ON THE SHRINKING TARGET PROPERTY

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Abstract. We show that some smooth measure preserving and mixing maps of the torus do not have the shrinking target property.

Given an ergodic dynamical system $\{M, T, \mu\}$ one knows that for $x_0 \in M$ fixed, and A a measurable subset of M containing x_0 and of measure $\mu(A) > 0$, there exist for almost every $x \in M$ infinitly many integers n such that $T^n(x) \in A$, that is the trajectory of almost every point x hits the target A infinitly often. In an attempt to quantify this recurrence one might ask what happens if the set A is not fixed, having for example its measure shrinking to zero when n goes to infinity. A general abstract definition exists:

Definition 1. *A sequence $\mathcal{A} = \{A_n\}_{n\in\mathbf{N}}$, of divergent sum of measures, $\sum_{n=0}^{\infty} \mu(A_n) = \infty$, is called a Borel Cantelli (BC) sequence for T if the sets*

$$N(x, \mathcal{A}) = \{n \in \mathbf{N} / T^n(x) \in A_n\}$$

are infinite for almost every point x in M.

Such typical points x are said to hit the shrinking target infinitly often. Initially, this definition was motivated by the classical theory of Diophantine approximations and was used in the study of discrete hyperbolic groups (for references, see [CK]). In another direction, many ergodic properties of a dynamical system can be characterised in terms of Borel Cantelli sequences. We restate some of the results that were summarised in the paper of Chernov and Kleinbock [CK].

Proposition 1. *Let T be a measure preserving transformation of a probability space (M, μ). Then:*

(i) *T is ergodic iff every constant sequence $A_n = A$, $\mu(A) > 0$, is BC;*

(ii) *T is weak mixing iff every sequence $\{A_n\}$ of divergent sum of measures that only contains finitely many distinct sets, none of them of measure zero, is BC;*

(iii) *T is lightly mixing iff every sequence $\{A_n\}$ of divergent sum of measures that only contains finitely many distinct sets, possibly of measure zero, is BC.*

Also in [CK], the authors recall the existence for any measure preserving system (M, μ, T) of sequences of subsets of M with divergent sum of measures that are not Borel Cantelli sequences. So in order to characterise dynamical systems using BC sequences one has to specify the type or shape of the subsets that form the target. In the context of ergodic continuous maps T on a metric space M, one might give the following definition:

Definition 2 (STP). *We say that T has the Shrinking Target Property if for any $x_0 \in M$, any sequence of centered balls around x_0 of divergent sum of measures is BC for T.*

In analogy with Diophantine approximations we can say that for sush a system, every point x_0 in the space is a "well approximated point". We can restrict even more the choice of the target to sequences of decreasing centered balls, $A_{n+1} \subset A_n$ for any n, and this would lead a second definition of the STP, call it monotone shrinking target property or mSTP.

Definition 3 (mSTP). *We say that T has the Monotone Shrinking Target Property if for any $x_0 \in M$, any sequence of decreasing centered balls around x_0 of divergent sum of measures is BC for T.*

Having made more restriction on the target the latter property would hold for larger classes of ergodic systems.

For instance, one can easily obtain the following lemmas:

Lemma 1. *There is no irrational translation on the circle that has the shrinking target property as this property is stated in Definition 2. Moreover, one can construct for any $x_0 \in \mathbf{T}^1$ a sequence of centered intervals around x_0, with divergent sum of measures, such that the set of points $x \in \mathbf{T}^1$ that hit this target infinitly often is reduced to the single point x_0.*

We will give the exact proof of this lemma in the next paragraph; the idea is to use the rigidity times[1] q_n of the rotation and take a lacunary serie for $\mathcal{A} = \{A_n\}_{n\in\mathbf{N}}$ with nonempty sets only when n is equal to some q_l. Since at those times every point on the circle remains almost unmoved, there will be no point except x_0 that hits the target infinitly often. The situation is more complex if we restrict to properly shrinking targets, where no lacunary series are allowed.

