

Math 212b, Spring 2009, Homework # 2

Banach Algebras

All vector spaces are over the field of complex numbers.

1. Let A be a Banach space on which multiplication is defined, with all the Banach algebra axioms satisfied except for

$$(*) \quad \|xy\| \leq \|x\|\|y\|,$$

instead of which it is assumed that the map $(x, y) \mapsto xy$ is continuous. Show that there exists $C > 0$ such that $\|xy\| \leq C\|x\|\|y\|$ for all $x, y \in A$, and therefore the norm is proportional to the one satisfying (*).

2. (Lang, Problem 5 on p. 413) Let A be a normed algebra, not necessarily with a unit element. Define $\tilde{A} = \mathbb{C} \oplus A$, with component-wise addition and multiplication and with the norm $\|(z, x)\| = |z| + \|x\|$, $z \in \mathbb{C}$, $x \in A$. Prove that \tilde{A} is a normed algebra with unit $(1, 0)$, and $A \subset \tilde{A}$ is an ideal.

3. (Lang, Problem 5 on p. 91) Let A be a Banach algebra. Suppose that there is a unit element $I \neq 0$ in A , but that we do not necessarily have $\|I\| = 1$.

(a) Show that $\|I\| \geq 1$.

(b) Define a new norm $||| \cdot |||$ on A by putting $|||x||| \stackrel{\text{def}}{=} \sup_{y \neq 0} \frac{\|xy\|}{\|y\|}$. Show that $||| \cdot |||$ is in fact a norm, and that $|||I||| = 1$.

(c) Prove that A is a Banach algebra under this new norm.

4. Let A be a Banach algebra with unit I and let $T \in A$. Recall that the *resolvent function* R_T of T is given by $R_T(\lambda) \stackrel{\text{def}}{=} (T - \lambda I)^{-1}$, where $\lambda \notin \sigma_T$. For $\lambda, \mu \notin \sigma_T$, prove *Hilbert's Identity*:

$$R_T(\lambda) - R_T(\mu) = (\mu - \lambda)R_T(\lambda)R_T(\mu).$$

5. Let A be a C^* -algebra with unit I . For a self-adjoint $T \in A$, define

$$C(T) \stackrel{\text{def}}{=} (T - iI)(T + iI)^{-1}$$

(the *Cayley transform* of T).

(a) Prove that $C(T)$ is unitary and $1 \notin \sigma_{C(T)}$.

(b) Prove that any unitary $U \in A$ with $1 \notin \sigma_U$ is of the form $C(T)$ for some self-adjoint $T \in A$.

6. (Lang, Problem 2 on p. 412) Let E be a Banach space and let E_1, E_2 be two closed subspaces such that $E = E_1 \oplus E_2$. Let $T \in B(E)$ be such that both E_1 and E_2 are T -invariant, and denote by T_i the restrictions of T to E_i . Show that $\sigma_T = \sigma_{T_1} \cup \sigma_{T_2}$.

7. Let $A = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{C} \right\}$. Identify the maximal ideals (equivalently, the characters) of A .