Math 20a Quiz 6 May 1, 2012

Instructions: Read the problems carefully before you start thinking or writing. No calculators allowed; leave any quantities such as $0.4e^5$ without evaluating numerically. There are a total of 30 points.

Let $C_1$ be the upper right quarter of the circle $\{x^2 + y^2 = 1, x, y \geq 0\}$, oriented from $M = (0, 1)$ to $N = (1, 0)$, and let $C_2$ be the straight line segment joining $M$ and $N$, also oriented from $M$ to $N$. Consider the vector field $\mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and the function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by
\[
\mathbf{F}(x, y) = (x + y, y - x) \quad \text{and} \quad h(x, y) = e^{x - y} \cos(\pi xy) .
\]

(a) Find $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$.

Solution: $C_1$ can be parameterized by $\mathbf{r}(t) = (\sin t, \cos t)$, $0 \leq t \leq \pi/2$. (A common mistake was to write $\mathbf{r}(t) = (\cos t, \sin t)$, which gave the opposite orientation, in which case the sign should be changed.) Then
\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (\sin t + \cos t, \cos t - \sin t) \cdot (\cos t, -\sin t) \, dt = \int_0^{\pi/2} (\sin^2 t + \cos^2 t) \, dt = \pi/2 .
\]

(b) Find $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

Solution: $C_2$ can be parameterized by $\mathbf{r}(t) = (t, 1 - t)$, $0 \leq t \leq 1$, thus
\[
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t + 1 - t, 1 - t - t) \cdot (1, -1) \, dt = \int_0^1 2t \, dt = 1 .
\]

(c) Is the vector field $\mathbf{F}$ conservative (that is, $\mathbf{F} = \nabla f$ for some $f$)? Justify your answer.

Solution: A comparison of answers in (a) and (b) is already enough to draw a conclusion: the two integrals are different even though the endpoints of the two curves are the same, which does not happen for conservative vector fields. Another way is to notice that $\partial Q/\partial x = -1$ is not equal to $\partial P/\partial y = 1$.

(d) Compute $\int_{C_1} \nabla h \cdot d\mathbf{r}$ and $\int_{C_2} \nabla h \cdot d\mathbf{r}$.

Solution: Here we are given a conservative vector field, so there is no need to compute $\nabla h$, parameterize the curves and integrate; it suffices to notice that
\[
\int_{C_1} \nabla h \cdot d\mathbf{r} = \int_{C_2} \nabla h \cdot d\mathbf{r} = h(N) - h(M) = e - 1/e .
\]