Lemma 2. *Let R_α be an irrational rotation on the circle; we have:*

- (i) *If α is of constant type, R_α has the mSTP.*
- (ii) *If α is not of constant type, R_α does not have the mSTP. Moreover one can construct for any $x_0 \in \mathbf{T}^1$ a sequence of decreasing centered intervals around x_0, with divergent sum of measures,*

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[1] For a dynamical system $\{M, T, \mu\}$, we say that q_n is a sequence of rigidity times if T^{q_n} tends to the identity when n goes to infinity, in the weak topology induced by μ. In the case of the rotation the convergence holds for the uniform topology of the circle.

such that the set of points $x \in \mathbf{T}^1$ that hit this shrinking target infinitly often is of zero measure.

First note that T does not have the STP (resp. mSTP) if and only if there exist $x_0 \in M$ and a sequence $\mathcal{A} = \{A_n(x_0)\}_{n\in\mathbf{N}}$ of centered balls around x_0 (resp. with in addition $A_{n+1} \subset A_n$ for any n) such that $\sum_{n=0}^{\infty} \mu(A_n) = +\infty$, while the corresponding sets

$$N(\mathcal{A}, x) = \{n \in \mathbf{N} / T^n(x) \in A_n\}$$

are finite for a positive measure set of points x. Or equivalently if

$$\mu\left(\bigcup_{n=n_0}^{\infty} T^{-n}(A_n)\right) < 1,$$

for a certain $n_0 \in \mathbf{N}$. Indeed, the points in the complementary of the latter set hit the target at most finitely many times and violate the shrinking target property. We say that they “miss” the target. If we have in addition that

$$\mu\left(\bigcup_{n=n_0}^{\infty} T^{-n}(A_n)\right) \longrightarrow 0,$$

when n_0 goes to infinity, it would follow that the points that “miss” the target are of full measure, which is exactely the opposite of the STP.

Proof of the lemmas. Here $M = \mathbf{T}^1$ is the circle $\mathbf{R}/\mathbf{Z}$, and μ is the Haar measure on it. For $x_0 \in M$ and $0 < r < \frac{1}{2}$, we will denote by $I(x_0, r)$ the interval of size $2r$ centered at x_0.

Proof of Lemma 1: Take $\alpha \in \mathbf{R} - \mathbf{Q}$ arbitrary and let q_l be the sequence of denominators of its rational approximations. One has

$$|||q_l \alpha||| < \frac{1}{q_{l+1}},$$

where $|||.|||$ denotes the distance to the closest integer. Hence, we have

$$R_{\alpha}^{q_l}\left(I(x_0, r)\right) \subset I(x_0, r + q_{l+1}^{-1}). \tag{1}$$

for any integer l and any $r > 0$. Define now $\{I_n(x_0)\}_{n\in\mathbf{N}}$ in the following way

- $I_n = I(x_0, \frac{1}{l})$, if $n = q_l$ for some integer $l > 0$,
- $I_n = \emptyset$ otherwise.

Clearely in this case

$$\sum_{n=0}^{\infty} \mu(I_n) = \sum_{l=1}^{\infty} \frac{1}{l} = +\infty;$$

while it follows from (1) and the definition of the intervals that for any $n \geq q_{l_0}$ (we only need to consider the n of the form q_l)

$$R_\alpha^{-n} I_n \subset I(x_0, \frac{1}{l_0} + \frac{1}{q_{l_0+1}}) \subset I(x_0, \frac{2}{l_0}),$$

hence

$$\bigcup_{n=q_{l_0}}^{\infty} \left(R_\alpha^{-n} I_n\right) \subset I(x_0, \frac{2}{l_0});$$

when $l_0 \to \infty$, this last set obviously shrinks down to the single point x_0 and the first lemma is proved.

Proof of (i) in Lemma 2: Suppose α is of constant type, i.e. there exists a constant $K > 0$ such that for any $l \in \mathbf{N}$

$$q_{l+1} \leq K q_l;$$

and suppose ... $\subset I_{n+1} \subset I_n \subset$ ... is any decreasing sequence of centered intervals around x_0 with a divergent sum of measures. We want to show that it is impossible to have

$$\mu\left(\bigcup_{n=n_0}^{\infty} R_\alpha^{-n}(I_n)\right) < 1, \tag{2}$$

for any choice of n_0 that we make. But as we will have to prove this fact for any sequence satisfying the above condition, it is enough to prove it for a given sequence when $n_0 = 1$. First we have:

$$\sum_{n=1}^{q_{l+1}-1} \mu(I_n) = \sum_{i=1}^{l} \sum_{n=q_i}^{q_{i+1}-1} \mu(I_n) \leq \sum_{i=1}^{l} q_{l+1}\mu(I_{q_l}),$$

so the serie $\sum q_{l+1}\mu(I_{q_l})$ is divergent, and from the arithmetical condition on α, this is equivalent to the divergence of $\sum q_l \mu(I_{q_{l+1}})$. This fact will be fundamental in our proof.

