Math 211a, Fall 2004, Homework # 4

Van der Waerden's Theorem and Multiple Recurrence

1. [Brin & Stuck, Exercises (2.8.1–3), 2.8.4].

2. Give an example of a finite partition $\mathbb{Z} = \cup_{i=1}^{r} A_i$ such that none of the sets A_i contains an infinite arithmetic progression.

3. Say that $A \subset \mathbb{Z}$ is *AP-rich* if it contains arbitrary long arithmetic progressions. Prove that syndetic sets are AP-rich.

4. Let $A = \cup_{i=1}^{r} A_i$ be a finite partition of an AP-rich set $A \subset \mathbb{Z}$; show that one of the sets A_i is AP-rich. (This is clearly an equivalent version of Van der Waerden's Theorem.)

5. Construct explicitly a sequence $\omega \in \{0,1\}^{\mathbb{N}}$ which is recurrent both for the left shift σ and for its square σ^2, but is not doubly recurrent for σ.

6. Let (X, f) be a topological dynamical system, and let a closed $D \subset X$ be homogeneous and *weakly recurrent* for f (that is, $\forall \varepsilon > 0\ \exists x, y \in D$ and $n \in \mathbb{N}$ with $d\big(x, f^n(y)\big) < \varepsilon$). Show that D contains a dense set of recurrent points. (Equivalently, if $G = \langle f_1, \ldots, f_\ell \rangle$ is commutative and acts minimally on X, then there is a dense set of multiply recurrent points.)