

**Math 211a, Fall 2004, Homework # 7**  
*Measure Preserving Transformations, Recurrence*

1. [Brin & Stuck, Exercises 4.1.1, (4.2.2) and 4.2.3].

In the remaining problems,  $(X, \mathcal{B}, \mu, T)$  is a *finite measure-preserving system*, i.e.  $(X, \mathcal{B}, \mu)$  is a measure space,  $\mu(X) < \infty$ , and  $T$  is a measurable self-map of  $X$  preserving  $\mu$ .

2. Prove that  $T$  preserves  $\mu$  (that is,  $\mu(T^{-1}(A)) = \mu(A)$  for all  $A \in \mathcal{B}$ ) if and only if the following equivalent conditions hold:

- (a)  $\mu(T^{-1}(A)) \leq \mu(A)$  for all  $A \in \mathcal{B}$ ;
- (b)  $\mu(T^{-1}(A)) \geq \mu(A)$  for all  $A \in \mathcal{B}$ ;
- (c)  $T(A) \in \mathcal{B}$  and  $\mu(T(A)) \geq \mu(A)$  for all  $A \in \mathcal{B}$ .

3. Suppose that  $\{S_x \mid x \in X\}$  is a family of measure-preserving maps of another measure space  $(Y, \nu)$ . Assume that  $S_x(y)$  is jointly measurable (as a map from  $X \times Y$  to  $Y$ ). Prove that the *skew product* transformation  $\widehat{T}(x, y) \stackrel{\text{def}}{=} (T(x), S_x(y))$  preserves  $\mu \times \nu$ .

4. Let  $f$  be a positive measurable function on  $X$ . Prove that the sum  $\sum_{n=1}^{\infty} f(T^n x)$  is infinite for  $\mu$ -a.e.  $x$ .

Say that  $R \subset \mathbb{N}$  is called a *set of measurable recurrence* (abbreviated by SMR) if for any  $(X, \mathcal{B}, \mu, T)$  and any  $A \in \mathcal{B}$  of positive measure there exists  $n \in R$  with  $\mu(A \cap T^{-n}(A)) > 0$ . (Equivalently, any subset of  $\mathbb{N}$  of positive upper (Banach) density has two elements whose difference is in  $R$ .)

The following are measurable analogues of topological recurrence problems from Homework # 5; feel free to try the new problems regardless of whether or not you have worked on the old ones!

5. Let  $R = \bigcup_{i=1}^r R_i$  be a finite partition of SMR  $R \subset \mathbb{N}$ ; show that at least one of the sets  $R_i$  is SMR.

6. Prove that any SMR contains a disjoint union of infinitely many SMRs.

7. Let  $\{a_n\} \subset \mathbb{N}$  be infinite. Prove that the following sets are SMR:

- (a) the difference set  $\{a_m - a_n \mid m, n \in \mathbb{N}, a_m > a_n\}$ ;
- (b) the set of the form  $\bigcup_{n=1}^{\infty} \{a_n, \dots, na_n\}$ .

8. Let  $R$  be SMR. Prove that for any  $k \in \mathbb{N}$ , (a)  $R \setminus \{1, \dots, k\}$ , (b)  $kR$  and

- (c)  $\frac{1}{k}R \stackrel{\text{def}}{=} \{n \in \mathbb{N} \mid kn \in R\}$  are SMRs.