Consider for any l the set

$$\mathcal{J}_l = \bigcup_{n=0}^{q_l-1} R_\alpha^{-n}(I_n),$$

and rearrange the intervals of this union in a successive order on the circle $J_0, ..., J_{q_l-1}$ of intervals centerd on the first q_l iterates of the rotation $R_{-\alpha}$ rearranged into $x_0, ..., x_{q_l-1}$. Assume that $K > 5$ so that between two successive points of this succession lie at least three intervals from

$$\mathcal{J}_{l+1} - \mathcal{J}_l = \bigcup_{n=q_l}^{q_{l+1}-1} R_\alpha^{-n}(I_n),$$

so either these intervals will fill the gap or one of them will fit completely in it. Now, there are many possible configurations for the intervals in $\mathcal{J}_l$, and between them two extreme cases can be encountered:

- The gap between any two successive intervals J_i, J_{i+1}, $i \leq q_l$, does not exist or is too small so that we can cover it by an interval of $\mathcal{J}_{l+1} - \mathcal{J}_l$.
- The gap between any two successive intervals J_i, J_{i+1}, $i \leq q_l$, is big enough so we can fit completely in it an interval of $\mathcal{J}_{l+1} - \mathcal{J}_l$.

In the first case there is nothing to do anymore, we covered the circle and (2) is impossible.
In the second case, we will have obviously

$$\mu\left(\mathcal{J}_{l+1}\right) \geq \mu\left(\mathcal{J}_l\right) + q_l \mu(I_{q_{l+1}}),$$

so we iterate untill we fall into the first case or we get a contradiction with (2) since $\sum q_l \mu(I_{q_{l+1}})$ is divergent. In fact our reasoning as it is held here is not exact because one can meet other sitations than the extreme ones, situations where at some places there is no gap and at other places the gap is big. But in this case we just consider the big gaps if the proportion of their occurency (over q_l) is bigger than a fixed constant $c > 0$ (otherwise we would cover the circle in measure). Then, as in the second argumentation, the divergence of $\sum c q_l \mu(I_{q_{l+1}})$ will lead to a contradiction with (2).

As for the proof of (ii), it is based on the proximity of R_α to rational rotations, i.e. periodic ones, under the arithmetical condition assumed on α. Roughly speaking, because of the periodicity the inverse images of the sets of the target, do not sum up in measure when we take their union and the measure in (2) is small.
Proof of (ii): Assume α is not of constant type, then there exists a sequence of integers q_{l_i} such that

$$|||q_{l_i}\alpha||| \leq \frac{1}{i^6 q_{l_i}}. \tag{3}$$

For any $x_0 \in \mathbf{T}$ define the sequence of intervals $I(x_0, r_n)$, centered at x_0 with r_n such that

$$r_n = \frac{1}{i^2 q_{l_i}}, \quad \text{for any } n \in [i^3 q_{l_{i-1}}, i^3 q_{l_i}].$$

Clearly r_n is decreasing while $\sum r_n = +\infty$.

Let us now look at the set

$$\begin{aligned} \bigcup_{n=i^3 q_{l_{i-1}}}^{i^3 q_{l_i}} R_\alpha^{-n}\left(I(x_0, r_n)\right) &= \bigcup_{n=i^3 q_{l_{i-1}}}^{i^3 q_{l_i}} R_\alpha^{-n}\left(I(x_0, \frac{1}{i^2 q_{l_i}})\right) \\ &\subset \bigcup_{n=0}^{i^3 q_{l_i}} R_\alpha^{-n}\left(I(x_0, \frac{1}{i^2 q_{l_i}})\right). \end{aligned} \tag{4}$$

