $A \in \mathcal{A}$ has positive measure, then $\mu(A) \geq c$.