Math 211a, Fall 2004, Homework # 5

More about Topological Recurrence, Transitivity, Mixing

In all of the problems below, a *dynamical system* (X, f) is a continuous map f of a compact metric space X with distance function d.

1. [Brin & Stuck, Exercises (2.2.1–3) and (2.3.1–4)].

2. [Brin & Stuck, Exercise 2.2.6]. (Hint: assume that there exists a proper minimal subset, and consider its intersections with fibers...)

3. Suppose that X has no isolated points and f is topologically transitive. Prove that there exists a dense G_δ set of points with dense orbits (G_δ stands for countable intersections of open sets).

Recall that $A \subset \mathbb{N}$ is called a *set of topological recurrence* (abbreviated by STR) if for any dynamical system (X, f) and any $\varepsilon > 0$ there exist $x \in X$ and $n \in A$ with $d\big(x, f^n(x)\big) < \varepsilon$. (Equivalently, for any finite coloring $\mathbb{N} = \bigcup_{i=1}^r A_i$ there exists a color $c \in \{1, \dots, r\}$ such that the difference between two elements of A_c is in A.)

4. Let $A = \bigcup_{i=1}^r A_i$ be a finite partition of STR $A \subset \mathbb{N}$; show that at least one of the sets A_i is STR.

5. Prove that any STR contains a disjoint union of infinitely many STRs.

6. Let $\{a_n\} \subset \mathbb{N}$ be infinite. Prove that the following sets are STR:

(a) the difference set $\{a_m - a_n \mid m, n \in \mathbb{N},\, a_m > a_n\}$;

(b) the set of the form $\bigcup_{n=1}^\infty \{a_n, \dots, na_n\}$.

7. Let A be STR. Prove that for any $k \in \mathbb{N}$, (a) $A \smallsetminus \{1, \dots, k\}$, (b) kA and (c) $\frac{1}{k}A \stackrel{\text{def}}{=} \{n \in \mathbb{N} \mid kn \in A\}$ are STRs.

8. Prove or disprove: any STR contains (a) a, b, c with $a + b = c$; (b) an arithmetic progression of length 3.

9. Recall that an increasing sequence $\{a_n\} \subset \mathbb{N}$ is called *lacunary* if

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1\,.$$

Prove that a lacunary sequence cannot be STR.

10. Let $\{a_n\} \subset \mathbb{N}$ be "very lacunary", that is, assume that $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 3\,.$ Prove that the set $\big\{\alpha \in \mathbb{R} \bigm| \{a_n\alpha \mod 1\} \text{ is not dense}\big\}$ is uncountable.