

Math 212b, Spring 2009, Homework # 3

Compactness and Local Compactness

1. (Lang, Problem 21(c) on p. 48) A topological space X is called *regular* if it is Hausdorff, and if for any closed subset A of X and any $x \notin A$ there exist disjoint open $U \ni x$ and $V \supset A$. Show that every locally compact Hausdorff space is regular. Give an example of a locally compact Hausdorff space which is not normal.

2. Let (X, d) be a compact metric space and let $\varphi : X \rightarrow X$ be such that

$$d(\varphi(x), \varphi(y)) < d(x, y)$$

for all $x, y \in X$. Show that φ has a unique fixed point.

In the next two problems X is a locally compact Hausdorff space, and the goal is to extend Stone-Weierstrass to algebras of continuous functions on X .

3. Let $A \subset C_c(X)$ be a subalgebra of the algebra of real-valued compactly supported continuous functions on X which separates points and is such that for any compact $K \subset X$ there exists $f \in A$ such that $f|_K \equiv 1$. Prove that A is dense in $C_c(X)$.

4. (Lang, Problem 18 on p. 62) Let $C_\infty(X)$ be the algebra of real-valued continuous functions on X vanishing at infinity (that is, for every ε there exists a compact $K \subset X$ such that $|f(x)| < \varepsilon$ if $x \notin K$). Let A be a subalgebra of $C_\infty(X)$ which separates points and with no common zero to all functions in A . Prove that A is dense in $C_\infty(X)$.

The next three problems deal with compact operators.

5. Let T be a map from l^p to itself ($1 \leq p \leq \infty$) given by $T(x_n) = a_n x_n$. Find a condition on the sequence (a_n) equivalent to the compactness of T .

6. Let E, F be Hilbert spaces. Prove that $T \in B(E, F)$ is compact if and only if it can be approximated by a sequence of operators $T_n : E \rightarrow F$ with $\dim(T_n(E)) < \infty$ (the “if” part is supposed to be proved in class).

7. Let $\Omega \subset \mathbb{R}^n$ be open and $X \subset \Omega$ be compact. Show that the restriction mapping $f \mapsto f|_X$ defines a compact operator $BC^{k+1}(\Omega) \rightarrow C^k(X)$ for any $k = 0, 1, \dots$