Math 211a, Fall 2004, Homework # 7

Measure Preserving Transformations, Recurrence

1. [Brin & Stuck, Exercises 4.1.1, (4.2.2) and 4.2.3].

In the remaining problems, $(X, \mathcal{B}, \mu, T)$ is a *finite measure-preserving system*, i.e. $(X, \mathcal{B}, \mu)$ is a measure space, $\mu(X) < \infty$, and T is a measurable self-map of X preserving μ.

2. Prove that T preserves μ (that is, $\mu\big(T^{-1}(A)\big) = \mu(A)$ for all $A \in \mathcal{B}$) if and only if the following equivalent conditions hold:

(a) $\mu\big(T^{-1}(A)\big) \le \mu(A)$ for all $A \in \mathcal{B}$;

(b) $\mu\big(T^{-1}(A)\big) \ge \mu(A)$ for all $A \in \mathcal{B}$;

(c) $T(A) \in \mathcal{B}$ and $\mu\big(T(A)\big) \ge \mu(A)$ for all $A \in \mathcal{B}$.

3. Suppose that $\{S_x \mid x \in X\}$ is a family of measure-preserving maps of another measure space (Y, ν). Assume that $S_x(y)$ is jointly measurable (as a map from $X \times Y$ to Y). Prove that the *skew product* transformation $\widehat{T}(x, y) \stackrel{\text{def}}{=} \big(T(x), S_x(y)\big)$ preserves $\mu \times \nu$.

4. Let f be a positive measurable function on X. Prove that the sum $\sum_{n=1}^{\infty} f(T^n x)$ is infinite for μ-a.e. x.

Say that $R \subset \mathbb{N}$ is called a *set of measurable recurrence* (abbreviated by SMR) if for any $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ of positive measure there exists $n \in R$ with $\mu\big(A \cap T^{-n}(A)\big) > 0$. (Equivalently, any subset of $\mathbb{N}$ of positive upper (Banach) density has two elements whose difference is in R.)

The following are measurable analogues of topological recurrence problems from Homework # 5; feel free to try the new problems regardless of whether or not you have worked on the old ones!

5. Let $R = \bigcup_{i=1}^{r} R_i$ be a finite partition of SMR $R \subset \mathbb{N}$; show that at least one of the sets R_i is SMR.

6. Prove that any SMR contains a disjoint union of infinitely many SMRs.

7. Let $\{a_n\} \subset \mathbb{N}$ be infinite. Prove that the following sets are SMR:

(a) the difference set $\{a_m - a_n \mid m, n \in \mathbb{N},\ a_m > a_n\}$;

(b) the set of the form $\bigcup_{n=1}^{\infty} \{a_n, \ldots, n a_n\}$.

8. Let R be SMR. Prove that for any $k \in \mathbb{N}$, (a) $R \setminus \{1, \ldots, k\}$, (b) kR and (c) $\frac{1}{k}R \stackrel{\text{def}}{=} \{n \in \mathbb{N} \mid kn \in R\}$ are SMRs.