Math 111a, Fall 2008, Homework # 8

Fubini, Radon-Nikodym, convolutions

In the first two problems, $(X, \mathcal{A}, \mu)$ is a complete measure space, $\mu(X) < \infty$, and E is a Hilbert space.

1. [Lang, p. 199, a proof left to the reader] Prove that the space of E-valued measures with the norm $\|\nu\| \overset{\text{def}}{=} |\nu|(X)$ is a Banach space.

2. Prove that for a measurable function f and $0 < p < \infty$,

$$\int_X |f|^p \, d\mu = p \int_0^\infty t^{p-1} \mu\big(\{x \in X \mid |f(x)| > t\}\big) \, dt \, .$$

In the next three problems, μ is Lebesgue measure on $\mathbb{R}$ or its subsets, $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets, and, unless otherwise specified, all the functions are real-valued.

3. Let ν be a positive finite measure on $\mathcal{M}$ such that for every $A \in \mathcal{M}$, the function $x \mapsto \nu(A + x)$ is continuous. Show that ν is μ-continuous.

4. [a modification of Lang, Problem 20 on p. 221] Let $X = [0, 1]$, let $E = L^2(X, \mu)$, and for each measurable $A \subset X$ let $\nu(A) = 1_A$. Clearly $\nu : \mathcal{M} \to E$ is countably additive. Is it an E-valued measure (that is, is $|\nu|(X) < \infty$)? is it μ-continuous?

5. If $f, g \in L^1(\mathbb{R}, \mu)$, define the convolution $f * g$ by

$$(f * g)(x) \overset{\text{def}}{=} \int_{\mathbb{R}} f(t) g(x - t) \, d\mu(t) \, ;$$

the definition is justified in [Lang, Chapter VIII, Theorem 1.1].

(a) [Lang, Problem 15(b) on p. 176] Prove the associativity of convolution:

$$f * (g * h) = (f * g) * h \text{ for any } f, g, h \in L^1 \, .$$

(b) Suppose that $f \in L^1$ and $g \in L^1 \cap L^\infty$; prove that $f * g$ is continuous.

(c) Find $f, g \in L^1$ such that $f * g$ is not continuous.