Math 211a, Fall 2004, Homework # 6

Topological Entropy, Rotation Numbers

1. [Brin & Stuck, Exercises (2.5.1–5), (2.5.7), (2.6.1–2)].

2. [Brin & Stuck, Exercise 2.5.6].

3. Prove the following generalization of Exercise 2.5.3 and its stronger version used in class: if there exist $k \in \mathbb{N}$ and $L \in \mathbb{R}$ such that $a_{m+n} \le a_m + a_{n+k} + L$ for all $m, n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n/n \in [-\infty, \infty)$ exists.

4. Prove the following strengthening of Exercise 2.5.7: if M is a compact Riemannian manifold of dimension at most n and $f : M \to M$ is Lipschitz with Lipschitz constant C, i.e. $d\big(f(x), f(y)\big) \le C \cdot d(x, y)$ for all $x, y \in M$, then $h(f) \le n \log C$.

5. Construct a dynamical system (X, f) with $h(f) = \infty$.

6. Show that $h(f|_{\mathrm{N}W(f)}) = h(f)$ (see [Brin & Stuck, §2.1] for the definition of the non-wandering set $\mathrm{N}W(f)$ of f).

7. [Brin & Stuck, Exercises (7.1.1–2)].

8. [Brin & Stuck, Exercise 7.1.3].

9. [Brin & Stuck, Exercise 7.1.4].

10. Let $f : S^1 \to S^1$ be a continuous (not necessarily injective) map of degree one, and let $F : \mathbb{R} \to \mathbb{R}$ be its lift.

(a) Prove that

$$\rho_+(F) \overset{\text{def}}{=} \lim_{n\to\infty} \max_x \frac{F^n(x) - x}{n} \qquad \text{and} \qquad \rho_-(F) \overset{\text{def}}{=} \lim_{n\to\infty} \min_x \frac{F^n(x) - x}{n}$$

both exist.

(b) Call

$$\mathcal{R}(F) \overset{\text{def}}{=} \left\{ \alpha \in \mathbb{R} \;\middle|\; \lim_{n\to\infty} \frac{F^n(x) - x}{n} = \alpha \text{ for some } x \in \mathbb{R} \right\}$$

the *rotation set* of F. Prove that $\mathcal{R}(F)$ is non-empty.