From (3) we have for any $k q_{l_i}$, multiple of q_{l_i}, such that $k q_{l_i} \leq i^3 q_{l_i}$

$$|||k q_{l_i}\alpha||| \leq \frac{1}{i^3 q_{l_i}},$$

consequently, for any r

$$\bigcup_{n=0}^{i^3 q_{l_i}} R_\alpha^{-n}\left(I(x_0, r)\right) \subset \bigcup_{n=0}^{q_{l_i}-1} R_\alpha^{-n}\left(I(x_0, r + \frac{1}{i^3 q_{l_i}})\right).$$

Back in (4) this imlplies

$$\bigcup_{n=i^3 q_{l_{i-1}}}^{i^3 q_{l_i}} R_\alpha^{-n}\left(I(x_0, r_n)\right) \subset \bigcup_{n=0}^{q_{l_i}-1} R_\alpha^{-n}\left(I(x_0, \frac{2}{i^2 q_{l_i}})\right).$$

Hence, clearly

$$\mu\left(\bigcup_{n=i^3 q_{l_{i-1}}}^{i^3 q_{l_i}} R_\alpha^{-n}\left(I(x_0, r_n)\right)\right) \leq \frac{4}{i^2},$$

and

$$\mu\left(\bigcup_{n=i_0^3 q_{l_{i_0-1}}}^{\infty} R_\alpha^{-n}\left(I(x_0, r_n)\right)\right) \leq \sum_{i=i_0}^{\infty} \frac{4}{i^2},$$

which goes to zero when $i_0 \to \infty$. The proof of (*ii*) is over. □

With this kind of topological definitions of the target it was proved that expanding maps on the circle (Philipp) and Anosov diffeomorphisms on topological spaces (Dolgopyat) had the shrinking target property. Also, Kleinbock and Margulis have many similar results concerning flows on homogeneous spaces and Diophantine approximations on manifolds, and drove from there many applications to geometry and number theory. All these results use more or less essentially the exponential decay of correlations of the corresponding transformations.

Than rose the problem of characterising those transformations that had the STP, and it was natural to ask wether there existed mixing transformations that did not have STP for regular sequences of target (ex. shrinking centered balls). Our goal here is to exhibit such examples, that moreover have polynomial decay of correlations.

In [**?**], a class of mixing flows is studied. These are special flows over an irrational translation of the circle, R_α, and under a ceiling function φ possessing a power-like singularity. In some cases (depending on the singularity), these flows can be seen as smooth flows on the torus preserving a smooth measure and having one fixed point. Kocergyn proves that such flows are mixing, and in [**?**] we prove that under some assumptions on the singularity and on the rotation number, they actually have polynomial decay of correlations between rectangles. Before we prove that these flows do not have the shrinking target property, we will recall some definitions and properties related to the special flows under consideration (for a complete description of the flows we refer to [**?**] or [**?**]):

First we give the definition of a special flow over an irrational rotation: Given a function $\varphi \in L^1(\mathbf{T}^1)$, $\varphi > c > 0$, *the special flow constructed over* R_α *and under the function* φ is the quotient flow of the action

$$\begin{array}{ccc} \mathbf{T}^1 \times \mathbf{R} & \longrightarrow & \mathbf{T}^1 \times \mathbf{R} \\ (x,s) & \longrightarrow & (x, s+t) \end{array}$$

by the relation $(x, s+\varphi(x)) \sim (R_\alpha(x), s)$. This flow acts on the space

$$M = \big\{(x,s) \in \mathbf{T}^1 \times \mathbf{R} /\ 0 \le s < \varphi(x)\big\}$$

and preserves the normalized Lebesgues measure on M. We can suppose without loss of generality that the *ceiling function* φ satisfies $\int \varphi = 1$.

We will also need the following definitions:

- Given $x_0 \in \mathbf{T}$ and $r > 0$ we denote by $I(x_0, r)$ the interval on the circle of size $2r$ centered at x_0.
- By $\Delta(x_0, r)$ we denote the region or "band" of M that lies between $I(x_0, r)$ and the ceiling function. By definition the measure of a band is given by $\mu\left(\Delta(x_0, r)\right) = \int_{I(x_0,r)} \varphi(x)dx$
- By centered rectangle $A(x_0, s_0, r_n, \delta)$ we denote the subset $\bigcup_{t=s_0-\delta}^{s_0+\delta} I(x_0, r)$.

