1. [16 points] Consider the surface $S$ described by the equation $3x^2 + y^2 + z^3 + 6xyz = 11$, and let $C$ be the section of this surface in the plane parallel to the $xz$ plane, given by $y = 1$. Find the slope of the curve $C$ at the point where $x = 1$ and $z = 1$ (and of course $y = 1$).

Solution: The problem asks for the derivative $\frac{\partial z}{\partial x}$ at the point $(1, 1, 1)$, because we are holding $y$ fixed and seeing how $z$ changes with $x$. Unless you're really good at solving cubic equations, the derivative is best found by implicit differentiation. Writing $F(x, y, z) = 3x^2 + y^2 + z^3 + 6xyz = 11$, one has

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}(1, 1, 1) = -\frac{6x + 6yz}{3z^2 + 6xy}(1, 1, 1) = -\frac{12}{9} = -\frac{4}{3}.$$ 

An alternative route is to take $\frac{\partial}{\partial x}$ of both sides of the equation, to get

$$6x + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$$

and then solve for $\frac{\partial z}{\partial x}$ at the given point.

Note: many people found the gradient of $F$ at $(1, 1, 1)$ but were not sure what to do with it. Some simply called it a slope which is quite bad since slope must be a number and not a vector.

2. [25 points] (a) Find all critical points of the function $T(x, y) = x^2 + 2y^2 - 2x + 2$ and classify each critical point as a local maximum, local minimum, or saddle point.

(b) A plate described by

$$\{(x, y) : x^2 + y^2 \leq 4\}$$

has a temperature distribution given by $T(x, y)$, the same function as in part (a). Find the hottest and the coolest points on this disc.

Solution: (a) $\nabla T = (2x - 2, 4y)$, which is zero only at the point $(1, 0)$. Applying the second derivative test, we get $T_{xx} = 2$, $T_{yy} = 4$, and $T_{xy} = 0$. So $D > 0$, and $T_{xx} > 0$, which means that $T$ has a local minimum at $(1, 0)$. For use in the next part, we note that $T(1, 0) = 1$.

(b) Since we've found all of the interior critical points in the disc in part (a), we should find the max and min on the boundary, which is given by the equation $x^2 + y^2 = 4$. This is quite similar to a problem we had on the midterm.

One way to do this is by expressing $T$ as a function of a single variable, for example writing $y^2 = 4 - x^2$; this way we want to find the max and min of the function $g(x) = x^2 + 2(4 - x^2) - 2x + 2 = -x^2 - 2x + 10$, with $-2 \leq x \leq 2$. To do this, first find all points (there’s only one) with $g'(x) = 0$, and then compute the value of $g$ there and at the endpoints $x = \pm 2$:

$$g'(x) = -2x - 2, \quad \text{giving} \quad g'(x) = 0 \quad \text{for} \quad x = -1.$$ 

We compute $g(-2) = 10$, $g(2) = 2$, $g(-1) = 11$, and already we have $T(1, 0) = 1$. So the minimum is 1, at the point $(1, 0)$, and the maximum is 11, at the points $(-1, \sqrt{3})$ and $(-1, -\sqrt{3})$. (There are two, because two $y$-values on the boundary correspond to $x = -1$.)

Another way is by using Lagrange multipliers:

$$\begin{cases} 2x - 2 = 2\lambda x \\ 4y = 2\lambda y \\ x^2 + y^2 = 4 \end{cases}$$
3. [16 points] Find the equation of the tangent plane to the surface parametrized by \\
\vec{r}(u, v) = (u \cos v) \mathbf{i} + (u \sin v) \mathbf{j} + (\sin u) \mathbf{k} \\

at the point \vec{r}(\pi/2, 0).

**Solution:** First we find the normal vector at the point, given by \( \vec{r}_u \times \vec{r}_v \). Here \( \vec{r}_u \) means the vector given by \((x_u, y_u, z_u)\). We compute that \( \vec{r}(\pi/2, 0) = (\pi/2, 0, 1) \), and that \\
\[ \vec{r}_u = (\cos v, \sin v, \cos u) = (1, 0, 0) \quad \text{at } u = \pi/2, \ v = 0 \] \\
\[ \vec{r}_v = (-u \sin v, u \cos v, 0) = (0, \pi/2, 0) \quad \text{at } u = \pi/2, \ v = 0 \] \\
\[ \vec{n} = \vec{r}_u \times \vec{r}_v = (0, 0, \pi/2). \]

So the plane is given by \( 0(x - \pi/2) + 0(y - 0) + \pi/2(z - 1) = 0 \). This simplifies to \( z = 1 \), which is a perfectly good equation of a plane.

This problem actually got here by mistake, since it is based on the material from section 11.4. But still this theme was not on the first midterm, and you should know how to approach such problems and should not get confused. The main issues here were not knowing how to calculate the normal vector from a parameterization, or not knowing how to get a plane given a point and normal vector. Also many people decided that an equation of the plane should relate variables \( u \) and \( v \) to each other, which makes no sense since those are simply parameters used to describe the surface which lies in the \( xyz \) space. Finally, people were trying to write down the gradient of \( \vec{r} \) but it is not quite clear what it means since \( \vec{r} \) is a vector-valued function!

4. [18 points] (a) Find the integral of the function \( f(x, y) = \frac{x}{(x^2 + y^2)^2} \) over the region \( R = [0, 1] \times [1, 2] \).

(b) Without any calculations, guess the value of \( \iint_D f(x, y) \, dA \), where \( D = [-1, 1] \times [1, 2] \) and \( f \) is as in part (a), and justify your guess.

**Solution:** The first step was to set up the integral as an iterated integral; as it turns out, both orders are just as easy (or as hard, depending on your viewpoint) so I’ll present both.

\[ \int_0^1 \int_1^2 \frac{x}{(x^2 + y)^2} \, dy \, dx = -\int_0^1 \left[ (x^2 + y)^{-1} \right]_1^2 \, dx = \int_0^1 (x^2 + 1)^{-1} - (x^2 + 2)^{-1} \, dx \]
\[ = \frac{1}{2} \left[ \ln(x^2 + 1) - \ln(x^2 + 2) \right]_0^1 = \frac{1}{2} (\ln 2 - \ln 1 - \ln 3 + \ln 2) = \frac{1}{2} \ln(\frac{4}{3}). \]

Doing the integral in the other order, we get

\[ \int_1^2 \int_0^1 \frac{x}{(x^2 + y)^2} \, dx \, dy = -\frac{1}{2} \int_1^2 \left[ (x^2 + y)^{-1} \right]_0^1 \, dy = \frac{1}{2} \int_1^2 \left( \frac{1}{y} - \frac{1}{1+y} \right) \, dy = \frac{1}{2} \left[ \ln y - \ln(1+y) \right]_1^2 \]
\[ = \frac{1}{2} (\ln 2 - \ln 1 - \ln 3 + \ln 2) = \frac{1}{2} \ln(\frac{4}{3}). \]

For part (b) it was enough to notice that \( f(-x, y) = -f(x, y) \), that is, \( f \) is odd in the variable \( x \), hence the integrals over \( R \) and over its mirror image cancel each other, and the integral over \( D \) is equal to zero. Some people argued that since the area of \( D \) is twice the area of \( R \), the integral should also be multiplied by 2, which was a mistake since the behavior of the function matters as well.