

Math 211a, Spring 2008, Homework # 2

Methods for computing Hausdorff dimension

Problems 4.2, 4.6, 4.9, 9.1, 9.3 from Falconer's book, and also:

6. Show that if $S_1, \dots, S_\ell : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a family of contractions, and its compact invariant set $K = \cup_{i=1}^\ell S_i(K)$ is such that different parts $S_i(K)$ are disjoint from each other, then the Open Set Condition is satisfied.

7. Recall that in class we discussed a Cantor-like set where each interval of stage k had 2^{2^k} children each of length 4^{-2^k} times the length of the parent. We proved that if the intervals are spread out uniformly, both Hausdorff and box dimensions of the limit set are equal to $1/2$. Consider the other extreme case, when all the intervals are bunched up at one of the sides, and compute Hausdorff and box dimensions of the limit set.

8. Modify the proof of full Hausdorff dimension of the set \mathcal{B} of badly approximable real numbers (where a tree-like set was constructed such that every interval had N^2 parents and at most 4 were thrown away) to show that for any $b_1, \dots, b_\ell \in \mathbb{R}$, $\dim(\cap_{i=1}^\ell (\mathcal{B} + b_i)) = 1$.

9. Recall that in class we considered the sets

$$A_p = \{x = 0.x_1x_2\cdots \mid \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n x_i = p\}$$

(here $0 \leq p \leq 1$ and $0.x_1x_2\cdots$ is a binary representation of $x \in [0, 1]$) and proved that $\dim(A_p)$ is equal to the base-2 entropy of (p, q) , that is, to $\frac{1}{\log 2}(p \log \frac{1}{p} + q \log \frac{1}{q})$, where $q = 1 - p$. Now consider

$$\tilde{A}_p = \{x = 0.x_1x_2\cdots \mid \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n x_i \leq p\}$$

and

$$\hat{A}_p = \{x = 0.x_1x_2\cdots \mid \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n x_i \leq p\}.$$

Compute $\dim(\tilde{A}_p)$ and $\dim(\hat{A}_p)$.

10. Show that for $A \subset \mathbb{R}^n$, $C_s(A) > 0$ (see p. 66) if and only if there exists a finite measure (mass distribution in Falconer's terminology) μ on A such that the s -potential ϕ_s , defined in (4.12), is bounded in \mathbb{R}^n .