We consider special flows over R_α with a ceiling function φ of class at least C^2 except at 0 where it has a power like singularity with exponent $-\gamma$, more precisely:

- $\lim\limits_{x\to 0^+} \frac{\varphi(x)}{x^{-\gamma}} = 1, \qquad \lim\limits_{x\to 0^-} \frac{\varphi(x)}{(-x)^{-\gamma}} = 1$
- $\lim\limits_{x\to 0^+} \frac{\varphi'(x)}{-\gamma x^{-\gamma-1}} = 1, \qquad \lim\limits_{x\to 0^-} \frac{\varphi'(x)}{\gamma(-x)^{-\gamma-1}} = 1$
- $\lim\limits_{x\to 0^+} \frac{\varphi''(x)}{\gamma(\gamma+1)x^{-\gamma-2}} = 1, \qquad \lim\limits_{x\to 0^-} \frac{\varphi''(x)}{\gamma(\gamma+1)(-x)^{-\gamma-2}} = 1.$

With the hypothesis $\gamma < \frac{2}{5}$ and a corresponding Diophantine condition on the rotation number α on the basis, we proved the following estimation for the special flow $\{R_\alpha, \varphi\}$, that clearly implies mixing:

Theorem: *There exists* $\eta > 0$ *such that, for any two rectangles* A *and* B*, we have for* t *large enough*

$$\left|\mu(A \bigcap T^{-t}B) - \mu(A)\mu(B)\right| \le \frac{1}{t^\eta}.$$

The time one map of this flow, T, is therefore mixing with power-like correlations. Still it does not have the STP, and in case the rotation number on the basis α is not of constant type, we will prove that T does not have the mSTP. The reason is the following: before that the trajectory of a rectangle A, under the action of the flow, passes too close to the singularity, A is translated as if under the action of an irrational flow on $\mathbf{T}^2$ with only slight deformations. Hence, the same phenomena of almost periodicity as for the irrational rotations is reproduced and the mSTP will be violated. To put this in other words we can say: although T is rapidly mixing, it takes a long time for a rectangle A (compared with the inverse of its measure) to be "spread" on the torus, i.e. for $T^n(A)$ to become "independent events".

From now on, we will continue to assume that α is ϵ-diophantine as is required in [?] to have polynomial correlations but we suppose in counterpart that α is not of constant type.

Proposition 2. *Let T be the time one map of the flow described above. There exists a decreasing sequence $\{r_n\}_{n\in\mathbf{N}}$ of positive numbers satisfying $\sum_{n=0}^{\infty} r_n = +\infty$, such that for any $(x_0, s_0) \in M$, the monotone shrinking target property (mSTP) is violated for T and the decreasing sequence of centered rectangles $A_n = A(x_0, s_0, r_n, \delta)$ by a set of full measure in M.*

Proof. We use the same notations as in the proof of (ii) of Lemma 2: the sequence l_i is such that

$$|||q_{l_i}\alpha||| \leq \frac{1}{i^6 q_{l_i}}$$

and

$$r_n = \frac{1}{i^2 q_{l_i}}, \quad \text{for any } n \in [i^3 q_{l_{i-1}}, i^3 q_{l_i}].$$

For every integer $i > 1$, we define

$$\mathcal{A}_{l_i} = \bigcup_{n=i^3 q_{l_{i-1}}}^{i^3 q_{l_i}} T^{-n}\left(A(x_0, s_0, \frac{1}{i^2 q_{l_i}}, .)\right);$$

clearly, if $(x, s) \in M - \bigcup_{i\geq i_0} \mathcal{A}_{l_i}$, one has $N(x, s, \mathcal{A}) \leq i_0^3 q_{l_{i_0-1}}$. So, as in the proof of Lemma 2 we need to prove that $\mu\left(\bigcup_{i\geq i_0} \mathcal{A}_{l_i}\right)$ goes to zero when i_0 goes to infinity. For this, *it is enough to prove that* $\sum \mu(\mathcal{A}_{l_i})$ *converges*, and the proof of the proposition will be over. (Here μ is the invariant normalized Lebesgue measure for the special flow $\{T^t\}$.)

