Math 212b, Spring 2009, Homework # 1

Topological Vector Spaces and Linear Operators

Notation: If E, F are normed spaces, $B(E, F)$ is the space of bounded linear maps from E to F, and $\mathrm{Aut}(E) \subset B(E, E)$ is the group of invertible self-maps of E.

1. Recall that a subset M of a topological vector space is called *bounded* if for any open $U \ni 0$ there exists $C > 0$ such that $M \subset CU$ (compare with the definition of *weakly bounded* sets given in class). Let E be the space of all sequences of real numbers with the topology of coordinate-wise congergence (that is, generated by the family of seminorms $\|x\|_n = |x_n|$ where $x = (x_n)_{n\in\mathbb{N}}$). Show that any bounded open subset of E is empty. Conclude that there is no norm on E which can generate the topology described above.

2. Show that the space $C_c^\infty(\mathbb{R}^n)$ of compactly supported infinitely differentiable functions is dense in $L^\infty(\mathbb{R}^n)$ when the latter is equipped with weak-$*$ topology (recall that $L^\infty(\mathbb{R}^n) = \big(L^1(\mathbb{R}^n)\big)'$).

3. Let $1 \le p, q \le \infty$.

(a) Find the norm of the identity operator mapping $L^p([a, b])$ into $L^q([a, b])$ when $p \ge q$.

(b) Identify, with proof, the set of all measurable functions φ on $[a, b]$ such that the operator of multiplication by φ is in $B\big(L^p([a, b]), L^q([a, b])\big)$.

4. Let E, F be normed spaces and let T be a linear operator from E to F which maps any strongly convergent sequence into a weakly convergent one. Prove that $T \in B(E, F)$.

5. Let E, F be Banach spaces. Show that the subset of $B(E, F)$ consisting of surjective maps is open in $B(E, F)$. [Hint: recall the proof of Open Mapping Theorem.]

6. Let E be a Banach space, let G be a locally compact topological group, and let π be a homomorphism $G \to \mathrm{Aut}(E)$. Show that π is continuous in the strong operator topology (that is, the map $g \mapsto \pi(g)x$ is continuous for every $x \in E$) if and only if the map $G \times E \to E$, $(g, x) \mapsto \pi(g)x$ is continuous. [Hint: Uniform Boundedness Theorem.]