If we recall the definition of rectangles and bands on M, we have clearly

$$\mathcal{A}_{l_i} \subset \bigcup_{n=0}^{i^3 q_{l_i}} T^{-n}(\Delta(x_0, \frac{1}{i^2 q_{l_i}})),$$

but it is easy to see (Cf. Lemma 2 in [**?**])[2] that because the ceiling function φ satisfies $\varphi > c > 0$ we have

$$\bigcup_{n=0}^{i^3 q_{l_i}} T^{-n}\left(\Delta(x_0, \frac{1}{i^2 q_{l_i}})\right) \subset \bigcup_{n=0}^{\frac{i^3 q_{l_i}}{c}} \Delta\left(R_\alpha^{-n}(x_0), \frac{1}{i^2 q_{l_i}}\right).$$

Now, reasoning exactly as in the proof of Lemma 2 we have

$$\bigcup_{n=0}^{\frac{i^3 q_{l_i}}{c}} \Delta\left(R_\alpha^{-n}(x_0), \frac{1}{i^2 q_{l_i}}\right) \subset \bigcup_{n=0}^{q_{l_i}-1} \Delta\left(R_\alpha^{-n}(x_0), \frac{2}{i^2 q_{l_i}}\right).$$

By definition of μ we have

$$\mu\left(\bigcup_{n=0}^{q_{l_i}-1} \Delta\left(R_\alpha^{-n}(x_0), \frac{2}{i^2 q_{l_i}}\right)\right) = \sum_{n=0}^{q_{l_i}-1} \int_{I\left(R_\alpha^{-n}(x_0), \frac{2}{i^2 q_{l_i}}\right)} \varphi(x)dx;$$

but since φ is of class C^3 except for the singularity at 0, the integral over an interval of the sum above is the larger when this interval is the closer to 0, nevertheless the intervals $I\left(R_\alpha^{-n}(x_0), \frac{2}{i^2 q_{l_i}}\right)$, $n = 0, ..., q_{l_i}-1$, are obviously disjoint. Hence, a very rough but sufficient bound on the integral can be obtained if we suppose all the intervals in the sum are adjacent and cover the interval $[-\frac{2}{i^2}, \frac{2}{i^2}]$. This worst situation is dynamically impossible but the bound we obtain, even if it is not the best one, is sufficient for our purpose, indeed:

$$\sum_{n=0}^{q_{l_i}-1} \int_{I\left(R_\alpha^{-n}(x_0), \frac{2}{i^2 q_{l_i}}\right)} \varphi(x)dx \quad \le \quad \int_{[-\frac{2}{i^2}, \frac{2}{i^2}]} \varphi(x)dx,$$

from the hypothesis on the singularity this leads to

$$\sum_{n=0}^{q_{l_i}-1} \int_{I\left(R_\alpha^{-n}(x_0), \frac{2}{i^2 q_{l_i}}\right)} \varphi(x)dx = O(\frac{1}{i^{2(1-\gamma)}}).$$

In conclusion

$$\mu(\mathcal{A}_{l_i}) = O(\frac{1}{i^{2(1-\gamma)}}),$$

since we assumed[3] $\gamma \le \frac{2}{5}$ the serie converges and the proposition is proved. □

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[2] Since the ceiling function is bounded from below by $c > 0$, we deduce that under the action of the special flow, up to time t, a point x_0 on the basis will have visited at most $\frac{t}{c}$ fibers.

[3] The fact that $\gamma < 1$ is less than 0.5 is not at all essential in the proof of the proposition since we could as well take l_i in (3) such that $|||q_{l_i}\alpha||| \le \frac{1}{10^i q_{l_i}}$,

Finally it is worth noticing while we compare the first point in Lemma 2 and the result of Proposition 2 that the shrinking target property and the mixing property, that can both be interpreted as a strengthening and a quantification of ergodicity, are actually independent.

I wish to thank Dmitry Kleinbock for his interest in this note.

References

[CK] N. Chernov and D. Kleinbock, *Dynamical Borel-Cantelli Lemmas for Gibbs measures*, Israel J. Math. **122** (2001), 1–27.

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and $r_n = \frac{1}{2^i q_{l_i}}$ for any $n \in [i^3 q_{l_{i-1}}, i^3 q_{l_i}]$. In this case we would have at the end $\mu(\mathcal{A}_{l_i}) = O(\frac{1}{2^{i(1-\gamma)}})$, and there will not be any problem of convergence.